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SOME REMARKS ON THE TOPOLOGY OF THE ELLIS SEMIGROUP OF A DISCRETE DYNAMICAL SYSTEM

S. GARCÍA-FERREIRA AND M. SANCHIS

Abstract. In this paper, we analyze some metric and structural properties of the Ellis semigroup $E(X, f)$ of a dynamical system (X, f), where X is a metric space and $f : X \to X$ is a continuous function. First, we discuss when $E(X, f)$ is a compactification of $\mathbb{N}$ with the discrete topology. We also study some metrizable properties of the Ellis semigroup $E(X, f)$. Also, necessary and sufficient conditions are given in order that the Ellis semigroup $E(X, f)$ be a topological semigroup. Our approach is based in the notion of a p-limit point of a sequence in X, where p is a free ultrafilter on $\mathbb{N}$. We apply our main results to the relevant cases of transitive and distal discrete dynamical systems.

1. Introduction and Preliminaries

By a (discrete) dynamical system, we understand that a pair (X, f) is a continuous function where X is a compact metric space and $f : X \to X$. It is well known that the study of a dynamical system (X, f) is equivalent to the study of the action of the semigroup $\mathbb{N}$ on the space X defined by $(n, x) \to f^n(x)$ where f^n stands for the n-th iterate of the

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function f. A useful tool for studying the behavior of a dynamical system (X, f) is the so-called *Ellis semigroup* or the *enveloping semigroup* which is the pointwise closure of $\{f^n \mid n \in \mathbb{N}\}$ in the compact space X^X, and its operation is the composition of functions, introduced by Robert Ellis in [10]. The Ellis semigroup of a dynamical system (X, f), usually denoted by $E(X, f)$, is equipped with the topology inherited from the product topology (pointwise convergence) of X^X.

Now some notation and basic concepts are in order. An open ball of X will be denoted by $B_\epsilon(x)$ (or $B^d_\epsilon(x)$ if we want to emphasize the metric d of X). If $x \in X$, then $\mathcal{N}(x)$ will denote the neighborhoods of x in the space X. For a space X, $C(X, X)$ will denote the set of all continuous functions from X to itself, and the symbol $C_\pi(X, X)$ will mean that $C(X, X)$ is equipped with the pointwise convergence topology. As usual, a subbasic open subset of X^X will be denoted by $[x, V] = \{h \in X^X : h(x) \in V\}$, where $x \in X$ and V is a nonempty open subset of X. If X and Y are compact metric spaces, then $C_u(X, Y)$ will denote the set of all continuous functions from X to Y equipped with the sup-metric d_u (where $d_u(f, g) = \sup_{x \in X} d(f(x), g(x))$). If (X, f) is a dynamical system, then the *orbit* of a point $x \in X$ is the set $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$. If $f : X \to Y$ is a continuous function between Tychonoff spaces, then $\overline{f} : \beta(X) \to \beta(Y)$ will stand for the Stone extension of f. The Stone-Čech compactification $\beta(\mathbb{N})$ of the natural numbers $\mathbb{N}$ with the discrete topology will be identified with the set of all ultrafilters on $\mathbb{N}$, and its remainder $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$ with the set of all free ultrafilters on $\mathbb{N}$. If $A \subseteq \mathbb{N}$, then $\hat{A} = cl_{\beta(\mathbb{N})} A = \{p \in \beta(\mathbb{N}) : A \in p\}$ is a basic clopen subset of $\beta(\mathbb{N})$, and $A^* = \hat{A} \setminus A = \{p \in \mathbb{N}^* : A \in p\}$ is a basic clopen subset of $\mathbb{N}^*$.

Let X be a space. Given $p \in \mathbb{N}^*$, a point $x \in X$ is said to be the *p-limit* point of a sequence $(x_n)_{n \in \mathbb{N}}$ in X ($x = p\text{-}\lim_{n \to \infty} x_n$) if for every neighborhood V of x, $\{n \in \mathbb{N} : x_n \in V\} \in p$. We remark that a point $x \in X$ is an accumulation point of a countable set $\{x_n : n \in \mathbb{N}\}$ if and only if there is $p \in \mathbb{N}^*$ such that $x = p - \lim_{n \to \infty} x_n$ and $x = x_n$ for finitely many n's. It is not hard to prove that each sequence of a compact space always has a p-limit point for every $p \in \mathbb{N}^*$. Indeed, if $f : \mathbb{N} \to X$ is an arbitrary function and X is compact and Hausdorff, then the value of the Stone-Čech extension of f at p is precisely the p-limit point of the sequence $(f(n))_{n \in \mathbb{N}}$. Thus, the p-limit point of a sequence in a compact space not only exists but is unique. The notion of the p-limit point has been introduced by several mathematicians, for example, Allen R. Bernstein [6], Harry Furstenberg [13, p. 179], and Ethan Akin [1, p. 5, 61]. The p-limit points play a very important role in the study of countably compact spaces and they are very useful in the study of some types of convergence in analysis and in topology.

Let us recall the definition of equicontinuity.

Definition 1.1. Let (X,d) be a metric space. We say that $Y \subseteq X^X$ is *equicontinuous at* $x \in X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $y \in X$ and $d(x,y) < \delta$ imply that $d(f(x), f(y)) < \epsilon$ for all $f \in Y$. $Y \subseteq X^X$ is said to be *equicontinuous* if it is equicontinuous at every point of X. A set $Y \subseteq X^X$ is called *uniformly equicontinuous* if, for every $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $d(x,y) < \delta$ imply that $d(f(x), f(y)) < \epsilon$ for all $f \in Y$.

In [18], the authors consider dynamical systems (X,f) where X is a compact metric space and $f : X \to X$ is a homeomorphism. They prove that the Ellis semigroup $E(X,f)$ is metrizable if and only if every closed subsystem of (X,f) is almost equicontinuous: $Y \subseteq X^X$ is said to be *almost equicontinuous* if it is equicontinuous at a dense subset of X. At the end of [18], the authors claim that the same result holds for any dynamical system $E(X,f)$.

Enveloping semigroups have played a very crucial role in topological dynamics and they are an active area of research (see, for instance, Eli Glasner's survey [16]). The behavior of the dynamical system (X,f) and the algebraic properties of the associated Ellis semigroup are closely related. For instance, (X,f) is *distal* if and only if its enveloping semigroup is a group [4, Theorem 5.6]. In [21], it is pointed out that the cardinality of the Ellis semigroup allows conclusions about certain chaotic behavior of (X,f).

In the present note we shall study the metrizability of the Ellis semigroup of a discrete dynamical system in the flavor of the results obtained (for group actions) in [14]. The dynamical system with the metric Ellis semigroup has been an interesting object to study. For instance, the weakly almost periodic flows with metrizable Ellis semigroups have been considered in [11]. For a compact metric G-space X, several conditions were given in [11, Theorem 9.14] which are equivalent to the hereditarily almost equicontinuity of X, and later, in [18], it was proved that these conditions are equivalent to the metrizability of the Ellis semigroup. These equivalences also hold when G is a right topological semigroup as it is mentioned in [18, Theorem 6.2]. We shall rewrite the proof of this result for the semigroup $\beta(\mathbb{N})$ (Theorem 3.13). Our purpose is to emphasize the connection between the dynamical properties of a discrete system and the combinatorial and topological properties of the semigroup $\beta(\mathbb{N})$.

Throughout this paper, we make use of the notion of the p-limit point to prove some properties of the Ellis semigroup of a dynamical system. The paper is organized as follows. Section 2 is devoted to some structural properties of $E(X,f)$. Mainly, we study the conditions that make $E(X,f)$

a compactification of $\mathbb{N}$ with the discrete topology. In the third section, we present several metrization theorems of the Ellis semigroup related to equicontinuity as it is done in [18]. In the last section, we also characterize when the Ellis semigroup is a topological semigroup. In particular, we show that, for topologically transitive dynamical systems, $E(X, f)$ is a topological semigroup if and only if it is metrizable with the supremum metric.

2. Basic Topological Properties of the Ellis Semigroup

Let us describe the elements of the Ellis semigroup by using ultrafilters on $\mathbb{N}$.

Definition 2.1. Let (X, f) be a dynamical system. For a free ultrafilter p on $\mathbb{N}$, the function $f^p : X \to X$ is defined by $f^p(x) = p\text{-}\lim_{n\to\infty} f^n(x)$ for every $x \in X$. For a point $x \in X$, the function $f_x := p \mapsto f^p(x) : \beta(\mathbb{N}) \to X$ denotes the Stone extension of the continuous function $n \mapsto f^n(x) : \mathbb{N} \to X$.

Observe that all functions f_x are continuous. However, the functions f^p may all be discontinuous for all $p \in \mathbb{N}^*$:

Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\}$ and define $f : X \to X$ as

$$f(x) = \begin{cases} x & \text{if } \ x \in \{0, 1\} \\ \frac{1}{n} & \text{if } \ x = \frac{1}{n+1} \ \text{ and } \ 1 \leq n \in \mathbb{N}. \end{cases}$$

If $p \in \mathbb{N}^*$, then $f^p(x) = 1$ for every $x > 0$, and $f^p(0) = 0$. Hence, it follows that f^p is discontinuous at 0 for all $p \in \mathbb{N}^*$.

For our convenience, we shall describe the Ellis semigroup by means of the p-limit points.

Theorem 2.2. *For every dynamical system (X, f), we have that*

$$E(X, f) = \{f^p : p \in \beta(\mathbb{N})\}$$

and $f^p = p\text{-}\lim_{n\to\infty} f^n$ for all $p \in \mathbb{N}^*$.

Proof. The function $\phi : \mathbb{N} \to X^X$ given by $\phi(n) = f^n$, for all $n \in \mathbb{N}$, is continuous. Let us consider its Stone extension $\hat{\phi} : \beta(\mathbb{N}) \to X^X$. We know from [19] that $\hat{\phi}(p) = p - lim_{n\to\infty}\phi(f^n) = f^p$ for all $p \in \mathbb{N}^*$. Then we obtain that $\hat{\phi}[\beta(\mathbb{N})] = E(X, f)$. Since $E(X, f)$ has the topology of the pointwise convergence, we must have that $f^p = p\text{-}\lim_{n\to\infty} f^n$ inside the compact space $E(X, f)$ for all $p \in \mathbb{N}^*$. $\square$

The proof of the previous theorem establishes the well-known fact that the semigroup of Ellis is a quotient of $\beta(\mathbb{N})$.

To extend the ordinary addition on the set of natural numbers to the whole $\beta(\mathbb{N})$, we may do the following:

For $p \in \beta(\mathbb{N})$ and $n \in \mathbb{N}$, we define $p + n = p\text{-}\lim_{m\to\infty}(m + n)$ and if $p, q \in \beta(\mathbb{N})$, then we define $p+q = q-\lim_{n\to\infty} p+n$. We know that $\beta(\mathbb{N})$ with this operation $+$ is a semigroup.

The next theorem is the link between the addition of $\beta(\mathbb{N})$ and the operation of the Ellis semigroup of an arbitrary dynamical system.

Theorem 2.3 ([7])**.** *Let (X, f) be a dynamical system. Then*

$$f^p \circ f^q = f^{q+p},$$

for every $p, q \in \beta(\mathbb{N})$.

Corollary 2.4 ([7])**.** *Let (X, f) be a dynamical system. If $f^p = f^q$ for some $p, q \in \beta(\mathbb{N})$, then $f^{p+r} = f^{q+r}$ and $f^{r+p} = f^{r+q}$ for all $r \in \beta(\mathbb{N})$.*

The Ellis semigroup is a semigroup compactification of $\mathbb{N}$ in the sense of [5], but, in general, is not a compactification of $\mathbb{N}$ with the discrete topology (that is, a compact space having an infinite countable dense discrete subset). For instance, if $f^n = f$ for some $2 \leq n \in \mathbb{N}$, then $E(X, f) = \{f^0, f^1, ..., f^n\}$ is finite, where $n \in \mathbb{N}$ is the least integer with the property $f^n = f$. In this case, the set $\{f^n : n \in \mathbb{N}\}$ is finite. More generally, if there are distinct $n, m \in \mathbb{N}$ such that $f^n = f^m$, then every point of X is an eventually periodic point. Indeed, all of them have the same period and so the behavior of the dynamical system (X, f) is trivial. If f is surjective and $f^n \circ f^m = f^m$ for some $m, n \in \mathbb{N}$, then $f^n = 1_X$ is the identity and so f is periodic. To avoid trivial situations, we shall always assume, in what follows, that if (X, f) is a dynamical system, then $f^n \neq f^m$ for distinct integers $n, m \in \mathbb{N}$ (these kinds of dynamical systems are called *aperiodic*). Hence, $\mathbb{N}$ can be identified (from an algebraic point of view) with $\{f^n : n \in \mathbb{N}\}$. Thus, in order that $E(X, f)$ be a compactification of $\mathbb{N}$ with the discrete topology, it suffices that the set $\{f^n : n \in \mathbb{N}\}$ contains an infinite dense discrete subset of $E(X, f)$. Since f is aperiodic, then every element of $E(X, f)^* := E(X, f) \setminus \{f^n : n \in \mathbb{N}\} = \{f^p : p \in \mathbb{N}^*\}$ is an accumulation point of $E(X, f)$. Notice that $E(X, f)^*$ is invariant. That is, if $f^p \in E(X, f)^*$ for some $p \in \mathbb{N}^*$, then $f^{p+n} \in E(X, f)^*$ for all $n \in \mathbb{N}$. Also, we have that if f^n is an accumulation point of $E(X, f)$ for some $n \in \mathbb{N}$, then f^{n+m} is an accumulation point of $E(X, f)$ for all $m \in \mathbb{N}$. Thus, we have that f^n is an isolated point of $E(X, f)$ for some $n \in \mathbb{N}$, if and only if $f^n \neq f^p$ for any $p \in \mathbb{N}^*$. We conclude that $E(X,f)$ is a compactification of $\mathbb{N}$ if and only if the set $\{f^n : n \in \mathbb{N}\}$ is infinite and discrete in $E(X,f)$.

Let us consider the space $X = \{0\} \cup \{\frac{1}{n+1} : n \in \mathbb{N}\}$ and a continuous function $f : X \to X$ with $f(0) = 0$. We know that $E(X,f)$ is a compactification of $\mathbb{N}$ if and only if there is $n \in \mathbb{N}$ such that $\mathcal{O}_f(\frac{1}{n})$ is infinite. Let us see some examples.

I. If $f : X \to X$ is the shift to the left, then it is an easy matter to see that $E(X,f)$ is a nontrivial convergent sequence with its limit point and X has just one periodic point which is a fixed point. Thus, $E(X,f)$ can be identified with the one-point compactification of $\mathbb{N}$

II. We define $f : X \to X$ as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2(n+1)+1} & \text{if } x = \frac{1}{2n+1} \\ \frac{1}{2(2n+1)+2} & \text{if } x = \frac{1}{2(2n)+2} \\ \frac{1}{2(2n)+2} & \text{if } x = \frac{1}{2(2n+1)+2}\,. \end{cases}$$

Then, $E(X,f)$ is a compactification of $\mathbb{N}$ and $E(X,f)$ has only two accumulation points which are the functions

$$g(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2n+1} \\ \frac{1}{2(2n+1)+2} & \text{if } x = \frac{1}{2(2n)+2} \\ \frac{1}{2(2n)+2} & \text{if } x = \frac{1}{2(2n+1)+2} \end{cases}$$

and

$$h(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2n+1} \\ \frac{1}{2n+2} & \text{if } x = \frac{1}{2n+2}\,. \end{cases}$$

For every $2 \leq k \in \mathbb{N}$, by an easy modification of this example, it is possible to define $f : X \to X$ so that each point of $X \setminus \{0\}$ has either infinite period or period k. Besides, we can define a dynamical system (X,f) so that $E(X,f)$ is a compactification of $\mathbb{N}$ and $E(X,f)$ has precisely k accumulation points.

III. Now, let us define $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ \frac{1}{2^k+i+1} & \text{if } x = \frac{1}{2^k+i}, 1 \leq i < 2^k \text{ and } 1 \leq k \in \mathbb{N} \\ \frac{1}{2^{k-1}+1} & \text{if } x = \frac{1}{2^k} \text{ and } 2 \leq k \in \mathbb{N}\,. \end{cases}$$

In this case, $E(X, f)$ has only two isolated points f^0 and f. By making small changes to this example, for each $3 \leq k \in \mathbb{N}$, we can get an Ellis semigroup $E(X, f)$ having only k-many isolated points.

In the next theorem, we shall see that when $E(X, f)$ has finitely many accumulation points, the dynamical systems (X, f) cannot have points with arbitrarily high period.

Theorem 2.5. *Let (X, f) be a dynamical system such that $E(X, f)$ is a compactification of $\mathbb{N}$ with the discrete topology and f is surjective. If $|E(X, f)| = k \in \mathbb{N}$, then (X, f) does not have any periodic point with period bigger than k*

Proof. Let $x \in X$ be a periodic point of (X, f). It follows from the hypothesis that for every $p \in \mathbb{N}^*$ there is $m \leq k$ such that $f^m \circ f^p = f^p$ and since $\mathcal{O}_f(x) = \{f^p(x) : p \in \mathbb{N}^*\}$, we must have that the period of x is at most k. □

It is well known that the Ellis semigroup of the shift on the Cantor space $\{0, 1\}^{\mathbb{N}}$ is the Stone-Čech compactification of the naturals. In [14], the author considers the Cantor set and a generalization of the shift which depends on a function from $\mathbb{N}$ to $\mathbb{N}$. In particular, using different functions $\mathbb{N}$ to $\mathbb{N}$, one can get as the Ellis semigroup either a convergent sequence, the Cantor set, or $\beta(\mathbb{N})$. Thus, the Ellis semigroup of a discrete dynamical system can be the smallest element of the lattice of all compactifications of $\mathbb{N}$ and also the biggest one. This motivates the following question.

Question 2.6. If K is a compactification of $\mathbb{N}$, is there a dynamical system (X, f) such that $E(X, f)$ and K are homeomorphic?

3. Metrizability of the Ellis Semigroup

We start this section by listing several conditions that are equivalent to the metrizability of the Ellis semigroup with the supremum metric.

If d is the metric on X and $D \subseteq X$ is dense, then a pseudometric on $E(X, f)$ relative to D is defined as

$$d_D(f^p, f^q) = \sup_{x \in D} d(f^p(x), f^q(x)) \qquad p, q \in \beta(\mathbb{N}).$$

Notice that being X compact, d is bounded and so d_D is well defined. If $X = D$, then $d_X = d_u$. If D is a dense subset of X, then we define $E_D(X) = \{f^p|_D : p \in \beta(\mathbb{N})\}$ and $\varphi_D : E(X, f) \to E_D(X)$ by $\varphi_D(f^p) = f^p|_D$ for all $p \in \beta(\mathbb{N})$. If we equipped $E_D(X)$ with the subspace topology inherited from the product X^D, then φ_D is continuous since it is a projection map, and hence we obtain that $E_D(X)$ is also compact, and separable as well. Following John L. Kelley [20], we say that $D \subseteq X$

distinguishes points of $Y \subseteq X^X$ if, for distinct $g, h \in Y$, there is $d \in D$ such that $g(d) \neq h(d)$. Then, if $D \subseteq X$ is dense and distinguishes points of $\{f^n : n \in \mathbb{N}\}$, then $E(X,f)$ and $E_D(X)$ are homeomorphic since, in this case, the function φ_D is one-to-one (see [20, p. 220, Theorem 2]).

The next lemma follows from the Lebesgue Covering Lemma and compactness.

Lemma 3.1. *Let X be a compact metric space and let $Y \subseteq X^X$. Then Y is equicontinuous if and only if it is uniformly equicontinuous.*

We shall also need the following lemma (see a more general theorem in [20, p. 232, Theorem 14]).

Lemma 3.2. *Let X be a metric space, $\mathcal{C} \subseteq C_\pi(X,X)$, and $\mathcal{F} \subseteq X^X$. If D is a dense subset of X, $\mathcal{C}$ is a dense subset of $\mathcal{F}$, and the family $\mathcal{C}$ is uniformly equicontinuous on D, then the family $\mathcal{F}$ is uniformly equicontinuous on X.*

Proof. Let $\epsilon > 0$. Choose $\delta > 0$ witnessing the uniform continuity of the family $\{f|_D : f \in \mathcal{F}\}$ for $\frac{\epsilon}{5}$. Let $x, y \in X$ such that $d(x,y) < \frac{\delta}{3}$. Fix two sequences $(d_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ in D such that $d_n \to x$ and $e_n \to y$. Let $f \in \mathcal{F}$. Choose $g \in \mathcal{C}$ and $l \in \mathbb{N}$ so that $g \in [x, B_{\frac{\epsilon}{5}}(f(x))] \cap [y, B_{\frac{\epsilon}{5}}(f(y))]$, $d(g(x), g(d_l)) < \frac{\epsilon}{5}$, $d(g(y), g(e_l)) < \frac{\epsilon}{5}$, and $d(g(d_l), g(e_l)) < \frac{\epsilon}{5}$. Hence, $d(f(x), g(x)) < \frac{\epsilon}{5}$ and $d(f(y), g(y) < \frac{\epsilon}{5}$. Thus, we obtain that

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), g(x)) \\ + d(g(x), g(d_l)) &+ d(g(d_l), g(e_l)) + d(g(e_l), g(y)) + d(g(y), f(y)) \\ &< \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} = \epsilon. \end{aligned}$$ □

The following notation is needed.

Let (X,f) be a dynamical system. Then $F : \beta(\mathbb{N}) \times X \to X$ will denote the action induced by (X,f) which is given by $F(p,x) = f^p(x)$ for all $(p,x) \in \beta(\mathbb{N}) \times X$, and $G : E(X,f) \times X \to X$ will denote the action $G(f^p, x) = f^p(x)$ for every $(f^p, n) \in E(X,f) \times X$. Observe that the following diagram commutes:

$$\begin{tikzcd} \beta(\mathbb{N}) \times X \arrow[r, "F"] \arrow[d, "\hat{\phi} \times Id"'] & X \\ E(X,f) \times X \arrow[ur, "G"'] & \end{tikzcd}$$

where $\hat{\phi}$ is the function defined in the proof of Theorem 2.2.

In the following theorem, we list several conditions that are equivalent to the equicontinuity of the family $\{f^n : n \in \mathbb{N}\}$.

Theorem 3.3. *For a dynamical system (X, f), the following are equivalent.*

(1) *The set $\{f^n : n \in \mathbb{N}\}$ is equicontinuous on X.*
(2) *The set $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous on X.*
(3) *The set $\{f^p : n \in \beta(\mathbb{N})\}$ is equicontinuous on X.*
(4) *The set $\{f^p : n \in \beta(\mathbb{N})\}$ is uniformly equicontinuous on X.*
(5) *The metric d_u induces the topology on $E(X, f)$.*
(6) *$G : E(X, f) \times X \to X$ is continuous.*
(7) *$F : \beta(\mathbb{N}) \times X \to X$ is continuous.*
(8) *d_D induces the topology of $E_D(X)$ for every dense subset D of X.*
(9) *There is a dense subset D of X such that d_D induces the topology of $E_D(X)$.*
(10) *The set $\{f^p|_D : p \in \beta(\mathbb{N})\}$ is uniformly equicontinuous for every dense subset D of X.*
(11) *There is a dense subset D of X such that the set $\{f^p|_D : p \in \beta(\mathbb{N})\}$ is uniformly equicontinuous.*
(12) *f^p is continuous for all $p \in \mathbb{N}^*$, and the topology of $E(X, f)$ coincides with the uniform convergence topology.*
(13) *f^p is continuous for all $p \in \mathbb{N}^*$, and the function $p \mapsto f^p : \beta(\mathbb{N}) \to C(X, X)$ is continuous, where $C(X, X)$ is equipped with the compact-open topology.*

Proof. The equivalences (1) ⇔ (2) ⇔ (4) follow from Theorem 3.1 and [15, Lemma 4.2]. The implications (5) ⇒ (8), (8) ⇒ (9), and (10) ⇒ (11) are evident. The equivalences (1) ⇔ (3) ⇔ (7) ⇔ (12) ⇔ (13) are proved in [3, Theorem 3.5]. The implication (11) ⇒ (4) is a particular case of Lemma 3.2. The equivalence (6) ⇔ (7) follows from the diagram:

$$\begin{tikzcd}
\beta(\mathbb{N}) \times X \arrow[r, "F"] \arrow[d, "\hat{\phi}\times Id"'] & X \\
E(X,f) \times X \arrow[ur, "G"'] &
\end{tikzcd}$$

(4) ⇒ (5). Let $V = (\bigcap_{i \leq k}[x_i, V_i]) \cap E(X, f)$ be a basic open subset of $E(X, f)$ and let $f^p \in V$. Then we can find $\epsilon > 0$ such that $B_\epsilon(f^p(x_i)) \subseteq V_i$ for each $i \leq k$. Suppose that $f^q \in B_\epsilon(f^p)$. Then $d_u(f^p, f^q) = sup_{x\in X} d(f^p(x), f^q(x)) < \epsilon$. In particular, we have that $d(f^p(x_i), f^q(x_i)) < \epsilon$ for each $i \leq k$. That is, $f^q(x_i) \in B_\epsilon(f^p(x_i)) \subseteq V_i$ for all $i \leq k$. This shows that $B_\epsilon(f^p) \subseteq V$. Now, let $\epsilon > 0$ and $p \in \beta(\mathbb{N})$. Choose $\delta > 0$ so that if $d(x, y) < \delta$, then $d(f^q(x), f^q(y)) < \frac{\epsilon}{3}$ for all $q \in \beta(\mathbb{N})$. Let $\{x_i : i \leq k\} \subseteq X$ be such that $X = \bigcup_{i \leq k} B_\delta(x_i)$. Put $U = (\bigcap_{i \leq k}[x_i, B_{\frac{\epsilon}{3}}(f^p(x_i))]) \cap E(X, f)$. Clearly, $f^p \in U$. Fix $f^q \in U$ and

$x \in X$. Then we know that $d(x, x_j) < \delta$ for some $j \leq k$. So

$$\begin{aligned} d(f^p(x), f^q(x)) &\leq d(f^p(x), f^p(x_j)) \\ &+ d(f^p(x_j), f^q(x_j)) + d(f^q(x_j), f^q(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, $d_u(f^p, f^q) = sup_{x \in X} d(f^p(x), f^q(x)) \leq \epsilon$. This shows that V is contained in the closed ball with center f^p and radius ϵ. This shows that d_u induces the topology on $E(X, f)$.

(5) $\Rightarrow$ (6). Fix $(f^p, x) \in E(X, f) \times X$ and let $\epsilon > 0$. Choose $\delta > 0$ so that $\frac{\delta}{2} < \frac{\epsilon}{3}$. Since $\{f^n : n \in \mathbb{N}\}$ is dense in $E(X, f)$, we can find $k \in \mathbb{N}$ so that $f^k \in B_{\frac{\delta}{4}}(f^p)$. Choose $\gamma > 0$ that witnesses the uniform continuity of f^k for $\frac{\epsilon}{3}$. Let $(f^q, y) \in B_{\frac{\delta}{4}}(f^p) \times B_\gamma(x)$. We know that

$$d_u(f^q, f^k) \leq d_u(f^q, f^p) + d_u(f^p, f^k) \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} < \frac{\epsilon}{3}.$$

Hence,

$$\begin{aligned} d(f^p(x), f^q(y)) &\leq d(f^p(x), f^k(x)) + d(f^k(x), f^k(y)) + d(f^k(y), f^q(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, $G : E(X, f) \times X \to X$ is continuous.

(6) $\Rightarrow$ (1). From the hypothesis it is easy to check that $E(X, f) \subseteq C_\pi(X, X)$. According to [22, Theorem 5.5.1], we obtain that $E(X, f)$ is metrizable. Now, by (6) and [15, Theorem 4.4], we conclude that $\{f^p : n \in \mathbb{N}\}$ is uniformly equicontinuous on X.

(8) $\Rightarrow$ (10). Let D be a dense subset of X and let $\epsilon > 0$. For $f^p \in E(X, f)$, we let $B^U_\epsilon(f^p|_D) = \{g \in E_D(X) : d_D(f^p|_D, g) < \epsilon\}$. Since d_D induces the topology of $E_D(X)$ and $E_D(X)$ is compact, there is $\{f^{p_i} : i \leq l\} \subseteq E(X, f)$ such that $E_D(X) = \bigcup_{i \leq l} B^u_{\frac{\epsilon}{5}}(f^{p_i}|_D)$. By the density of $\{f^n|_D : n \in \mathbb{N}\}$ in $E_D(X)$, for each $i \leq l$, we can find $n_i \in \mathbb{N}$ so that $f^{n_i}|_D \in B^u_{\frac{\epsilon}{5}}(f^{p_i}|_D)$. Let δ witness the uniform continuity of all f^{p_i}'s for $\frac{\epsilon}{5}$. Fix $q \in \beta(\mathbb{N})$. We choose $j \leq l$ so that $f^q|_D \in B^U_{\frac{\epsilon}{5}}(f^{p_j}|_D)$. Suppose that $x, y \in D$ satisfy that $d(x, y) < \delta$. Then we have that

$$\begin{aligned} d(f^q(x), f^q(y)) &\leq d(f^q(x), f^{p_j}(x)) + d(f^{p_j}(x), f^{n_j}(x)) \\ &+ d(f^{n_j}(x), f^{n_j}(y)) + d(f^{n_j}(y), f^{p_j}(y)) + d(f^{p_j}(y), f^q(y)) \\ &< \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} = \epsilon. \end{aligned}$$

Therefore, the set $\{f^p|_D : p \in \beta(\mathbb{N})\}$ is uniformly equicontinuous.

(9) $\Rightarrow$ (11). The proof is completely similar to that of the previous implication. $\square$

Several equivalences of the previous theorem are known. Indeed, the equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (7) $\Leftrightarrow$ (12) $\Leftrightarrow$ (13) are proved, in a more general setting, in [3, Theorem 3.5]. The implication (6) $\Leftrightarrow$ (1) is a consequence of [15, Theorem 4.4] and (2) $\Leftrightarrow$ (4) is precisely [15, Lemma 4.2].

A dynamical system satisfying the first condition of Theorem 3.3 is called *equicontinuous*.

We want to remark on the next well-known result used in the proof of the implication (6) $\Rightarrow$ (1).

Corollary 3.4. *Let (X, f) be a dynamical system. If $E(X, f) \subseteq C_\pi(X, X)$, then $E(X, f)$ is metrizable.*

For a G-space X, the dynamical system (G, X) is *weakly almost periodic* if and only if $E(G, X) \subseteq C_\pi(X, X)$ (see [9]).

Our next task is to reformulate some results and their proofs from [18] in the context of discrete dynamical systems acting on the semigroup $\mathbb{N}$. In particular, let us consider the following metric that has already been considered in [17].

Now, for each $x, y \in X$, we define

$$d_{\mathbb{N}}(x, y) = \sup\{d(f^n(x), f^n(y)) : n \in \mathbb{N}\}.$$

It is evident that $d_{\mathbb{N}}$ is a metric on X such that

$$d(x, y) \leq d_{\mathbb{N}}(x, y), \tau_d \subseteq \tau_{d_N},$$

and

$$d_{\mathbb{N}}(x, y) = \sup\{d(f^p(x), f^p(y)) : p \in \beta(\mathbb{N})\}.$$

It follows from the definition that if the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous at a point $x \in X$, then the topologies τ_d and $\tau_{d_{\mathbb{N}}}$ coincide at x.

We omit the proof of the following easy lemma.

Lemma 3.5. *Let (X, f) be a dynamical system X and let $x \in X$. Then the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous at x if and only if the function $f : (X, d) \to (X, d_{\mathbb{N}})$ is continuous at x.*

Lemma 3.6. *Let (X, f) be a dynamical system and let $D \subseteq X$. If $B \subseteq D$ is $\tau_{d_{\mathbb{N}}}$-dense in D, then $f^p|_D = f^q|_D$ whenever $f^p|_B = f^q|_B$ for each $p, q \in \beta(\mathbb{N})$.*

Proof. Assume that $B \subseteq D$ is $\tau_{d_{\mathbb{N}}}$-dense in D and that $f^p|_B = f^q|_B$ for some $p, q \in \beta(\mathbb{N})$. Fix $x \in D$ and $\epsilon > 0$. Then there exists $y \in B$ such that

$$d_{\mathbb{N}}(x, y) = \sup\{d(f^s(x), f^s(y)) : s \in \beta(\mathbb{N})\} < \frac{\epsilon}{2}.$$

Hence, we obtain that

$$d(f^p(x), f^q(x)) \leq d(f^p(x), f^p(y)) + d(f^p(y), f^q(y)) + d(f^q(y), f^q(x))$$
$$= d(f^p(x), f^p(y)) + d(f^q(y), f^q(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ was taken arbitrarily, $f^p(x) = f^q(x)$. Therefore, $f^p|_D = f^q|_D$. $\square$

Lemma 3.7. *Let (X, f) be a dynamical system. If $\{f^n : n \in \mathbb{N}\}$ is equicontinuous at each point of a subset $D \subseteq X$, then $E_D(X)$ is metrizable.*

Proof. By equicontinuity, the topologies τ_d and $\tau_{d_{\mathbb{N}}}$ coincide at each point of D. Since X is compact and metrizable, D contains a countable dense subset E. It follows from Lemma 3.6 that the projection map $\eta : E_D(X) \to X^E$ is an embedding and since X^E is metrizable, we conclude that $E_D(X)$ is metrizable. $\square$

If X and Y are topological spaces and $\mathcal{F} \subseteq Y^X$ is a subspace, then the evaluation map $e : X \to Y^{\mathcal{F}}$ satisfies that $E(X, f) \in C(\mathcal{F}, Y)$ for all $x \in X$. Hence, if (X, f) is a dynamical system, then $e[X] \subseteq C(E(X, f), X)$ and $E(X, f) = f_x$ for every $x \in X$.

Lemma 3.8. *Let (X, f) be a dynamical system and let D be a G_δ-subset of X. If $E_D(X)$ is metrizable, then there is a τ_d-dense G_δ-subset D' of D such that $(D', \tau_{d_{\mathbb{N}}}|_{D'})$ is separable.*

Proof. Assume that $E_D(X)$ is metrizable. Let $e : D \to C_u(E_D(X), X)$ be the evaluation map. By [18, Proposition 2.4], there is a dense G_δ-subset D' of D such that e is continuous at every point of D'. Hence, we have that the family $\{f^p : p \in \beta(\mathbb{N})\}$ is equicontinuous at each point of D'. So the topologies τ_d and $\tau_{d_{\mathbb{N}}}$ coincide on D'. Since $(D', \tau_d|'_D)$ is separable, $(D', \tau_{d_{\mathbb{N}}}|_{D'})$ is also separable. $\square$

Lemma 3.9. *Let (X, f) be a dynamical system and let $D \subseteq X$. If $(D, \tau_{d_{\mathbb{N}}}|_D)$ is separable, then $E_D(X)$ is metrizable.*

Proof. Let $D \subseteq X$ be a τ_d-dense subset D of X such that $(D, \tau_{d_{\mathbb{N}}}|_D)$ is separable. Let E be a countable $\tau_{d_{\mathbb{N}}}$-dense subset of D. By Lemma 3.6, the projection map $\eta : E_D(X) \to X^E$ is an embedding and since X^E is metrizable, we obtain that $E_D(X)$ is metrizable. $\square$

In the next theorem, we list some conditions which are equivalent to the equicontinuity of the set $\{f^n : n \in \mathbb{N}\}$ at each point of a dense G_δ-subset of X. We shall need the following two lemmas.

We omit the proof of the next easy and well-known result.

Lemma 3.10. *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, let $\mathcal{F} \subseteq Y^X$, and let $x \in X$. Then the family $\mathcal{F}$ is equicontinuous at x if and only if the evaluation map $e : X \to C_u(\mathcal{F}, Y)$ is continuous at x.*

In [18], the authors show that if (X, f) is a dynamical system with X a compact metric space and $f : X \to X$ is a homeomorphism, then $E(X, f)$ is metrizable if and only if $(X, \tau_{d_{\mathbb{N}}})$ is separable. For a continuous function $f : X \to X$, they state without proof that $E(X, f)$ is metrizable if and only if (X, f) is hereditarily almost equicontinuous if and only if (X, f) is a Radon-Nikodým system. In this context, we have the following theorem.

Theorem 3.11. *For a dynamical system (X, f), the following are equivalent.*

(1) *The set $\{f^n : n \in \mathbb{N}\}$ is equicontinuous at each point of a dense G_δ-subset of X.*
(2) *The set $\{f^p : p \in \beta(\mathbb{N})\}$ is equicontinuous at each point of a dense G_δ-subset of X.*
(3) *There is a dense G_δ-subset D of X such that $E_D(X)$ is metrizable.*
(4) *There is a dense G_δ-subset D of X such that $G : E(X, f) \times D \to X$ is continuous.*
(5) *There is a dense G_δ-subset D of X such that $F : \beta(\mathbb{N}) \times D \to X$ is continuous.*
(6) *There is a dense G_δ-subset D of X such that $E_D(X) \subseteq C_\pi(D, X)$.*
(7) *There is a dense G_δ-subset D of X such that $(D, \tau_{d_{\mathbb{N}}}|_D)$ is separable.*

Proof. The implication (4) ⇒ (6) is trivial; the implication (1) ⇒ (2) is a particular case of Lemma 3.2; the implication (2) ⇒ (3) is a direct consequence of Lemma 3.7; the implication (7) ⇒ (3) follows from Lemma 3.9. The equivalence (4) ⇔ (5) can be deduced from the following commutative diagram:

$$\begin{tikzcd} \beta(\mathbb{N}) \times D \arrow[r, "F"] \arrow[d, "\hat{\phi}\times Id"'] & X \\ E(X,f) \times D \arrow[ur, "G"'] & \end{tikzcd}$$

(3) ⇒ (1). Suppose that $D \subseteq X$ is a dense G_δ-set of X and $E_D(X)$ is metrizable. Let us consider the evaluation map $e : X \to C_u(E_D(X), X)$ given by $E(X, f)(f^p) = f^p(x)$ for each $x \in X$. By [18, Proposition 2.4], we can find a dense G_δ-subset E of D such that the evaluation map $e : D \to C_u(E_D(X), X)$ is continuous at each point of E. From Lemma

3.10, the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous at each point of E and, since D is a dense G_δ-set of X, E is also a dense G_δ-set of X.

(3) $\Rightarrow$ (4). Assume that there is a dense G_δ-subset D' of X such that $E_{D'}(X)$ is metrizable. Consider the function $\sigma : D' \to C_u(E_{D'}(X), X)$ defined by $\sigma(d)(f^p|_{D'}) = f^p(d)$ for each $d \in D'$. For each $n \in \mathbb{N}$, we know that the function $d \mapsto \sigma(d)(f^n|_{D'}) = f^n(d)$ is continuous for all $n \in \mathbb{N}$. By virtue of [18, Proposition 2.4], there is a dense G_δ-subset D of D' such that the function $\sigma|_D : D \to C_u(E_{D'}(X), X)$ is continuous. Let us prove that $G : E(X, f) \times D \to X$ is continuous. Given $d \in D$ and $\epsilon > 0$ there is $\delta > 0$ such that $\sigma[B_\delta(d) \cap D] \subseteq B_\epsilon(\sigma(d))$. Fix $q \in \beta(\mathbb{N})$. If $(f^p, z) \in E(X, f) \times (B_\delta(d) \cap D)$, then

$$d(f^p(z), f^q(d)) = d(\sigma(z)(f^p|_D), \sigma(d)(f^q|_D) \leq d_u(\sigma(z), \sigma(d)) < \epsilon.$$

This shows that $G : E(X, f) \times D \to X$ is continuous at (f^q, d) and it is clear that D is a dense G_δ-subset of X.

(6) $\Rightarrow$ (3). Let D be a dense G_δ-subset of X such that $E_D(X) \subseteq C_\pi(D, X)$. Since D is metric and second countable, by [22, Exercise IV. 12(a)], we must have that $C_\pi(D, X)$ is a σ-space and hence $E_D(X)$ is a compact σ-space. According to [23, Theorem 9] (see also [12, Exercise 5.4.I]), we obtain that $E_D(X)$ is metrizable.

(3) $\Rightarrow$ (7). Assume that D' is a dense G_δ-subset of X such that $E_{D'}(X)$ is metrizable. By Lemma 3.8, there is a dense G_δ-subset D of D' such that $(D, \tau_{d_{\mathbb{N}}}|_D)$ is separable. It is evident that D is also a dense G_δ-subset of X. □

The equivalence (1) $\Leftrightarrow$ (2) of the previous theorem is a particular case of a theorem of Akin, J. Auslander, and K. Berg [2].

Let us consider the space $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\}$ and let $f : X \to X$ be the function defined after Definition 2.1. We know that $E(X, f)$ is a convergent sequence, and hence it is metrizable. Put $D = \{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\}$. Since D is discrete, we have that the family of functions $\{f^p : n \in \mathbb{N}\}$ is equicontinuous at D which is a dense open subset of X. We also know that f^p is not continuous for any $p \in \mathbb{N}^*$. Thus, the dynamical system (X, f) satisfies that $E(X, f)$ is metric, but it does not satisfy any condition of Theorem 3.3.

The following theorem is due to Eli Glasner, Michael Megrelishvili, and Vladimir V. Uspenskij [18, Theorem 1.2]; we would like to include some details of the original proof for the semigroup $\beta(\mathbb{N})$. We need the following easy lemma.

Lemma 3.12. *Let (X, f) be a dynamical system with X a metric compact space. If D is a $\tau_{d_{\mathbb{N}}}$-dense subset of X, then the projection map $\eta : E(X, f) \to E_D(X)$ is a homeomorphism.*

Proof. Let $f^p, f^q \in E(X, f)$ be such that $f^p|_D = f^q|_D$. Fix $x \in X$. Let $\epsilon > 0$. Choose $y \in D$ such that $d_{\mathbb{N}}(x, y) < \frac{\epsilon}{2}$. Then

$$d(f^p(x), f^q(x)) \leq d(f^p(x), f^p(y)) + d(f^p(y), f^q(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $f^p = f^q$. This shows that η is one-to one and since $E(X, f)$ and $E_D(X)$ are compact, it follows that η is a homeomorphism. □

In the *Sufficiency* part of the following theorem we give a specific description of a compatible metric on $E(X, f)$ in the case that it is metrizable.

Theorem 3.13. *Let (X, f) be a dynamical system with X a compact metric space. Then $E(X, f)$ is metrizable if and only if $(X, d_{\mathbb{N}})$ is separable.*

Proof. Necessity. Suppose that $(X, d_{\mathbb{N}})$ is not separable. Then there are $\epsilon > 0$ and an uncountable subset A of X such that $d_{\mathbb{N}}(x, y) > \epsilon$ for distinct $x, y \in A$. Since A is second countable as a subset of X, by the Cantor-Bendixson Theorem (see [12, Problem 1.7.11]), we can find a perfect subset $M \subseteq A$ such that $A \setminus M$ is countable. Let $Y = cl_X(\{f^n(a) : a \in M \text{ and } n \in \mathbb{N}\})$. It is clear that Y is an invariant closed subset of X and that $E(Y)$ is also metrizable. By Theorem 3.11, there is a point $x \in Y$ such that $\{f^n|_Y : n \in \mathbb{N}\}$ is equicontinuous. Let V be an open subset of Y that contains x such that if $y \in V$, then $d(f^m(x), f^m(y)) < \epsilon$ for all $m \in \mathbb{N}$. We know that $V \cap \{f^n(a) : a \in M \text{ and } n \in \mathbb{N}\} \neq \emptyset$. Hence, there is $n \in \mathbb{N}$ such that $f^{-n}(V) \cap M \neq \emptyset$. Choose $\delta > 0$ witnessing the uniform continuity of the functions f, f^2,...., and f^{n-1} for ϵ. Let U be an open subset of X with diameter $< \delta$ such that $U \cap f^{-n}(V) \cap M \neq \emptyset$. Since M has no isolated points, there are two distinct points $y, z \in U \cap f^{-n}(V) \cap M \neq \emptyset$. Then $d(f^i(x), f^i(y)) < \epsilon$ for all $i < n$, and $d(f^{n+m}(y), f^{n+m}(z)) < \epsilon$ for all $m \in \mathbb{N}$, but this implies that $d_{\mathbb{N}}(x, y) < \epsilon$, which is a contradiction.

Sufficiency. This follows from Lemma 3.9, but we would like to give the metric that makes $E(X, f)$ be metrizable. Let $D = \{d_n : n \in \mathbb{N}\}$ be a $\tau_{d|_{\mathbb{N}}}$-dense countable subset of X. By Lemma 3.12, the projection $\eta : E(X, f) \to E_D(X)$ is a homeomorphism. We know that $\rho(g, h) = \sum_{n=0}^{\infty} \frac{d(g(d_n), h(d_n))}{2^n}$, for each $g, h \in X^D$, is a metric compatible with the topology of X^D. Hence, we deduce that, for $p, q \in \beta(\mathbb{N})$,

$$\rho(f^p, f^q) = \sum_{n=0}^{\infty} \frac{d(f^p(d_n), f^q(d_n))}{2^n}$$

is a metric on $E(X, f)$ compatible with its topology. □

In connection with Corollary 3.4, we have the following.

Theorem 3.14. *Let (X, f) be a dynamical system. If $E(X, f) \subseteq C_\pi(X, X)$, then there is a dense G_δ-subset D of $E(X, f)$ such that the topology of $E(X, f)$ and the topology induced by the metric d_u coincide at each point of D.*

Proof. It is clear every basic open subset of $E(X, f)$ contains an open ball of the metric space $(E(X, f), d_u)$. According to Corollary 3.4, we have that $E(X, f)$ is metrizable and, by the assumption, $G : E(X, f) \times X \to X$ is separately continuous. By a generalization of Namioka's theorem given in [8], there exists a dense G_δ-subset D of $E(X, f)$ such that $G : E(X, f) \times X \to X$ is continuous at each point of $D \times X$. Fix $g \in D$ and $\epsilon > 0$. For each $x \in X$ we can find $\delta_x > 0$ and an open neighborhood V_x of g in $E(X, f)$ such that $G[V_x \times B_{\delta_x}(x)] \subseteq B_{\frac{\epsilon}{2}}(g)$. By the compactness of X, we can find $x_0, ..., x_k \in X$ such that $X = \bigcup_{i \leq k} B_{\delta_{x_i}}(x_i)$. Let $U_g = \bigcap_{i \leq k} V_{x_i}$. Fix $(h, x) \in U_g \times X$. Then, $x \in B_{\delta_{x_i}}(x_i)$ for some $i \leq k$, and so $G((h, x)) = h(x) \in B_{\frac{\epsilon}{2}}(g)$. Thus, $d(g(x), h(x)) < \frac{\epsilon}{2}$ for all $(h, x) \in U_g \times X$. That is, $d_u(g, h) \leq \frac{\epsilon}{2} < \epsilon$ for all $h \in U_g$. So $g \in U_g \subseteq B^{d_u}_\epsilon(g)$. Therefore, the topology of $E(X, f)$ and the topology induced by the metric d_u coincide at each point of D. □

Corollary 3.15. *Let (X, f) be a dynamical system. If $E(X, f) \subseteq C_\pi(X, X)$, then there is a dense G_δ-subset of $E(X, f)$ that is metrizable with the sup-metric d_u.*

4. Topological semigroup structure

In this section, we shall give some conditions for the Ellis semigroup $E(X, f)$ to be a topological semigroup.

Given a compact metric space (X, d), we can extend the metric d_u to the whole product X^X by defining $d_u(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$. As the metric d is bounded, d_u is well defined and it is a metric. We also know that $C_u(X, X)$ is a topological semigroup with the composition of functions. In this context, we have the following theorem.

Theorem 4.1. *Let (X, f) be a dynamical system. If $f^p \in C(X, X)$ for some $p \in \beta(\mathbb{N})$ and τ_{d_u} is the topology on $E(X, f)$ induced by the sup-metric d_u, then the operation on $E(X, f)$ is τ_{d_u}-continuous at (f^p, g) for all $g \in E(X, f)$.*

Proof. Fix $q \in \beta(\mathbb{N})$ and $\epsilon > 0$. We can find $\delta > 0$ witnessing the uniform continuity of f^p for $\frac{\epsilon}{2}$ such that $\delta < \frac{\epsilon}{2}$. Let $(f^s, f^t) \in B^{d_u}_\delta(f^p) \times B^{d_u}_\delta(f^q)$. Then we have that $d(f^p(y), f^s(y)) \leq d_u(f^p, f^s) < \delta$ and $d(f^q(y), f^t(y)) \leq$

$d_u(f^q, f^t) < \delta$ for every $y \in X$, and so

$$d(f^{q+p}(x), f^{t+s}(x)) = d(f^p(f^q(x)), f^s(f^t(x)))$$
$$\leq d(f^p(f^q(x)), f^p(f^t(x))) + d(f^p(f^t(x)), f^s(f^t(x))) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that the operation on $E(X, f)$ is τ_{d_u}-continuous at (f^p, f^q). $\square$

Let (X, f) be an arbitrary dynamical system. It follows from the previous theorem that if $E(X, f) \subseteq C(X, X)$, then $E(X, f)$ is a topological semigroup with the topology inherited from $C_u(X, X)$. As a direct application of this remark and of Theorem 3.3, we have that if the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous on X, then $E(X, f)$ is a topological semigroup.

For a dynamical system (X, f), we let $\mathcal{O} = \{cl_X \mathcal{O}_f(x) : x \in X\}$. It is evident that $\mathcal{O}$ covers X. For each $x \in X$, we define

$$\psi_x : E(X, f) \to cl_X \mathcal{O}_f(x)^{cl_X \mathcal{O}_f(x)}$$

by the requirement $\psi_x(f^p) = f^p|_{cl_X \mathcal{O}_f(x)}$ for every $p \in \beta(\mathbb{N})$. Let $\tau_{\mathcal{O}}$ be the weak topology on $E(X, f)$ that makes the function

$$\psi_x : E(X, f) \to (cl_X \mathcal{O}_f(x)^{cl_X \mathcal{O}_f(x)}, \tau_{d_u})$$

continuous, where d_u is the uniform metric for every $x \in X$. If τ_π is the original topology of $E(X, f)$, then it is evident that $\tau_\pi \subseteq \tau_{\mathcal{O}}$. Let us see a condition equivalent to the equality $\tau_\pi = \tau_{\mathcal{O}}$.

Theorem 4.2. *For any dynamical system (X, f), the following are equivalent.*

1. *The Ellis semigroup $E(X, f)$ is a topological semigroup.*
2. *$E(cl_X \mathcal{O}_f(x), f|_{cl_X \mathcal{O}_f(x)})$ is a subspace of $C_u(cl_X \mathcal{O}_f(x), cl_X \mathcal{O}_f(x))$ for all $x \in X$.*
3. *$\tau_\pi = \tau_{\mathcal{O}}$.*

Proof. We remind the reader that $\mathcal{O}_f(x) = \{f^p(x) : p \in \beta(\mathbb{N})\}$ for each $x \in X$.

(1) $\Rightarrow$ (2). Consider the functions $\varphi : E(X, f) \times E(X, f) \to E(X, f) \times cl_X \mathcal{O}_f(x)$ and $\sigma : E(X, f) \times cl_X \mathcal{O}_f(x) \to X$ defined, respectively, as

$$\varphi(f^p, f^q) = (f^p, f^q(x)) \quad \text{for all } f^p, f^q \in E(X, f),$$

and

$$\sigma(f^p, y)) = f^p(y) \qquad \text{for all } f^p \in E(X, f),\ y \in cl_X \mathcal{O}_f(x).$$

Observe that φ is continuous and onto. The space $E(X, f) \times E(X, f)$ being compact, φ is a quotient mapping [12, Corollary 2.4.8]. Now, since

$E(X,f)$ is a topological semigroup, the composition $\sigma \circ \varphi$ is continuous which implies that σ is a continuous function [12, Proposition 2.4.2]. It is not hard to prove that the continuity of σ implies that

$$E(cl_X\mathcal{O}_f(x), f\mid_{cl_X\mathcal{O}_f(x)}) \subseteq C(cl_X\mathcal{O}_f(x), cl_X\mathcal{O}_f(x))$$

and, by Theorem 3.3, the metric d_u induces the topology of

$$E(cl_X\mathcal{O}_f(x), f\mid_{cl_X\mathcal{O}_f(x)})$$

for every $x \in X$.

(2) $\Rightarrow$ (3). We know that $\tau_\pi \subseteqq \tau_{\mathcal{O}}$. To see the reverse inclusion, observe from our assumption that

$$\psi_x : (E(X,f), \tau_\pi) \to C_u(cl_X\mathcal{O}_f(x), cl_X\mathcal{O}_f(x))$$

is continuous for every $x \in X$. Therefore, $\tau_{\mathcal{O}} \subseteq \tau_\pi$.

(3) $\Rightarrow$ (1). Let f^p and f^q be two elements of $E(X,f)$, and let $\{f^{p_\lambda}\}_{\lambda\in\Delta}$ and $\{f^{q_\lambda}\}_{\lambda\in\Delta}$ be two nets pointwise converging to f^p and f^q, respectively. We shall prove that the net $\{f^{p_\lambda} \circ f^{q_\lambda}\}_{\lambda\in\Delta}$ pointwise converges to $f^p \circ f^q$. To do that we fix $x \in X$. Since the metric d_u induces the topology on $E(cl_X\mathcal{O}_f(x), f\mid_{cl_X\mathcal{O}_f(x)})$ and each function belonging to this set is continuous, given an arbitrary $\epsilon > 0$ there exists $\lambda_0 \in \Delta$ such that

$$d(f^{p_\lambda}(y), f^p(y)) < \epsilon \quad \text{and} \quad d(f^{q_\lambda}(y), f^q(y)) < \epsilon,$$

for every $y \in cl_X\mathcal{O}_f(x)$ and for every $\lambda_0 \leq \lambda \in \Delta$. As $f|_{cl_X\mathcal{O}_f(x)}$ is continuous, we may choose λ_0 satisfying the additional condition

$$d(f^p(f^{q_\lambda}(x), f^p(f^q(x)) < \epsilon$$

for all $\lambda_0 \leq \lambda \in \Delta$. Then

$$\begin{aligned} &d(f^{p_\lambda}(f^{q_\lambda}(x)), f^p(f^q(x))) \\ &\leq d(f^{p_\lambda}(f^{q_\lambda}(x)), f^p(f^{q_\lambda}(x))) + d(f^p(f^{q_\lambda}(x)), f^p(f^q(x))) \leq 2\epsilon \end{aligned}$$

for every $\lambda_0 \leq \lambda \in \Delta$, which completes the proof. $\square$

It is a well-known result that topological transitivity is equivalent to the existence of dense orbit in the case that the phase space is compact (more generally, a Baire space).

Corollary 4.3. *If (X,f) is topologically transitive, then the following conditions are equivalent.*

(1) *$E(X,f)$ is a topological semigroup.*
(2) *$E(X,f)$ is metrizable with the uniform metric d_u.*

An abstract group G endowed with a topology τ is said to be a *paratopological group* if the operation on G is τ-continuous. As we said in the introduction, a dynamical system (X,f) is distal if and only if $E(X,f)$ is a group. We have the following corollary.

Corollary 4.4. *If (X, f) is distal, then the following conditions are equivalent.*

(1) $E(X, f)$ *is a paratopological group.*
(2) $E(cl_X \mathcal{O}_f(x), f\mid_{cl_X \mathcal{O}_f(x)})$ *is a subspace of* $C_u(cl_X \mathcal{O}_f(x), cl_X \mathcal{O}_f(x))$ *for all* $x \in X$.
(3) $\tau_\pi = \tau_\mathcal{O}$.

The previous results suggest the following problem.

Problem 4.5. Determine the dynamical systems satisfying that the Ellis semigroup of their orbit closures are metrizable.

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