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by

Neil Hindman and Dona Strauss

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# THE CENTER AND EXTENDED CENTER OF THE MAXIMAL GROUPS IN THE SMALLEST IDEAL OF $\beta\mathbb{N}$

NEIL HINDMAN AND DONA STRAUSS

Abstract. A good deal is known about the maximal groups in the smallest ideal $K(\beta\mathbb{N})$ of the compact right topological semigroup $(\beta\mathbb{N}, +)$. For example they are pairwise isomorphic and highly noncommutative – they contain a copy of the free group on $2^{\mathfrak{c}}$ generators. If $q$ is an idempotent in $K(\beta\mathbb{N})$, then $\mathbb{Z}+q$ is contained in the center of the maximal group $q+\beta\mathbb{N}+q$. We do not know whether that center is equal to $\mathbb{Z}+q$. In this paper we investigate the center of $q+\beta\mathbb{N}+q$ and the *extended center* consisting of all elements of $\beta\mathbb{N}$ that commute with every element of $q+\beta\mathbb{N}+q$. This extended center trivially includes all idempotents $r$ of $\beta\mathbb{N}$ such that $q \leq r$, as well as elements of the form $n+r$ for such $r$ and for $n \in \mathbb{Z}$. We show, for example, that if those are the only elements of the extended center, then there are no nontrivial continuous homomorphisms from $\beta\mathbb{N}$ to $\beta\mathbb{N}\setminus\mathbb{N}$. This would answer a long standing open question. We include several other open questions.

## 1. Introduction

Addition on the set $\mathbb{N}$ of positive integers extends to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ making $(\beta\mathbb{N}, +)$ a right topological semigroup (meaning that for each $p \in \beta\mathbb{N}$, the function $\rho_p : \beta\mathbb{N} \to \beta\mathbb{N}$ is continuous, where $\rho_p(q) = q + p$) with $\mathbb{N}$ contained in its topological center (meaning that for each $n \in \mathbb{N}$, the function $\lambda_n : \beta\mathbb{N} \to \beta\mathbb{N}$ is continuous,

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where $\lambda_n(q) = n + q$). As with any compact Hausdorff right topological semigroup, $(\beta\mathbb{N}, +)$ has a smallest two sided ideal

$$\begin{aligned} K(\beta\mathbb{N}) &= \bigcup\{L : L \text{ is a minimal left ideal of } \beta\mathbb{N}\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } \beta\mathbb{N}\}\,. \end{aligned}$$

Any left ideal contains a minimal left ideal, which is closed, and any right ideal contains a minimal right ideal. If $L$ is a minimal left ideal and $R$ is a minimal right ideal, then $L \cap R$ is a group and $L \cap R = q + \beta\mathbb{N} + q$ where $q$ is the unique idempotent in $L \cap R$. Any two such groups are isomorphic. If $q$ and $r$ are idempotents in the same minimal right ideal, then the restriction of $\rho_r$ to $q + \beta\mathbb{N} + q$ is an isomorphism and a homeomorphism onto $r + \beta\mathbb{N} + r$.

The facts just mentioned about $K(\beta\mathbb{N})$ are true in any compact Hausdorff right topological semigroup. Many additional facts are known about $K(\beta\mathbb{N})$ that do not hold in all such semigroups. We know, for example, that there are $2^{\mathfrak{c}}$ minimal right ideals and $2^{\mathfrak{c}}$ minimal left ideals and the maximal groups in $K(\beta\mathbb{N})$ each contain a copy of the free group on $2^{\mathfrak{c}}$ generators. We also know that the center of $\beta\mathbb{Z}$ is $\mathbb{Z}$ so if $q$ is an idempotent in $K(\beta\mathbb{N})$, then $\mathbb{Z} + q$ is contained in the center of $q + \beta\mathbb{N} + q$. We do not know whether the center of $q + \beta\mathbb{N} + q$ is equal to $\mathbb{Z} + q$. It is this question which is the primary focus of this paper.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. Given $A \subseteq \mathbb{N}$, the closure $\overline{A}$ of $A$ in $\beta\mathbb{N}$ is $\{p \in \beta\mathbb{N} : A \in p\}$ and $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a basis for the open sets of $\beta\mathbb{N}$. See [4] for an elementary introduction to the topology and the algebraic structure of $\beta S$ where $S$ is an infinite discrete semigroup, as well as for proofs of all of the facts mentioned in the paragraphs above. (The original references for these facts are [1], [2], [3], [5], [6], and [7].)

**Definition 1.1.** Let $q$ be an idempotent in $K(\beta\mathbb{N})$. $G_q = q + \beta\mathbb{N} + q$ and $D_q = \{u \in \mathbb{N}^* : (\forall v \in G_q)(u + v = v + u)\}$.

Of course, the center $Z(G_q) = D_q \cap G_q$. We call $D_q$ the *extended center* of $G_q$.

**Definition 1.2.** (1) $I = \bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$.
(2) $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$.
(3) For $A \subseteq \beta\mathbb{N}$, $E(A) = \{q \in \beta\mathbb{N} : q + q = q\}$.

In section 2 of this paper we present some basic results, including the fact that for any $q \in E(\beta\mathbb{N})$, $D_q \subseteq \mathbb{Z} + I$.

In section 3 we investigate more deeply the structure of $D_q$, establishing the fact that either there is no nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, or there is a member of $D_q \cap I$ which is not an idempotent. We also include in this section a proof that $D_q$ contains a decreasing sequence of idempotents of order type $(\omega+1)^*$, that is, the reverse of $\omega+1$.

Section 4 consists primarily of a derivation of the fact that if the center of $G_q$ is not trivial, then $G_q$ contains a copy of $\mathbb{Z}\times\mathbb{Z}$.

## 2. Basic Facts about the Extended Center

We begin by observing that the only elements in the extended center that are not in the center lie outside of the smallest ideal.

**Theorem 2.1.** *Let $q\in E\big(K(\beta\mathbb{N})\big)$. Then $Z(G_q)=D_q\cap K(\beta\mathbb{N})$.*

*Proof.* Trivially $Z(G_q)\subseteq D_q\cap K(\beta\mathbb{N})$. For the reverse inclusion, let $x\in D_q\cap K(\beta\mathbb{N})$. Since $q\in K(\beta\mathbb{N})$, $\beta\mathbb{N}+q$ is a minimal left ideal and $q+\beta\mathbb{N}$ is a minimal right ideal, so $G_q=(\beta\mathbb{N}+q)\cap(q+\beta\mathbb{N})$. Thus, if $x\notin G_q$, then either $x\notin\beta\mathbb{N}+q$ or $x\notin q+\beta\mathbb{N}$. So either $x$ and $q$ are in different minimal left ideals of $\beta\mathbb{N}$ or $x$ and $q$ are in different minimal right ideals of $\beta\mathbb{N}$. In either case, $x+q\neq q+x$. □

The idempotents of $\beta\mathbb{N}$ are partially ordered by the relation $\leq$, defined by $e\leq f$ if and only if $e=e+f=f+e$. By [4, Theorem 2.9], $e$ is minimal with respect to this order if and only if $e\in K(\beta\mathbb{N})$. Further, given any non minimal idempotent $e$ in $\beta\mathbb{N}$, by [4, Theorem 1.60], there is a minimal idempotent $q\in K(\beta\mathbb{N})$ with $q\leq e$. So the following lemma shows that for at least some $q\in K(\beta\mathbb{N})$, $D_q\neq Z(G_q)$.

**Lemma 2.2.** *Let $q\in E\big(K(\beta\mathbb{N})\big)$. Then $\{e\in E(\beta\mathbb{N}):q\leq e\}=E(D_q)$.*

*Proof.* Let $e\in E(\beta\mathbb{N})$ such that $q\leq e$ and let $x\in G_q$. Then $e+x=e+q+x=q+x=x+q=x+q+e=x+e$ and so $e\in D_q$.

Conversely, let $e\in E(D_q)$. Since $e+q=q+e$, $e+q$ is an idempotent in the same minimal right ideal and the same minimal left ideal as $q$. So $e+q=q+e=q$ and $q\leq e$. □

The remainder of this section will be devoted to a proof, as a consequence of a more general theorem, that $D_q\subseteq\mathbb{Z}+I$.

It is well known, and routine to verify, that each member of $\mathbb{N}$ has a unique factorial representation, that is, a representation of the form $\sum_{n\in H}a_n\cdot n!$ where $H$ is a finite nonempty subset of $\mathbb{N}$ and for each $n\in H$, $a_n\in\{1,2,\ldots,n\}$.

**Definition 2.3.** Define $d:\mathbb{N}\to\times_{n=1}^{\infty}\{0,1,\ldots,n\}$ by $y=\sum_{n=1}^{\infty}d(y)(n)\cdot n!$ for each $y\in\mathbb{N}$. For $y\in\mathbb{N}$, let $\operatorname{supp}_f(y)=\{n\in\mathbb{N}:d(y)(n)\neq 0\}$ and let $c(y)=|\operatorname{supp}_f(y)|$. Let $\widetilde{d}:\beta\mathbb{N}\to\times_{n=1}^{\infty}\{0,1,\ldots,n\}$ and $\widetilde{c}:\beta\mathbb{N}\to\beta\mathbb{N}$ be the continuous extensions of $d$ and $c$, respectively.

**Lemma 2.4.** *Let $x \in \mathbb{N}^*$. If $x \notin \mathbb{Z} + I$, then*

(1) $\{n \in \mathbb{N} : \widetilde{d}(x)(n) \neq 0\}$ *is infinite,*
(2) $\{n \in \mathbb{N} : \widetilde{d}(x)(n) < n\}$ *is infinite, and*
(3) $\{n \in \mathbb{N} :$ *either* $0 < \widetilde{d}(x)(n) < n$ *or both* $\widetilde{d}(x)(n) = n$ *and* $\widetilde{d}(x)(n+1) = 0\}$ *is infinite.*

*Proof.* (1) Suppose that $\{n \in \mathbb{N} : \widetilde{d}(x)(n) \neq 0\}$ is finite and let $k = \max\{n \in \mathbb{N} : \widetilde{d}(x)(n) \neq 0\}$. Let $m = \sum_{n=1}^{k} \widetilde{d}(x)(n) \cdot n!$. We claim that $x \in m + I$. To see this, let $l > k$. To see that $x \in -m + \mathbb{N}l!$, pick $A \in x$ such that $\widetilde{d}(x)[\overline{A}\,] \subseteq \times_{n=1}^{l} \pi_n^{-1}[\{\widetilde{d}(x)(n)\}]$ and let $y \in A$. Pick $j > l$ such that $j! > y$. Then

$$\begin{aligned} y - m &= \textstyle\sum_{n=l+1}^{j} d(y)(n) \cdot n! + \sum_{n=1}^{l} \widetilde{d}(x)(n) \cdot n! - \sum_{n=1}^{k} \widetilde{d}(x)(n) \cdot n! \\ &= \textstyle\sum_{n=l+1}^{j} d(y)(n) \cdot n! \end{aligned}$$

since $\widetilde{d}(x)(n) \cdot n! = 0$ for $n > k$.

(2) Suppose that $\{n \in \mathbb{N} : \widetilde{d}(x)(n) < n\}$ is finite and pick $k \in \mathbb{N}$ such that for all $n > k$, $\widetilde{d}(x)(n) = n$. Let $m = 1 + \sum_{n=1}^{k} \big(n - \widetilde{d}(x)(n)\big) \cdot n!$. We claim that $x \in -m + I$. To see this, let $l > k$. To see that $x \in m + \mathbb{N}l!$, pick $A \in x$ such that $\widetilde{d}(x)[\overline{A}\,] \subseteq \times_{n=1}^{l} \pi_n^{-1}[\{\widetilde{d}(x)(n)\}]$ and let $y \in A$. Pick $j > l$ such that $j! > y$. Then $m + y - 1 = \sum_{n=1}^{k} \big(n - \widetilde{d}(x)(n)\big) \cdot n! + \sum_{n=1}^{l} \widetilde{d}(x)(n) \cdot n! + \sum_{n=l+1}^{j} d(y)(n) \cdot n! = \sum_{n=l+1}^{j} d(y)(n) \cdot n! + \sum_{n=1}^{l} n \cdot n!$ and so $(l+1)!$ divides $m + y$.

(3) Assume that $\{n \in \mathbb{N} : 0 < \widetilde{d}(x)(n) < n\}$ is finite. Pick $k$ such that for all $n > k$, $\widetilde{d}(x)(n) \in \{0, n\}$. Then by (1) and (2), both $\{n \in \mathbb{N} : \widetilde{d}(x)(n) = 0\}$ and $\{n \in \mathbb{N} : \widetilde{d}(x)(n) = n\}$ are infinite, and consequently $\{n \in \mathbb{N} : \widetilde{d}(x)(n) = n$ and $\widetilde{d}(x)(n+1) = 0\}$ is infinite. □

**Lemma 2.5.** *Let $x \in \mathbb{N}^*$ and let $q, r \in I$. If $q + x + r \in \mathbb{Z} + I$, then $x \in \mathbb{Z} + I$.*

*Proof.* Assume that $m \in \mathbb{Z}$, $z \in I$, and $q + x + r = m + z$. Given any $n \in \mathbb{N}$, $\{k \in \mathbb{N} : k \equiv m \pmod{n!}\} \in q + x + r$ and $\{k \in \mathbb{N} : k \equiv 0 \pmod{n!}\} \in q \cap r$, so $\{k \in \mathbb{N} : k \equiv m \pmod{n!}\} \in x$. □

**Lemma 2.6.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$ and let $x \in Z(G_q)$. Then $x \in \mathbb{Z} + I$.*

*Proof.* Suppose that $x \notin \mathbb{Z} + I$ and let

$$\begin{aligned} E = \{n \in \mathbb{N} : \ & \text{either } 0 < \widetilde{d}(x)(n) < n \\ & \text{or both } \widetilde{d}(x)(n) = n \text{ and } \widetilde{d}(x)(n+1) = 0\}\,. \end{aligned}$$

Then, by Lemma 2.4(3), $E$ is infinite, so pick $p \in \mathbb{N}^*$ such that $\{n! : n \in E\} \in p$. We shall show that

(a) $\widetilde{c}(x+p+q) = \widetilde{c}(x+q)+1$ and
(b) $\widetilde{c}(q+p+x) = \widetilde{c}(q+x)$.

This will suffice because (using the fact that $x+q = q+x = x$) we will then have that $\widetilde{c}(x+q+p+q) = \widetilde{c}(x+p+q) = \widetilde{c}(x+q)+1 = \widetilde{c}(x)+1 \neq \widetilde{c}(x) = \widetilde{c}(q+x) = \widetilde{c}(q+p+x) = \widetilde{c}(q+p+q+x)$.

To verify (a), it suffices that $\widetilde{c} \circ \rho_{p+q}$ and $\rho_1 \circ \widetilde{c} \circ \rho_q$ agree on $\mathbb{N}$, so let $y \in \mathbb{N}$ be given. To see that $\widetilde{c}(y+p+q) = \widetilde{c}(y+q)+1$, it suffices that $\widetilde{c} \circ \lambda_y \circ \rho_q$ is constantly equal to $\widetilde{c}(y+q)+1$ on $\{n! : n \in \mathbb{N}$ and $n > \max \operatorname{supp}_f(y)\}$, so let $n \in \mathbb{N}$ such that $n > \max \operatorname{supp}_f(y)$. To see that $\widetilde{c}(y+n!+q) = \widetilde{c}(y+q)+1$, it suffices that $\widetilde{c} \circ \lambda_{y+n!}$ and $\rho_1 \circ \widetilde{c} \circ \lambda_y$ agree on $\mathbb{N}(n+1)!$, so let $z \in \mathbb{N}(n+1)!$. Then $\operatorname{supp}_f(y+n!+z) = \operatorname{supp}_f(y) \cup \{n\} \cup \operatorname{supp}_f(z)$ and $\operatorname{supp}_f(y+z) = \operatorname{supp}_f(y) \cup \operatorname{supp}_f(z)$.

To verify (b), it suffices that $\widetilde{c} \circ \rho_{p+x}$ and $\widetilde{c} \circ \rho_x$ agree on $\mathbb{N}$, so let $y \in \mathbb{N}$ and pick $a \in E$ such that $a > \max \operatorname{supp}_f(y)$. To see that $\widetilde{c}(y+p+x) = \widetilde{c}(y+x)$, it suffices that $\widetilde{c} \circ \lambda_y \circ \rho_x$ is constantly equal to $\widetilde{c}(y+x)$ on $\{n! : n \in E$ and $n > a+2\}$ so let $b \in E$ such that $b > a+2$. Pick $A \in x$ such that $\widetilde{d}[\overline{A}] \subseteq \bigcap_{n=1}^{b+1} \pi_n^{-1}[\{\widetilde{d}(x)(n)\}]$. To see that $\widetilde{c}(y+b!+x) = \widetilde{c}(y+x)$, it suffices that $\widetilde{c} \circ \lambda_{y+b!}$ agrees with $\widetilde{c} \circ \lambda_y$ on $A$, so let $z \in A$. We need to show that $c(y+b!+z) = c(y+z)$.

Pick $l > b+1$ such that $l! > z$. Then $z = \sum_{n=b+2}^{l} d(z)(n) \cdot n! + \sum_{n=1}^{b+1} \widetilde{d}(x)(n) \cdot n!$. Since $b > a+2 > a > \max \operatorname{supp}_f(y)$, there is no carrying beyond position $a+1$ when the factorial representations of $z$ and $y$ are added. (Either $\widetilde{d}(x)(a) < n$, in which case there is no carrying beyond position $a$, or $\widetilde{d}(x)(a) = n$ and $\widetilde{d}(x)(a+1) = 0$.) Thus, $z+y = \sum_{n=b+2}^{l} d(z)(n) \cdot n! + \sum_{n=a+2}^{b+1} \widetilde{d}(x)(n) \cdot n! + \sum_{n=1}^{a+1} d(z+y)(n) \cdot n!$.

Assume first that $0 < \widetilde{d}(x)(b) < n$. Then

$$y+b!+z = \sum_{n=b+2}^{l} d(z)(n) \cdot n! + \widetilde{d}(x)(b+1) \cdot (b+1)! + (\widetilde{d}(x)(b)+1) \cdot b! + \sum_{n=a+2}^{b-1} \widetilde{d}(x)(n) \cdot n! + \sum_{n=1}^{a+1} d(z+y)(n) \cdot n!\,;$$

so $\operatorname{supp}_f(y+b!+z) = \operatorname{supp}_f(y+z)$.

Now assume that $\widetilde{d}(x)(b) = n$ and $\widetilde{d}(x)(b+1) = 0$. Then $y+b!+z = \sum_{n=b+2}^{l} d(z)(n) \cdot n! + (b+1)! + \sum_{n=a+2}^{b-1} \widetilde{d}(x)(n) \cdot n! + \sum_{n=1}^{a+1} d(z+y)(n) \cdot n!\,;$ so $\operatorname{supp}_f(y+b!+z) = (\operatorname{supp}_f(y+z) \setminus \{b\}) \cup \{b+1\}$. □

**Theorem 2.7.** *Let $u, v, x \in \beta\mathbb{N}$. If $x$ commutes with every member of $u+\beta\mathbb{N}+v$, then either $x \in \mathbb{N}$ or $x \in \mathbb{Z}+I$.*

*Proof.* Assume that $x \notin \mathbb{N}$. Pick a minimal right ideal $R$ and a minimal left ideal $L$ of $\beta\mathbb{N}$ such that $R \subseteq u+\beta\mathbb{N}$ and $L \subseteq \beta\mathbb{N}+v$. Let $q$ be the identity of $R \cap L$. Then $q+\beta\mathbb{N}+q \subseteq u+\beta\mathbb{N}+v$ and so $x$ commutes with every member of $q + \beta\mathbb{N} + q$, and consequently so does $q + x + q$. (Let $p \in q + \beta\mathbb{N} + q$. Then $p + q = q + p = p$ so $q + x + q + p = q + x + p + q = q + p + x + q = p + q + x + q$.) By Lemma 2.6, $q + x + q \in \mathbb{Z} + I$ and so by Lemma 2.5, $x \in \mathbb{Z} + I$. □

The following corollary is an immediate consequence of Theorem 2.7.

**Corollary 2.8.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $D_q \subseteq \mathbb{Z} + I$.*

## 3. The Structure of the Extended Center

Section 1.7 of [4] has a large number of results whose hypothesis asserts the existence of a minimal left ideal with an idempotent. The following theorem puts all of those results at our disposal.

**Theorem 3.1.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $D_q$ is a semigroup and $D_q \cap G_q$ is both a minimal left ideal of $D_q$ with an idempotent and a minimal right ideal of $D_q$.*

*Proof.* Trivially, $D_q$ is a semigroup. To see that $D_q \cap G_q$ is an ideal of $D_q$, let $x \in D_q \cap G_q$ and let $y \in D_q$. Then $y + x = y + q + x = q + y + x = q + y + x + q \in G_q$ and $x + y = x + q + y = x + y + q = q + x + y + q \in G_q$. Since $D_q \cap G_q = Z(G_q)$, $D_q \cap G_q$ is a group and is therefore a minimal left ideal and a minimal right ideal and has an idempotent. □

Among the consequences of the existence of a minimal left ideal in a semigroup is the fact that the smallest ideal exists.

**Corollary 3.2.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $D_q \cap G_q = K(D_q)$.*

*Proof.* Since $K(D_q)$ is the union of all of the minimal left ideals of $D_q$, by Theorem 3.1, we have that $D_q \cap G_q \subseteq K(D_q)$. Since $D_q \cap G_q$ is a two sided ideal of $D_q$, we have that $K(D_q) \subseteq D_q \cap G_q$. □

**Lemma 3.3.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$ and let $x \in D_q$. There is some $y \in D_q \cap G_q$ such that $y + x = x + y = q$.*

*Proof.* Since $D_q$ contains a minimal left ideal with an idempotent, by [4, Corollary 1.47 and Theorem 1.56], we have that $D_q + x$ contains a minimal left ideal of $D_q$ with an idempotent, and this idempotent is in $K(D_q) = D_q \cap G_q$. Since $q$ is the only idempotent in $G_q$, $q \in D_q + x$. Pick $w \in D_q$ such that $q = w + x$. Let $y = q + w + q$. Then $y \in D_q \cap G_q$. Also, $y + x = q + w + q + x = q + w + x + q = q + q + q = q$. Since $y \in D_q$, we also have that $x + y = q$. □

**Theorem 3.4.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$ and let $x \in D_q$. Then $G_q = x + G_q + x = x + G_q = G_q + x$.*

*Proof.* Pick, by Lemma 3.3, some $y \in D_q \cap G_q$ such that $y+x = x+y = q$. Since $x \in D_q$, we have that $x + G_q = G_q + x$. We shall show that $G_q \subseteq x + G_q + x \subseteq x + G_q \subseteq G_q$. Let $w \in G_q$. Then $w = q + w + q = x + y + w + y + x \in x + G_q + x$.

To see that $x + G_q + x \subseteq x + G_q$, let $w \in G_q$ and pick $z \in G_q$ such that $z + y = w$. Then $x + w + x = x + z + y + x = x + z + q = x + z$.

To see that $x + G_q \subseteq G_q$, let $w \in G_q$, and pick $z \in G_q$ such that $y + z = w$. Then $x + w = x + y + z = q + z = z$. □

**Corollary 3.5.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$. For any distinct $x, y \in D_q$, $x \in \beta\mathbb{N} + y$ or $y \in \beta\mathbb{N} + x$.*

*Proof.* By Theorem 3.4, $q \in (G_q + x) \cap (G_q + y)$. So our claim follows from [4, Corollary 6.21]. □

**Corollary 3.6.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$. If $M$ is a $G_\delta$ subset of $\mathbb{N}^*$, then $D_q \cap M$ is nowhere dense in $M$. In particular, $D_q \cap I$ is nowhere dense in $I$ and $D_q \cap \mathbb{H}$ is nowhere dense in $\mathbb{H}$.*

*Proof.* We first observe that $\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$ contains a dense open subset $U$ of $\mathbb{N}^*$. This follows from the fact that, if $p \in \mathbb{N}^*$ and $B \in p$, we can choose a sequence $\langle x_n \rangle_{n=1}^{\infty}$ contained in $B$ such that $x_{n+1} - x_n > n$ for every $n \in \mathbb{N}$. So, if $A = \{x_n : n \in \mathbb{N}\}$, $\overline{A} \subseteq \overline{B}$ and, by [4, Exercise 4.1.7], $\overline{A} \cap (\mathbb{N}^* + \mathbb{N}^*) = \emptyset$.

If there exists an element $x \in D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*))$, then, by Corollary 3.5, for any element $y \in D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*))$, $x \in \mathbb{N} + y$ or $y \in \mathbb{N} + x$. So $D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)) = \mathbb{Z} + x$. Since $\mathbb{Z} + x$ is countable, $\mathbb{Z} + x$ is nowhere dense in $\mathbb{N}^*$ [4, Corollary 3.37]. Put $V = U$ if no such element $x$ exists; otherwise, put $V = U \setminus cl(\mathbb{Z} + x)$. Then $V$ is a dense open subset of $\mathbb{N}^*$ and $V \cap D_q = \emptyset$.

It follows from [4, Theorem 3.36] that $\text{int}_{\mathbb{N}^*}(M)$ is dense in $M$. So $V \cap M$ is a dense open subset of $M$ disjoint from $D_q$. □

Given $q \in E\big(K(\beta\mathbb{N})\big)$, we know $\big(\mathbb{Z} + E(D_q)\big) \subseteq D_q$ and we know, by Lemma 2.2, that $E(D_q) = \{e \in E(\beta\mathbb{N}) : q \leq e\}$. So the only things that we know are in $D_q \cap I$ are the idempotents above $q$. It is a longstanding open problem as to whether there are any nontrivial continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$. The following theorem connects our lack of knowledge about these two issues.

**Theorem 3.7.** *If for all $q \in E\big(K(\beta\mathbb{N})\big)$, $D_q \cap I \subseteq E(\beta\mathbb{N})$, then there is no nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$.*

*Proof.* Let $q \in E\big(K(\beta\mathbb{N})\big)$ and suppose that $D_q \cap I \subseteq E(\beta\mathbb{N})$ and that there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$. By [4, Corollary 10.20], pick $e \in E(\beta\mathbb{N})$ and $p \neq e$ such that $p + p = p + e =$

$e+p=e$. Pick $q \in E\big(K(\beta\mathbb{N})\big)$ such that $q \leq e$. Then $p+q=p+e+q=e+q=q=q+e=q+e+p=q+p$. We claim that $p \in D_q$, so let $w \in G_q$. Then $p+w=p+q+w=q+w=w=w+q=w+q+p=w+p$. Thus, $p \in D_q$. Since $p+q=q$, we also have that $p \in I$. But $p$ is not an idempotent. □

We do not know whether there are any maximal idempotents in $\beta\mathbb{N}$ or even whether there are maximal idempotents in $K(\beta\mathbb{N})$, so, as far as we know, it is possible that, for some $q \in E\big(K(\beta\mathbb{N})\big)$, the extended center $D_q$ of $G_q$ is equal to the center of $G_q$. We shall show in Theorem 3.9 that for many $q \in E\big(K(\beta\mathbb{N})\big)$, $D_q \cap c\ell K(\beta\mathbb{N})$ contains an infinite decreasing chain of idempotents. As a corollary, we obtain the fact that $c\ell K(\beta\mathbb{N})$ contains a decreasing sequence of idempotents of reverse order type $\omega+1$. (It was previously known that it contains such a sequence of reverse order type $\omega$.)

**Lemma 3.8.** *Let $R$ be a minimal right ideal of $\beta\mathbb{N}$. There is an injective sequence $\langle q_n\rangle_{n=1}^{\infty}$ of idempotents in $R$ such that if $p$ is an accumulation point of $\langle q_n\rangle_{n=1}^{\infty}$, then $p \notin \mathbb{Z}^*+\mathbb{Z}^*$. In particular, any accumulation point of $\langle q_n\rangle_{n=1}^{\infty}$ is right cancelable in $\beta\mathbb{Z}$.*

*Proof.* Pick an injective sequence $\langle v_n\rangle_{n=1}^{\infty}$ in

$$\{2^n : n \in \mathbb{N}\}^* = \overline{\{2^n : n \in \mathbb{N}\}} \setminus \mathbb{N}\,.$$

We claim that $(\beta\mathbb{N}+v_n)\cap(\beta\mathbb{N}+v_m)=\emptyset$ when $n \neq m$. To see this, define $\phi : \mathbb{N} \to \omega$ by $\phi(n) = \max\big(\mathrm{supp}(n)\big)$, where $\mathrm{supp}(n)$ is the binary support of $n$, that is, $n=\sum_{t\in \mathrm{supp}(n)} 2^t$. Let $\widetilde{\phi} : \beta\mathbb{N} \to \beta\omega$ be the continuous extension of $\phi$. By [4, Exercise 3.4.1], $\widetilde{\phi}$ is injective on $\overline{\{2^n : n \in \mathbb{N}\}}$ and by [4, Lemma 6.8], for each $n \in \mathbb{N}$, $\widetilde{\phi}[\beta\mathbb{N}+v_n] = \{\,\widetilde{\phi}(v_n)\,\}$, so the claim is established.

For each $n \in \mathbb{N}$ choose an idempotent $q_n \in R\cap(\beta\mathbb{N}+v_n)$ and note that $q_n \neq q_m$ if $n \neq m$. Let $p$ be an accumulation point of $\langle q_n\rangle_{n=1}^{\infty}$ and suppose that $p=x+y$ for some $x,y \in \mathbb{Z}^*$. By [4, Exercise 4.3.5], $y \in \mathbb{N}^*$. Note that there is at most one $n \in \mathbb{Z}$ such that $n+y \in \mathbb{H}$. (If $n<m$ and $2^k>m-n$, then $(-n+2^k\mathbb{N})\cap(-m+2^k\mathbb{N})=\emptyset$.) Let $X=\{n \in \mathbb{Z} : n+y \notin \mathbb{H}\}$. Then $X \in x$. If $n \neq m$, we have that $\widetilde{\phi}(q_n)=\widetilde{\phi}(v_n) \neq \widetilde{\phi}(v_m)=\widetilde{\phi}(q_m)$, so there are at most three values of $n \in \mathbb{N}$ for which

$$\widetilde{\phi}(q_n) \in \{\widetilde{\phi}(y)-1, \widetilde{\phi}(y), \widetilde{\phi}(y)+1\}\,.$$

Let $M=\big\{n \in \mathbb{N} : \widetilde{\phi}(q_n) \notin \{\widetilde{\phi}(y)-1, \widetilde{\phi}(y), \widetilde{\phi}(y)+1\}\big\}$. Then

$$p \in c\ell\{q_n : n \in M\} \cap c\ell(X+y);$$

so, by [4, Theorem 3.40], either there is some $n \in X$ such that $n+y \in c\ell\{q_n : n \in M\}$ or there is some $n \in M$ such that $q_n \in c\ell(X+y)=\overline{X}+y$.

Suppose first that we have $n \in X$ such that $n + y \in c\ell\{q_n : n \in M\}$. By [4, Lemma 6.8], $\{q_n : n \in M\} \subseteq \mathbb{H}$, so $n + y \in \mathbb{H}$, contradicting the fact that $n \in X$.

Now assume that we have $n \in M$ such that $q_n \in \overline{X} + y$ and pick $z \in \overline{X}$ such that $q_n = z + y$. Then $\widetilde{\phi}(q_n) \notin \{\widetilde{\phi}(y) - 1\widetilde{\phi}(y), \widetilde{\phi}(y) + 1\}$, so pick $A \in \widetilde{\phi}(q_n)$ such that $\mathbb{N} \setminus A \in \widetilde{\phi}(y) - 1$, $\mathbb{N} \setminus A \in \widetilde{\phi}(y)$, and $\mathbb{N} \setminus A \in \widetilde{\phi}(y) + 1$. Pick $B \in z$ such that $\widetilde{\phi}[\,\overline{B} + y] \subseteq \overline{A}$ and pick $k \in B$. Then $\widetilde{\phi}(k + y) \in \overline{A}$, so pick $C \in y$ such that $\widetilde{\phi}[k + \overline{C}\,] \subseteq \overline{A}$. Since $\mathbb{N} \setminus A \in \widetilde{\phi}(y) - 1$, $\mathbb{N} \setminus A \in \widetilde{\phi}(y)$, and $\mathbb{N} \setminus A \in \widetilde{\phi}(y) + 1$, pick $D \in y$ such that $\widetilde{\phi}[\,\overline{D}\,] - 1 \subseteq \overline{\mathbb{N} \setminus A}$, $\widetilde{\phi}[\,\overline{D}\,] \subseteq \overline{\mathbb{N} \setminus A}$, and $\widetilde{\phi}[\,\overline{D}\,] + 1 \subseteq \overline{\mathbb{N} \setminus A}$. Pick $r \in C \cap D$ such that $r > k$. Then $\phi(k + r) = \phi(r) - 1$, $\phi(k + r) = \phi(r)$, or $\phi(k + r) = \phi(r) + 1$. Since $\phi(k + r) \in A$, this says that $\phi(r) - 1 \in A$, $\phi(r) \in A$, or $\phi(r) + 1 \in A$, a contradiction.

The "in particular" conclusion follows from [4, Theorem 8.18]. □

**Theorem 3.9.** *Let $R$ be a minimal right ideal of $\beta\mathbb{N}$. There is a decreasing sequence $\langle p_n \rangle_{n=1}^{\infty}$ of idempotents in $c\ell K(\beta\mathbb{N})$ such that*

$$|\{q \in E(R) : \{p_n : n \in \mathbb{N}\} \subseteq D_q\}| = 2^{\mathfrak{c}}\,.$$

*Proof.* Choose a sequence $\langle q_n \rangle_{n=1}^{\infty}$ as guaranteed by Lemma 3.8 and pick an accumulation point $x$ of this sequence. Then $x$ is right cancelable in $\beta\mathbb{Z}$. Since each $q_n$ is in $R$ and each idempotent in $R$ is a right identity for $R$, we have that for each $n \in \mathbb{N}$ and each $p \in E(R)$, $q_n + p = p$, and consequently, for each $p \in E(R)$, $x + p = p$. Let $M = \bigcap\{C \subseteq \beta\mathbb{Z} : C$ is a compact subsemigroup of $\beta\mathbb{Z}$ and $x \in C\}$. Note that $M \subseteq \beta\mathbb{N}$. For each $p \in E(R)$, $\{z \in \beta\mathbb{N} : z + p = p\}$ is a compact subsemigroup of $\beta\mathbb{Z}$ with $x$ as a member, so we have that for all $z \in M$ and all $p \in E(R)$, $z + p = p$.

By [4, Corollary 8.54], pick a decreasing sequence $\langle p_n \rangle_{n=1}^{\infty}$ in $M$ and let $w$ be a cluster point of $\langle p_n \rangle_{n=1}^{\infty}$. By [4, Lemma 9.22], $w$ is right cancelable in $\beta\mathbb{Z}$ and for each $n \in \mathbb{N}$, $w \in \beta\mathbb{Z} + p_n$. By [4, Theorem 6.56], $\beta\mathbb{N} + w$ contains $2^{\mathfrak{c}}$ pairwise disjoint left ideals. Let $L$ be one of these and pick an idempotent $q \in R \cap L$. To complete the proof, we show that for each $n \in \mathbb{N}$, $q \leq p_n$ (so, by Corollary 2.2, $\{p_n : n \in \mathbb{N}\} \subseteq D_q$). Let $n \in \mathbb{N}$. Then $L \subseteq \beta\mathbb{N} + w \subseteq \beta\mathbb{Z} + p_n$ so $q + p_n = q$. Also, $p_n \in M$ and $q \in E(R)$, so $p_n + q = q$. □

**Corollary 3.10.** *There exist decreasing chains of idempotents in $c\ell K(\beta\mathbb{N})$ of reverse order type $\omega + 1$.*

*Proof.* Pick a minimal right ideal $R$ of $\beta\mathbb{N}$, pick a sequence $\langle p_n \rangle_{n=1}^{\infty}$ as guaranteed by Theorem 3.9, and pick $q \in E(R)$ such that

$$\{p_n : n \in \mathbb{N}\} \subseteq D_q\,.$$

By Corollary 2.2, given $n \in \mathbb{N}$, $q \leq p_n$ and, since $p_{n+1} < p_n$, $q < p_n$. $\square$

## 4. Copies of $\mathbb{Z} \times \mathbb{Z}$ in $G_q$

We know, of course, that if $q \in E\big(K(\beta\mathbb{N})\big)$, then the center of $G_q$ contains $\mathbb{Z} + q$. We show in this section that if it is not equal to $\mathbb{Z} + q$, then $G_q$ contains an algebraic copy of $\mathbb{Z} \times \mathbb{Z}$.

**Definition 4.1.** Let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, and let $m, x \in \mathbb{N}$.

(a) $\mathrm{supp}(x)$ is the binary support of $x$.
(b) $c_{B_1}(x) = |\mathrm{supp}(x) \cap B_1|$.
(c) $c_{B_1,\ldots,B_k}(x) =$
$|\{(i_1, i_2, \ldots, i_k) \in \big(\mathrm{supp}(x)\big)^k : i_1 < \ldots < i_k \text{ and each } i_t \in B_t\}|$.
(d) $c_{B_1,m}(x) \in \mathbb{Z}_m$ and $c_{B_1,m}(x) \equiv c_{B_1}(x) \pmod{m}$.
(e) $c_{B_1,\ldots,B_k,m}(x) \in \mathbb{Z}_m$ and $c_{B_1,\ldots,B_k,m}(x) \equiv c_{B_1,\ldots,B_k}(x) \pmod{m}$.

**Lemma 4.2.** *Let $u, v \in \mathbb{H}$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, and let $m \in \mathbb{N}$.*

(1) $\widetilde{c}_{B_1,m}(u + v) = \widetilde{c}_{B_1,m}(u) + \widetilde{c}_{B_1,m}(v)$.
(2) *If $k > 1$, then* $\widetilde{c}_{B_1,\ldots,B_k,m}(u+v) = \widetilde{c}_{B_1,\ldots,B_k,m}(u)+\widetilde{c}_{B_1,\ldots,B_k,m}(v)+ \sum_{t=1}^{k-1} \widetilde{c}_{B_1,\ldots,B_t,m}(u) \cdot \widetilde{c}_{B_{t+1},\ldots,B_k,m}(v)$.

*Proof.* (1) It suffices that $\widetilde{c}_{B_1,m} \circ \rho_v$ and $\rho_{\widetilde{c}_{B_1,m}(v)} \circ \widetilde{c}_{B_1,m}$ agree on $\mathbb{N}$, so let $x \in \mathbb{N}$. Let $k = \max \mathrm{supp}(x) + 1$. It suffices to observe that $\widetilde{c}_{B_1,m} \circ \lambda_x$ and $\lambda_{c_{B_1,m}(x)} \circ \widetilde{c}_{B_1,m}$ agree on $\mathbb{N}2^k$.

(2) Note that singletons are open in $\mathbb{Z}_m$. Pick $C \in u$ such that for all $x \in C$, $\widetilde{c}_{B_1,\ldots,B_k,m}(x + v) = \widetilde{c}_{B_1,\ldots,B_k,m}(u + v)$ and for $t \in \{1, 2, \ldots, k\}$, $\widetilde{c}_{B_1,\ldots,B_t,m}(x) = \widetilde{c}_{B_1,\ldots,B_t,m}(u)$. Pick $x \in C$ and let $l = \max \mathrm{supp}(x) + 1$. Pick $D \in v$ such that for all $y \in D$, $c_{B_1,\ldots,B_k,m}(x+y) = \widetilde{c}_{B_1,\ldots,B_k,m}(x+v)$ and for $t \in \{1, 2, \ldots, k-1\}$, $c_{B_{t+1},\ldots,B_k,m}(y) = \widetilde{c}_{B_{t+1},\ldots,B_k,m}(v)$. Pick $y \in D \cap \mathbb{N}2^l$. Then $c_{B_1,\ldots,B_k,m}(x + y) = c_{B_1,\ldots,B_k,m}(x) + c_{B_1,\ldots,B_k,m}(y) + \sum_{t=1}^{k-1} c_{B_1,\ldots,B_t,m}(x) \cdot c_{B_{t+1},\ldots,B_k,m}(y)$. $\square$

**Lemma 4.3.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, and let $m \in \mathbb{N}$. Then $\widetilde{c}_{B_1,\ldots,B_k,m}(q) = 0$.*

*Proof.* This follows immediately by induction on $k$ from Lemma 4.2. $\square$

**Lemma 4.4.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of $\omega$, let $m \in \mathbb{N}$, and let $u \in \mathbb{H} \cap D_q$. Then $\widetilde{c}_{B_1,\ldots,B_k,m}(u) = 0$.*

*Proof.* We show first that it suffices to show this under the additional assumption that $\mathbb{N}\setminus\bigcup_{i=1}^{k}B_i$ is infinite. Suppose we have done this and let $B'_k$ and $B''_k$ be disjoint infinite subsets of $B_k$ with $B'_k\cup B''_k=B_k$. Note that for all $x\in\mathbb{N}$, $c_{B_1,\ldots,B_k,m}(x)=c_{B_1,\ldots,B'_k,m}(x)+c_{B_1,\ldots,B''_k,m}(x)$ so $\widetilde{c}_{B_1,\ldots,B_k,m}(u)=\widetilde{c}_{B_1,\ldots,B'_k,m}(u)+\widetilde{c}_{B_1,\ldots,B''_k,m}(u)=0+0$.

So assume that $B_{k+1}\mathbb{N}\setminus\bigcup_{i=1}^{k}B_i$ is infinite. Pick $p\in\{2^n:n\in B_{k+1}\}^*$. Note that for all $n\in B_{k+1}$, $c_{B_{k+1},m}(2^n)=1$, and if $t\in\{1,2,\ldots,k\}$, then $c_{B_1,\ldots,B_t,m}(2^n)=c_{B_t,\ldots,B_{k+1},m}(2^n)=0$. Therefore, $\widetilde{c}_{B_{k+1},m}(p)=1$, and if $t\in\{1,2,\ldots,k\}$, then $\widetilde{c}_{B_1,\ldots,B_t,m}(p)=\widetilde{c}_{B_t,\ldots,B_{k+1},m}(p)=0$. Since all terms of the expansions given in Lemma 4.2 except one involve $q$ and are therefore 0, we have that $\widetilde{c}_{B_{k+1},m}(p+q)=1$, and if $t\in\{1,2,\ldots,k\}$, then $\widetilde{c}_{B_1,\ldots,B_t,m}(p+q)=\widetilde{c}_{B_t,\ldots,B_{k+1},m}(p+q)=0$ and $\widetilde{c}_{B_{k+1},m}(q+p)=1$, and if $t\in\{1,2,\ldots,k\}$, then $\widetilde{c}_{B_1,\ldots,B_t,m}(q+p)=\widetilde{c}_{B_t,\ldots,B_{k+1},m}(q+p)=0$.

Next note that

$$\begin{aligned}
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+u+p+q)=\\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q)+\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q)+\\
&\textstyle\sum_{t=1}^{k}\widetilde{c}_{B_1,\ldots,B_t,m}(q)\cdot\widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(u+p+q)=\\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q)\text{ and}\\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u+q)=\\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u)+\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q)+\\
&\textstyle\sum_{t=1}^{k}\widetilde{c}_{B_1,\ldots,B_t,m}(q+p+u)\cdot\widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(q)=\\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u)\,.
\end{aligned}$$

Since

$$\begin{aligned}
\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+u+p+q)&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+q+p+q)\\
&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+q+u)\\
&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u+q),
\end{aligned}$$

we, therefore, have that $\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q)=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u)$.

Now

$$\begin{aligned}
\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q)&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)+\widetilde{c}_{B_1,\ldots,B_{k+1},m}(p+q)\\
&\quad+\textstyle\sum_{t=1}^{k}\widetilde{c}_{B_1,\ldots,B_t,m}(u)\cdot\widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(p+q)\\
&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)+\widetilde{c}_{B_1,\ldots,B_k,m}(u)\text{ and}\\
\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u)&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p)+\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)\\
&\quad+\textstyle\sum_{t=1}^{k}\widetilde{c}_{B_1,\ldots,B_t,m}(q+p)\cdot\widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(u)\\
&=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)\,.
\end{aligned}$$

Consequently, $\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)+\widetilde{c}_{B_1,\ldots,B_k,m}(u)=\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)$, so $\widetilde{c}_{B_1,\ldots,B_k,m}(u)=0$. □

**Lemma 4.5.** *Let $q\in E\big(K(\beta\mathbb{N})\big)$, let $p\in\{2^n:n\in\mathbb{N}\}^*$, and let $\psi_p:\mathbb{Z}\to G_q$ be the homomorphism such that $\psi_p(1)=q+p+q$. Then for all $n\in\mathbb{Z}\setminus\{0\}$, $\psi_p(n)\notin D_q$.*

*Proof.* Pick infinite $B \subseteq \mathbb{N}$ such that $\{2^n : n \in B\} \in p$. Then for each $m \in \mathbb{N} \setminus \{1\}$, $\widetilde{c_{B,m}}(q+p+1) = 1$, so for all $n \in \mathbb{N}$ and all $m > n$, $\widetilde{c_{B,m}}\big(\psi_p(n)\big) = n$, and thus $\psi_p(n) \notin D_q$ by Lemma 4.4. Now $D_q \cap G_q$ is a group, so if $n \in \mathbb{N}$ and $\psi_p(-n) \in D_q$, so is $\psi_p(n)$. □

**Theorem 4.6.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $p \in \{2^n : n \in \mathbb{N}\}^*$, and let $\psi_p : \mathbb{Z} \to G_q$ be the homomorphism such that $\psi_p(1) = q + p + q$. Assume that $u \in \mathbb{H} \cap G_q \cap D_q \setminus \{q\}$ and let $\varphi : \mathbb{Z} \to G_q$ be the homomorphism such that $\varphi(1) = u$. Define $\tau : \mathbb{Z} \times \mathbb{Z} \to G_q$ by $\tau(m,n) = \varphi(m) + \psi_p(n)$. Then $\tau$ is an injective homomorphism.*

*Proof.* Given $(m,n)$ and $(k,l)$ in $\mathbb{Z} \times \mathbb{Z}$, one has that $\tau\big((m,n)+(k,l)\big) = \tau(m,n) + \tau(k,l)$ because $\varphi(k) \in D_q$. Now assume that $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $\tau(m,n) = q$. Then $\varphi(m) + \psi_p(n) = q$, so $\varphi(m) = \psi_p(-n)$, and thus $\psi_p(-n) \in D_q$ so that $n = 0$ by Lemma 4.5. Therefore, $\varphi(m) = q$. By Zelenyuk's Theorem [8] (or see [4, Theorem 7.17]), $\beta\mathbb{N}$ contains no nontrivial finite groups. If one had $m \neq 0$, then $\varphi[\mathbb{Z}]$ would be a nontrivial finite group, so $m = 0$. □

**Corollary 4.7.** *Let $q \in E\big(K(\beta\mathbb{N})\big)$. If the center of $G_q$ is not equal to $\mathbb{Z} + q$, then $G_q$ contains an algebraic copy of $\mathbb{Z} \times \mathbb{Z}$.*

*Proof.* Assume we have $x \in Z(G_q) \setminus (\mathbb{Z} + q)$. Then, by Lemma 2.6, $x \in \mathbb{Z} + I$, so pick $n \in \mathbb{Z}$ and $u \in I$ such that $x = n + u$. Then $u \in \mathbb{H} \cap G_q \cap D_q \setminus \{q\}$. Pick any $p \in \{2^n : n \in \mathbb{N}\}^*$. Define $\tau$ as in Theorem 4.6. Then $\tau$ is an injective homomorphism. □

We conclude by listing some of the tantalizing open questions that have arisen in the study of the center and extended center of $G_q$.

**Question 4.8.** (1) Let $q \in E\big(K(\beta\mathbb{N})\big)$. Does $Z(G_q) = \mathbb{Z} + q$?
(2) Let $q \in E\big(K(\beta\mathbb{N})\big)$. Is $D_q \subseteq \mathbb{Z} + E(\beta\mathbb{N})$?
(3) Does there exist $q \in E\big(K(\beta\mathbb{N})\big)$ for which $E(D_q)$ is finite?
(4) Does there exist $q \in E\big(K(\beta\mathbb{N})\big)$ for which $E(D_q)$ is uncountable?
(5) Let $q_1, q_2 \in E\big(K(\beta\mathbb{N})\big)$. Are $D_{q_1}$ and $D_{q_2}$ isomorphic?

## References


[1] J. W. Baker and P. Milnes, *The ideal structure of the Stone-Čech compactification of a group*, Math. Proc. Cambridge Philos. Soc. **82** (1977), no. 3, 401–409.

[2] Ching Chou, *On a geometric property of the set of invariant means on a group*, Proc. Amer. Math. Soc. **30** (1971), 296–302.

[3] Neil Hindman and John Pym, *Free groups and semigroups in* $\beta\mathbb{N}$, Semigroup Forum **30** (1984), no. 2, 177–193.

[4] Neil Hindman and Dona Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications*. 2nd ed. de Gruyter Expositions in Mathematics, 27 Berlin: Walter de Gruyter & Co., 2012.

[5] D. Rees, *On semi-groups*, Proc. Cambridge Philos. Soc. **36** (1940), 387–400.

[6] Wolfgang Ruppert, *Rechtstopologische halbgruppen*, J. Reine Angew. Math. **261** (1973), 123–133.

[7] Anton Suschkewitsch, *Über die endlichen gruppen ohne das gesetz der eindeutigen umkehrbarkeit*, Math, Ann. **99** (1928), 30–50.

[8] E. G. Zelenjuk, *Finite groups in* $\beta\mathbb{N}$ *are trivial*, Semigroup Forum, **55** (1997), no. 1, 131–132.

(Hindman) Department of Mathematics; Howard University; Washington, DC 20059, USA

*E-mail address*: nhindman@aol.com

(Strauss) Department of Pure Mathematics; University of Leeds; Leeds LS2 9J2, UK

*E-mail address*: d.strauss@hull.ac.uk