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ON CONVERGENT SEQUENCES AND COPIES OF βN IN THE STONE SPACE OF ONE BOOLEAN ALGEBRA

A. A. GRYZLOV

Abstract. We consider the Stone space of one Boolean algebra constructed by Murray G. Bell (see Example 2.1 in *Compact ccc nonseparable spaces of small weight*, Topology Proc. **5** (1980), 11–25 (1981)), which is a compactification, BN, of a countable discrete space N.

We get a necessary condition for a set $A \subseteq N$ to be a convergent sequence in BN and a necessary condition for a set $A \subseteq N$ to be such that $\overline{A}$ is homeomorphic to βN.

1. Introduction

We consider the Stone space of the Boolean algebra constructed by Murray G. Bell [1, Example 2.1]. This space, BN, is a compactification of a countable discrete space, N, with ccc non-separable remainder.

In [5], it was proved that for any infinite chain A of N, one gets $|\overline{A} \setminus A| = 1$, i.e., A is a convergent sequence [2, Theorem 3.9]. We also showed [2, Lemma 3.5] that if $A \subseteq N$ is a strict anti-chain (see the definition in §2), then $\overline{A}$ is homeomorphic to βN.

Here we get the following.

Theorem 3.1. *If a set $A \subseteq N$ is such that $|\overline{A} \setminus A| = 1$, then $A \setminus K$ is a chain for some finite $K \subseteq A$.*

Theorem 3.2. *If the closure $\overline{A}$ of a set $A \subseteq N$ is a copy of βN, then A is a union of finitely many anti-chains.*

Example 3.3. *There are two anti-chains A', $A'' \subseteq N$ such that $\overline{A'}$ and $\overline{A''}$ are copies of βN, but $\overline{A' \cup A''}$ is not a copy of βN.*

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2. Preliminaries

Bell's construction of the compactification BN [1] follows.

Let $P = \{f \in \omega^\omega : 0 \leqslant f(n) \leqslant n+1 \text{ for all } n \in \omega\}$ and
$N = \{f|_n : f \in P,\, n \in \omega\}$.
Define $T = \{\pi \in N^\omega : \operatorname{dom}\pi(n) = n+1 \text{ for all } n \in \omega\}$.
For every $s \in N$, let $C_s = \{t \in N : t|_{\operatorname{dom} s} = s\}$.
For every $\pi \in T$, let $C_\pi = \cup\{C_{\pi(n)} : n \in \omega\}$.
Let B be the Boolean algebra, generated by

$$B' = \{C_\pi : \pi \in T\}$$

in the power set of N.

Note that $\{\{s\} : s \in N\} \cup \{C_s : s \in N\} \subseteq B$.

Denote by BN the Stone space of B. We are identifying each $s \in N$ with the ultrafilter $\xi_x \in BN$ such that $\{s\} \in \xi_x$. So BN is a compactification of the countable discrete space N. We will denote points of N as $f|_n$, s, t.

There is an order on N: $s \leqslant t$ if t is an extension of s for s, $t \in N$. We write $s < t$ if $s \leqslant t$ and $s \neq t$.

Recall that a subset A of the ordered set N is called an anti-chain if A consists of incomparable points of N.

We call an anti-chain $A \subseteq N$ a *strict* anti-chain, if $\operatorname{dom} x \neq \operatorname{dom} y$ for all x, $y \in A$, $x \neq y$.

For $\pi \in T$ and $M \subseteq \omega$, define $C_{\pi|M} = \cup\{C_{\pi(n)} : n \in M\}$.

It was proved in [5, Lemma 3.1] that $C_{\pi|M} \in B$.

The following results were proved in [5]

Theorem 2.1. *The family*

$$\tilde{B} = \{\overline{C_{\pi|M} \setminus \bigcup_{i<n} C_{\pi_i}} : M \subseteq \omega,\, n \in \omega \text{ and } \{\pi\} \cup \{\pi_i : i < n\} \subseteq T\}$$

is a base for the topology of BN.

Theorem 2.1 follows from [5, Lemma 3.1 and Lemma 3.2] and from the fact that $\{s\} \in \tilde{B}$ for all $s \in N$.

The following result is [5, Lemma 3.5].

Theorem 2.2. *Let $\{s_i : i \in \omega\} \subseteq N$ be a strict anti-chain. If $x_i \in \overline{C_{s_i}}$ for each $i \in \omega$, then $\overline{\{x_i : i \in \omega\}}$ is homeomorphic to βN.*

Theorem 2.3. *Let $A = \{s_i : i \in \omega\}$ be an infinite chain in N. Then $|\overline{A} \setminus A| = 1$, i.e., A is a convergent sequence in BN.*

More results about BN can be found in [4], [2], and [3].

3. Main Results

Theorem 3.1. *If a set $A \subseteq N$ is such that $|\overline{A} \setminus A| = 1$, then $A \setminus K$ is a chain for some finite $K \subseteq A$.*

Proof. Let $A \subseteq N$ be such that $\overline{A} \setminus A = \{x\}$. Then $|A| = \omega$ and $x \in BN \setminus N$. We claim that for all $s \in N$ the following is true:

$$(*) \qquad \text{If } A' \subseteq A \text{ is infinite, then either } C_s \cap A' \text{ is finite or } (N \setminus C_s) \cap A' \text{ is finite.}$$

Otherwise, for some infinite $A' \subseteq A$, we have $\overline{C_s \cap A'} \ni x$ and $\overline{(N \setminus C_s) \cap A'} \ni x$, but $\overline{C_s} \cap \overline{N \setminus C_s} = \emptyset$.

Denote $A_1 = \{s \in A : |(N \setminus C_s) \cap A| = \omega\}$.

We will prove that A_1 is a finite set. Suppose $|A_1| = \omega$. We construct a strict anti-chain $\{s_i : i \in \omega\} = A_2 \subseteq A_1$.

Let $s_0 = s$ for some $s \in A_1$. Assume $\{s_i : i \leqslant n\}$ has been chosen. Since $s_i \in A_1$, by $(*)$, we have $|C_{s_i} \cap A| < \omega$, and therefore $|C_{s_i} \cap A_1| < \omega$.

Then $|(\bigcup_{i \leqslant n} C_{s_i}) \cap A_1| < \omega$, and we choose $s_{n+1} \in A_1 \setminus \bigcup_{i \leqslant n} C_{s_i}$ such that $\operatorname{dom} s_{n+1} > \operatorname{dom} s_i$ $(i \leqslant n)$.

Note that s_{n+1} is not an extension of s_i for all $i \leqslant n$.

By construction, the set $A_2 = \{s_i : i < \omega\}$ is a strict anti-chain. By Theorem 2.2, $\overline{A}_2$ is homeomorphic to βN, but $\overline{A}_2 \setminus A_2 = \{x\}$, a contradiction. So A_1 is a finite set.

Denote $\widetilde{A} = A \setminus A_1$.

We will prove that $\widetilde{A}$ is a chain. Note that the following holds:

$$(**) \qquad |\widetilde{A} \setminus C_s| < \omega \text{ for all } s \in \widetilde{A}.$$

Let $s, s' \in \widetilde{A}$ so that $s \neq s'$. If s and s' are not comparable, we obtain $C_s \cap C_{s'} = \emptyset$. Then $C_{s'} \cap \tilde{A} \subseteq \tilde{A} \setminus C_s$. But $C_{s'} \cap \tilde{A}$ is infinite and this contradicts (**). So $\widetilde{A}$ is a chain. □

Theorem 3.2. *If the closure $\overline{D}$ of a set $D \subseteq N$ is homeomorphic to βN, then D is the union of finitely many anti-chains.*

Proof. Let $D = \{s_n : n \in \omega\}$ be a subset of N such that $\overline{D}$ is homeomorphic to βN. First, we prove that

$$(*) \qquad \text{there is a natural number } K \text{ such that the cardinality of every chain in } D \text{ is not more than } K.$$

For $s_n \in D$, denote

$$P(s_n) = \{s_k \in D : s_k \leqslant s_n\}, \qquad h(s_n) = |P(s_n)|.$$

Note that each set $P(s_n)$ is a chain and $s_n = \max P(s_n)$. Note also that D contains no infinite chain. Indeed, an infinite chain is a converging sequence by Theorem 2.3 and this contradicts that $\overline{D}$ is homeomorphic to βN. Thus, every chain in D is a subset of $P(s_n)$ of some s_n.

We prove that there is a number K such that $h(s_n) \leqslant K$ for all $s_n \in D$.

Assume otherwise, i.e., that $\{h(s_n) : s_n \in D\}$ is unbounded. As a first step, we construct a sequence $\{s_{n_k} : k \in \omega\}$ in the following way.

Let $s_{n_1} \in D$ be such that $h(s_{n_1}) = \min\{h(s_n) : s_n \in D\}$. Assume we have chosen $\{s_{n_i} : i \leqslant k\}$. Choose $s_{n_{k+1}}$ such that $h(s_{n_{k+1}}) \geqslant \sum_{i=1}^{k} \operatorname{dom} s_{n_i} + k$.

Consider the sequence $\{s_{n_k} : k \in \omega\}$. For all s_{n_k} $(k \in \omega)$, define a finite set $H(s_{n_k})$ as follows:

$$H(s_{n_k}) = P(s_{n_k}) \setminus \cup\{P(s_{n_i}) : i < k\}.$$

Note that the family $\{H(s_{n_k}) : k \in \omega\}$ satisfies the following:

$$|H(s_{n_k})| \geqslant k \text{ and } H(s_{n_k}) \cap H(s_{n_m}) = \emptyset \text{ if } s_{n_k} \neq s_{n_m}.$$

For every set $H(s_{n_k})$, let $\{H_1(s_{n_k}),\, H_2(s_{n_k})\}$ be a partition of $H(s_{n_k})$ such that

(**) for every two elements s and t of one member of the partition satisfying $s < t$, there is an element r of the other member of the partition such that $s < r < t$.

Denote $H_i = \cup\{H_i(s_{n_k}) : k \in \omega\}$ $(i = 1,\, 2)$. We have $H_i \subseteq D$ $(i = 1,\, 2)$ and $H_1 \cap H_2 = \emptyset$. We will prove that $\overline{H}_1 \cap \overline{H}_2 \neq \emptyset$ in order to get a contradiction with the condition that $\overline{D}$ is homeomorphic to βN.

Assume that $\overline{H}_1 \cap \overline{H}_2 = \emptyset$. There is a finite cover $\lambda = \{O_i : i = 1, \dots, n\}$ of H_1 such that $H_2 \cap (\bigcup_{i \leqslant n} O_i) = \emptyset$ where O_i are basic open sets of the form described in Theorem 2.1.

Let k be a natural number such that $k > 2n + 1$. Consider the sets $H_1(s_{n_k})$ and $H_2(s_{n_k})$ and note that $|H_i(s_{n_k})| > n$ $(i = 1,\, 2)$.

We claim that if $s,\, t \in H_1(s_{n_k})$ are distinct, then there is no $O_i \in \lambda$ such that $s,\, t \in O_i$.

Assume otherwise and let $s,\, t \in H_1(s_{n_k})$ and $O_i \in \lambda$ be such that $s < t$ and $s,\, t \in O_i$.

By (**), there is $r \in H_2(s_{n_k})$ such that $s < r < t$. Since $O_i = \overline{C_{\pi|M} \setminus \bigcup_{j<\ell} C_{\pi_j}}$ for some $\{\pi\} \cup \{\pi_j : j < \ell\} \subseteq T$ and $s \in O_i$, we have $s \in C_{\pi|M}$. Since $s < r$, we have $r \in C_{\pi|M}$.

On the other hand, $r \in H_2(s_{n_k})$ implies $r \notin O_i$. Since $r \in C_{\pi|M}$, we obtain $r \in \bigcup_{j<\ell} C_{\pi_j}$. But $r < t$ and so $t \in \bigcup_{j<\ell} C_{\pi_j}$; therefore, $t \notin O_i$.

So we proved that distinct elements of the set $H_1(s_{n_k})$ are in distinct elements of the cover $\lambda = \{O_i : i = 1, \ldots, n\}$, but this contradicts $|H_1(s_{n_k})| > n$.

So we proved that $\overline{H}_1 \cap \overline{H}_2 \neq \emptyset$, but this contradicts the fact that $\overline{D}$ is a copy of βN. This contradiction proves that our assumption that $\{h(s_n) : s_n \in D\}$ is unbounded is not true.

Note that since D contains no infinite chain, the following holds:

(m) Every nonempty subset $D' \subset D$ has an element which is maximal in D'.

Here an element $s \in D'$ is called *maximal* in D' if there is no $t \in D'$, such that $s < t$.

We will construct the family of subsets of D, $\{D_i : i \in \omega\}$, as follows.

Let $D_0 \subseteq D$ be the set of all elements of D which are maximal in D. Assume $\{D_i : i \leqslant n\}$ has been constructed. If $D \setminus \cup\{D_i : i \leqslant n\} \neq \emptyset$, let D_{n+1} be the set of all maximal elements of $D \setminus \cup\{D_i : i \leqslant n\}$.

Note that D_n is an anti-chain for all n and $D_n \cap D_m = \emptyset$ if $n \neq m$. Note also that by (m), $D_n = \emptyset$ if and only if $D \setminus \cup\{D_i : i < n\} = \emptyset$ and therefore, if $D_n = \emptyset$ for some $n \in \omega$, then $D_m = \emptyset$ for all $m > n$.

Let $D_{n+1} \neq \emptyset$ for some n. We will prove that

$$(***) \qquad \text{for every } s \in D_{n+1}, \text{ there is } q \in D_n \text{ such that } s < q.$$

Let $s \in D_{n+1}$. There is an element $t \in D$ such that $s < t$, otherwise $s \in D_0$. Moreover, if $s < t$, then $t \in \cup\{D_i : i \leqslant n\}$, because otherwise s is not maximal in $D \setminus \cup\{D_i : i \leqslant n\}$, and $s \notin D_{n+1}$. Let

$$(\ell) \qquad \ell = \max\{n : \text{there is } t \in D_n \text{ such that } s < t\}.$$

Let us show that s is maximal in $D \setminus \cup\{D_i : i \leqslant \ell\}$. Assume that the set $A = \{r \in D \setminus \cup\{D_i : i \leqslant \ell\} : s < r\}$ is nonempty. By (m), A has a maximal element r_0. Then r_0 is maximal in $D \setminus \cup\{D_i : i \leqslant \ell\}$, and therefore $r_0 \in D_{\ell+1}$, but this contradicts (ℓ). So s is maximal in $D \setminus \cup\{D_i : i \leqslant \ell\}$, and therefore $s \in D_{\ell+1}$. Then $\ell = n$, because otherwise, since $D_{\ell+1} \cap D_{n+1} = \emptyset$, we get $s \notin D_{n+1}$. So (***) is proved.

Let K be a natural number such that the cardinality of every chain in D is not more than K.

We will prove that $D_K = \emptyset$ and therefore $D = \cup\{D_i : i < K\}$. Assume otherwise; then $D_i \neq \emptyset$ for all $i = 0, \ldots, K$. By (***), for each $i = 0, \ldots K$, there exists $s_i \in D_{K-i}$ such that $\{s_i : i = 0, \ldots K\}$ is a chain. □

Example 3.3. There are two anti-chains A', $A'' \subseteq N$ such that $\overline{A'}$ and $\overline{A''}$ are homeomorphic to βN, but $\overline{A' \cup A''}$ is not homeomorphic to βN.

Let $A = \{s_n : n \in \omega\}$ be an infinite strict anti-chain. Let $\lambda = \{A_m : m \in \omega\}$ be a partition of A such that

(1) $\mathrm{dom}(s_\ell) < \mathrm{dom}(s_{\ell'})$ for all $s_\ell \in A_m$, $s_{\ell'} \in A_{m'}$ and $m < m'$,
(2) $|A_m| = m$.

Define $d_m = \max\{\mathrm{dom}\, s_n : s_n \in A_m\}$.

We construct an anti-chain A' as follows. For every $s_n \in A_m$, fix an element $s'_n \in N$ such that $\mathrm{dom}\, s'_n = d_m$ and $s_n \leq s'_n$.

Let $A'_m = \{s'_n : s_n \in A_m\}$, $A' = \cup\{A'_m : m \in \omega\}$, and $\lambda' = \{A'_m : m \in \omega\}$.

Now we will define an anti-chain A''. For every $s'_n \in A'$, let s''_n be a successor of s'_n, i.e., $s'_n < s''_n$ and $\mathrm{dom}\, s''_n = \mathrm{dom}\, s'_n + 1$. Let $A'' = \{s''_n : s'_n \in A'\}$. We prove that

(1) A' and A'' are homeomorphic to βN,
(2) $\overline{A'} \cap \overline{A''} \neq \emptyset$ and therefore $\overline{A' \cup A''}$ is not homeomorphic to βN.

For the strict anti-chain $A = \{s_n : n \in \omega\}$, there is an associated family $\{C_{s_n} : s_n \in A\}$ (see the definition of the Boolean algebra B). Note that $A' = \{s'_n : n \in \omega\}$ and $A'' = \{s''_n : n \in \omega\}$ are such that $s'_n, s''_n \in C_{s_n}$ for all $n \in \omega$. Therefore, by Theorem 2.2, $\overline{A'}$ and $\overline{A''}$ are homeomorphic to βN.

Let us regard $A' = \{s'_n : n \in \omega\}$.

Note that

(∗) there is an ultrafilter ξ' on A' such that for every $F' \in \xi'$ and for each natural number k, there is $A'_m \in \lambda'$ such that $|A'_m \cap F'| > k$.

Indeed, denote $\eta_k = \{E \subseteq A' : |E \cap A'_m| \leq k \text{ for all } m \in \omega\}$ for every $k \in \omega$ and $\tilde{\eta} = \cup\{\eta_k : k \in \omega\}$. Since $|A_m| = m$ for all $m \in \omega$, the family $\theta = \{A' \setminus E : E \in \tilde{\eta}\}$ has the finite intersection property. An ultrafilter ξ' such that $\theta \subseteq \xi'$ is as required.

Let $\{x(\xi')\} = \cap\{\overline{F'} : F' \in \xi'\}$, then $x(\xi') \in \overline{A'}$.

We will show that $x(\xi') \in \overline{A''}$. Assume that there is a neighbourhood $Ox(\xi') = \overline{C_{\pi|M} \setminus \bigcup_{i<k_0} C_{\pi_i}}$ of $x(\xi')$ for some $\{\pi\} \cup \{\pi_i : i < k_0\} \subseteq T$, such that $Ox(\xi') \cap A'' = \emptyset$.

Since $x(\xi') \in \overline{A'}$ and $\overline{A'}$ is homeomorphic to βN, $Ox(\xi') \cap A' \in \xi'$. Then, by (∗), there is $A'_m \in \lambda'$ such that $|A'_m \cap Ox(\xi') \cap A'| = |A'_m \cap Ox(\xi')| > k_0$. By the construction of the sets A' and A'', it follows that

(∗∗) if $s'_n \in Ox(\xi') \cap A'_m$, then $s''_n \in C_{\pi|M}$, and hence $s''_n \in C_{\pi|M} \cap A''_m$.

Define $D = \{s''_n : s'_n \in Ox(\xi') \cap A'_m\}$. From (∗∗), it follows that

$$D \subseteq C_{\pi|M} \cap A''_m \quad \text{and} \quad |D| = |Ox(\xi') \cap A'_m| > k_0.$$

From $Ox(\xi') \cap A''_m = \emptyset$ and $(**)$, it follows that $D \subseteq \bigcup_{i<k_0} C_{\pi_i}$. Recall also that $\operatorname{dom} s''_n = \operatorname{dom} s'_n + 1 = d_m + 1$ and $s'_n < s''_n$ for all $s''_n \in A''_m$.

We claim that if $s''_n \in D \cap C_{\pi_i}$ for some $i < k_0$, then $s''_n \in C_{\pi_i(d_m)}$ and $\pi_i(d_m) = s''_n$. Indeed, if $s''_n \in C_{\pi_i(l)}$, then $l \leq d_m$. If $l < d_m$, then $s'_n \in C_{\pi_i(l)}$, but since $s''_n \in D$, we have $s'_n \in Ox(\xi') \cap A'_m$, and therefore $s'_n \notin \bigcup_{i<k_0} C_{\pi_i}$. It is a contradiction. So $l = d_m$.

From this it follows that if $s''_{n_1}, s''_{n_2} \in D$ are distinct and $s''_{n_1} \in C_{\pi_{i_1}}$ and $s''_{n_2} \in C_{\pi_{i_2}}$, then $i_1 \neq i_2$.

We get that $k_0 \geqslant |D|$, but $|D| > k_0$. It is a contradiction.

So $x(\xi') \in \overline{A''}$ and $\overline{A'} \cap \overline{A''} \neq \emptyset$.

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