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by

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SUPER-REMOTE POINTS

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Abstract. We construct a new type of point in Čech-Stone compactifications of metrizable crowded spaces. Every point $p \in X^*$ of this type is a remote point and a weak P-point simultaneously. Moreover, if a metrizable locally compact space X satisfies the countable chain condition (ccc), then p is a P-point in

$$X^* \setminus \bigcup\{D^* : D \text{ is closed and discrete in } X\}.$$

If a metrizable space X nowhere satisfies the ccc, then p is not in the closure of any subset of $\beta X \setminus \{p\}$ satisfying the ccc. This provides a partial answer to a question of Jan van Mill.

1. Introduction

In this paper we show that the Čech-Stone remainder $X^* = \beta X \setminus X$ of every non-compact metrizable space X contains points that we call super-remote. To define these points, we need the notion of a β-nowhere dense set of βX.

A subset F of βX is said to be *β-nowhere dense (β-nd)* if for any discrete family $\mathcal{U}$ of open subsets of X there is a choice $\{V_U : U \in \mathcal{U}\}$ of non-empty open sets, with $V_U \subset U$, such that $F \cap \operatorname{Ex} \bigcup_U V_U = \emptyset$, where $\operatorname{Ex} O = \beta X \setminus [X \setminus O]_{\beta X}$ is the largest open set in βX whose trace on X equals O (and $[A]_Y$ denotes the closure of a set A in the space Y).

We call a point p of X^* *super-remote* if it is not in the closure of any σ-compact and β-nd subset $\beta X \setminus \{p\}$.

Theorem 1.1. *Every non-compact metrizable space X has super-remote points in its Čech-Stone remainder X^*.*

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Key words and phrases. β-nowhere dense set, metrizable space, remote point, super-remote point, weak P-point.

This theorem will be proved in section 3.

Denote by $X^*_{\mathcal{R}}$ all remote points of X^*, let $X^*_{\mathcal{N}} = X^* \setminus X^*_{\mathcal{R}}$ and let $X^*_{\mathcal{L}} = X^* \setminus \bigcup\{D^* : D \text{ is closed and discrete in } X\}$. Assume X to be σ-compact and locally compact satisfying the countable chain condition (ccc), and assume $F \subset X^*_{\mathcal{R}}$ to be σ-compact. Then, by Lemma 1.4, $[F]_{\beta X} \setminus F$ consists of remote points which are not super-remote.

Question 1.2. Is every super-remote point remote?

Obviously, the answer is positive for realcompact spaces.

There are no non-empty β-nd sets in $\beta\omega$. The set B_x from Lemma 1.3 is nowhere dense in βX, but not β-nd. Every remote point of space without isolated points provides an example of a β-nd set which is not in the closure of any nowhere dense subset of X.

Lemma 1.3. *Let $F \subset \beta X$. Then F is β-nd if and only if, for any discrete family of non-empty open sets $\{U_\alpha\}_{\alpha\in A}$ in X and for any ultrafilter x on A, the following holds: $B_x = \bigcap_{C\in x}[\bigcup_{\alpha\in C} U_\alpha]_{\beta X}$ is not in the closure of F.*

Proof. If F is β-nd, then there are non-empty open $V_\alpha \subset U_\alpha$ with $\mathcal{E}x(\bigcup_{\alpha\in A} V_\alpha) \cap F = \emptyset$. But $\mathcal{E}x(\bigcup_{\alpha\in A} V_\alpha) \cap B_x$ is non-empty.

Conversely, we identify A with a discrete space. For any $x \in \beta A$ we choose open $O_x \subset \beta X$ so that $O_x \cap B_x \neq \emptyset$ and $[O_x]_{\beta X} \cap F = \emptyset$. Then

$$A_x = \{\alpha \in A : O_x \cap U_\alpha \neq \emptyset\}$$

belongs to x. Indeed, otherwise, $A\setminus A_x \in x$ implies $B_x \subset [\bigcup_{\alpha\in A\setminus A_x} U_\alpha]_{\beta X}$. But the last set is disjoint from O_x. The cover $\{A^\varepsilon_x : x \in \beta A\}$ of βA has a finite subcover $\{A^\varepsilon_{x_i}\}_{i\leq k}$. Then every $V_\alpha = \bigcup_{i\leq k} O_{x_i} \cap U_\alpha$ is non-empty and the closure of $\bigcup_{\alpha\in A} V_\alpha \subset \bigcup_{i\leq k} O_{x_i}$ does not meet F. □

Lemma 1.4. *Let a crowded locally compact space X satisfy the ccc. Then $F \subset X^*$ is β-nd in each of the following cases:*

(1) $c(F) < \mathfrak{c}$;
(2) *$F \subset X^*_{\mathcal{L}}$ is Lindelöf;*
(3) *$F \subset X^*_{\mathcal{N}}$ is Lindelöf and, additionally, X is metrizable.*

Proof. Let $\{U_n\}_{n\in\omega}$ be a discrete family of open sets having compact closures in X.

(1) We fix non-empty open sets $V_n(t) \subset U_n$ for all $t \in 2^n$, with disjoint closures, and put $V_\alpha = \bigcup_{n\in\omega} V_n(\alpha/2^n)$ for every $\alpha \in 2^\omega$. Since the family $\{\mathcal{E}xV_\alpha \cap X^*\}_{\alpha\in 2^\omega}$ is cellular, some of its members do not meet F.

(2) Let $D = \{d_n : n \in \omega\}$, where $d_n \in U_n$. Every $x \in F$ is not in the closure of some neighborhood $O_xD \subset \beta X$. For $Ox = \beta X \setminus [O_xD]_{\beta X}$ the

cover $\{Ox : x \in F\}$ of F has a countable subcover $\{Ox_n : n \in \omega\}$. Then $V_n = \bigcap_{i \le n} O_{x_i} D \cap U_n$ satisfies the definition of β-nd sets.

(3) Every $x \in F$ is in the closure of some nowhere dense $K_x \subset X$. By [7], for each $n \in \omega$, we can find open $V_{xn} \subset U_n$ so that $[K_x] \cap [V_{xn}] = \emptyset$ and $\{V_{xn} : x \in F\}$ is n-centered. For $O_x = X^* \setminus [\bigcup_{n\in\omega} V_{xn}]_{\beta X}$, the cover $\{O_x : x \in F\}$ of F has a countable subcover $\{O_{x_n} : n \in \omega\}$. Then $O_n = \bigcap_{i \le n} V_{x_i n}$ is as required. □

Corollary 1.5. *Let a metrizable crowded locally compact space X satisfy the ccc and let $p \in X^*$ be a super-remote point. Then p is not in the closure of $F \subset X^* \setminus \{p\}$ if at least one of the following conditions holds:*

(1) *F is σ-compact and $c(F) < \mathfrak{c}$.*
(2) *$F \subset X^*_{\mathcal{L}}$ is σ-compact.*
(3) *$F \subset X^*_{\mathcal{N}}$ is σ-compact.*

Corollary 1.6. *Let X be as in Corollary 1.5. Then every super-remote point $p \in X^*$ is a P-point in $X^*_{\mathcal{L}}$.*

For normal X, we call $B \subset X^*$ a *van Douwen set* if there is a countable and discrete family of non-empty open sets $\{U_n\}_{n\in\omega}$ in X and a family of subsets $\mathcal{F}_n$ in every U_n with the following properties:

(1) $\mathcal{F}_n$ is at least n-centered;
(2) for any nowhere dense $G \subset X$, there is $F \in \mathcal{F}_n$ with $G \cap [F]_X = \emptyset$;
(3) $B = \bigcap\{[F]_{\beta X} : F \subset X$ and $\forall n \in \omega (F \cap U_n \in \mathcal{F}_n)\}$.

It is easy to see that every van Douwen set B consists of remote points and $c(B) \ge \mathfrak{c}$. To obtain a remote point in a normal non-pseudocompact space X, we have to build every $\mathcal{F}_n$.

Corollary 1.7. *Let X satisfy the conditions of Corollary 1.5. Then every super-remote point $p \in X^*$ is not in the closure of any subset of $X^* \setminus \{p\}$ that can be represented as a countable union of van Douwen sets.*

In 1982, Jan van Mill [11] stated that $p \in X^*$ could be a weak P-point and a remote point simultaneously and raised the following question:

> Let a non-pseudocompact space X nowhere satisfy the ccc. Does X^* have a point p which is not in the closure of any subset of $\beta X \setminus \{p\}$ satisfying the ccc?

There has not been any significant progress in this area up to now.

Lemma 1.8. *Let a space X be nowhere ccc and let $p \in X^*$. Then every $F \subset \beta X \setminus \{p\}$ satisfying the ccc is in the closure of some β-nd σ-compact $G \subset \beta X \setminus \{p\}$.*

Proof. Let $\{U_i : i \in \omega\}$ be an everywhere dense in F countable family of open sets $U_i \subset F$ with $p \notin [U_i]_{\beta X}$ for all $i \in \omega$. Then $G = \bigcup_{i\in\omega}[U_i]_{\beta X}$ is as required. □

Corollary 1.9. *Let a metrizable space X be nowhere ccc and let $p \in X^*$ be a super-remote point. Then p is not in the closure of any ccc subset of $\beta X \setminus \{p\}$.*

2. Preliminaries

In our paper, $\omega = \{0, 1, 2...\}$, $\mathfrak{c}$ is the cardinality of real line $\mathbb{R}$ and 2^A is the set of all functions from A to $2 = \{0, 1\}$. If $B \subset A$ and $f \in 2^A$, then $f \restriction B$ is a restriction of f to B. By $Ox \subset X$, we denote any open neighborhood of x in X, and by $[\ ]$ and $[\ ]_{\beta X}$, the closure operators in X and βX, respectively. If $H \subset X$ is closed, then $H^* = [H]_{\beta X} \setminus X$. By $c(X)$, we denote the *Souslin number* of X, i.e., the supremum of the cardinalities of cellular (i.e., pairwise-disjoint) families of non-empty open sets. If $c(X) = \omega$, then the ccc holds. If no open set $U \subset X$ satisfies the ccc, then X is *nowhere ccc*. A family $\mathcal{B}$ is called *n-centered* if each of its no more than n-members subfamily has non-empty intersection. A family of non-empty open sets $\mathcal{B}$ is called a *π-base* if any non-empty open subset of X contains some member of $\mathcal{B}$. The minimum of cardinalities of π-bases is called *π-weight* of X, denoted by $\pi w(X)$. A space X is *crowded* if X has no isolated points.

If $p \in X^*$ is not in the closure of any nowhere dense subset of X, then p is called a *remote point*. A non-pseudocompact space X has remote points, for instance, in each of the following cases: If X has a countable π-base (Eric K. van Douwen [2] and, independently, Soo Bong Chae and Jeffrey H. Smith [1]); if X has a σ-locally finite π-base (M. Henriksen and T. J. Peters [7]); if X has π-weight ω_1 and either X satisfies the ccc or $cf(\omega^\omega, <_a) = \omega_1$ for some $a \in \omega^*$ (Alan Dow [5]). However, there are conditional examples of both non-pseudocompact separable spaces and non-pseudocompact spaces of π-weight ω_1 having no remote points [5]. Numerous applications of remote points appeared. Totally non-remote points were defined and constructed under CMA (van Douwen [4]) and then constructed in ZFC (Sergei Logunov [10]).

If $p \in X^*$ is not in the closure of any σ-compact (countable) subset of $X^* \setminus \{p\}$, then p is called a *P-point* (*weak P-point*). Under some additional axioms, like CH, ω^* has P-points (Walter Rudin [13]), but, by Shelah, it is unprovable in ZFC (Edward L. Wimmers [14]). By van Douwen [3], locally compact non-pseudocompact (and some other) spaces have P-points simultaneously with ω.

There are a few types of weak P-points "naively" constructed in $\beta\omega$ and then transferred to other spaces. A point p of ω^* is called an *R-point* (van Mill [12]) if p is in the closure of some open σ-compact $U \subset \omega^* \setminus \{p\}$, but p is not in the closure of any $V \subset U$ with $|V| < c$. A point p of ω^* is called a $\mathfrak{c}$-OK-point (K. Kunen [8]), if, for any sequence of neighborhoods $\{U_n : n \in \omega\}$ of p in ω^*, there is another sequence of its neighborhoods $\{V_\alpha : \alpha \in 2^\omega\}$ with the following property: If $n \in \omega$ and $\alpha_0, ..., \alpha_n \in 2^\omega$ are pairwise different, then $\bigcap\{V_{\alpha_i} : i \le n\} \subset U_n$. If p is a matrix point of ω^* (A. Gryzlov [6]), then p has a rather technical definition and the following properties: p is an R-point and a $\mathfrak{c}$-OK-point, $\omega^* \setminus \{p\}$ is not normal, and p is not in the closure of any σ-compact $F \subset \omega^* \setminus \{p\}$ with $c(F) < \mathfrak{c}$.

3. Proof of Theorem 1.1

Now we show that a weak P-point in a crowded space may be constructed as a kind of remote point, without any use of independently linked families on ω or transfer from $\beta\omega$. This makes the number of inductive steps unrestricted in our construction and allows us to answer van Mill's question for metrizable spaces.

We shall use the following simple facts in the main construction.

Lemma 3.0.1. *Let $U_0, ..., U_n$ be open in X and let $p \in X^*$ be remote. If $p \in [U_i]_{\beta X}$ for all $i \le n$, then $p \in [\bigcap_{i \le n} U_i]_{\beta X}$.*

Proof. We shall check it for two sets. If $p \notin [U \cap V]_{\beta X}$, then there is a neighborhood $Op \subset \beta X$ with $Op \cap [U] \cap [V] = \emptyset$ because p is remote. But then $[Op]_{\beta X} \cap [U]$ and $[Op]_{\beta X} \cap [V]$ are disjoint and have p in their closures. □

Lemma 3.0.2. *If every non-compact metrizable crowded space has a remote super-remote point, then every non-compact metrizable space has a super-remote point.*

Proof. If there is a closed infinite discrete $D \subset X$ containing isolated points, then every $p \in D^*$ is super-remote. Otherwise, there is an infinite discrete family of non-empty open crowded sets $\{U_\alpha\}_{\alpha\in\tau}$. The space $Y = \bigcup_{\alpha\in\tau}[U_\alpha]$ has a remote super-remote point p in $\beta Y = [Y]_{\beta X}$ by the lemma assumption. If $F \subset \beta X \setminus \{p\}$ is σ-compact and β-nd in βX, then $F \bigcap \beta Y$ has the same properties in βY. Hence, $p \notin [F \bigcap \beta Y]_{\beta Y}$. Since $p \in \operatorname{Ex} \bigcup_{\alpha\in\tau} U_\alpha$, p is as required. □

From now on X is non-compact metrizable and crowded. By the previous lemma, Theorem 1.1 follows if we prove that X has a remote super-remote point, which we now proceed to do.

3.1. The *Cell* operation and special bases.

Let π and σ be any families of open sets. We write $\pi \succ \sigma$ if $U \in \pi$ is a proper subset of $V \in \sigma$ whenever $U \cap V \neq \emptyset$. We say that π is *nice* if π is maximal cellular and locally finite in X; $U \in \pi$ is a maximal member of π if U is not a proper subset of any other member of π.

The notion of cellular refinement of π was introduced in [9]:

$$Cel(\pi) = \{\bigcap \phi \setminus [\bigcup(\pi \setminus \phi)] : \phi \subset \pi\}.$$

Lemma 3.1.1. *If π is a locally finite open cover of X, then $Cel\,(\pi)$ is nice.*

Proof. Let $\varphi, \psi \subset \pi$.

If φ has non-empty intersection, then φ is finite. So $\cap\varphi - [\bigcup(\pi \setminus \varphi)]$ is open.

If $U \in \varphi \setminus \psi$, then $\bigcap \varphi \subset U$ and $\bigcap \psi \cap U = \emptyset$.

If $\psi = \{U \in \pi : U \cap Ox \neq \emptyset\}$ is finite for some $Ox \subset X$, then $\{\varphi \subset \pi : \bigcap \varphi \cap Ox \neq \emptyset\} \subset \exp \psi$ is finite as well.

Let $x \notin [U] \setminus U$ for any $U \in \pi$ and $\varphi = \{U \in \pi : x \in U\}$. Then $x \in \bigcap \varphi \setminus [\bigcup(\pi \setminus \varphi)]$ and $Cel(\pi)$ is maximal. □

Let $\mathcal{P}_0$ be any open locally finite infinite cover of X and $\mathcal{B}_0 = Cel(\mathcal{P}_0)$. If $\mathcal{B}_i$ has been constructed for some $i \in \omega$, we can choose an open locally finite infinite cover $\mathcal{P}_{i+1}$ so that for every $U \in \mathcal{P}_{i+1}$, diam $U \leq \frac{1}{i+1}$ and $V \setminus U \neq \emptyset$ for each $V \in \mathcal{B}_i$. We put $\mathcal{B}_{i+1} = Cel(\mathcal{B}_i \bigcup \mathcal{P}_{i+1})$.

Then it is folklore and easy to see that $\mathcal{P} = \bigcup_{i\in\omega} \mathcal{P}_i$ is a regular base of Arhangel'skiĭ; i.e., $\mathcal{P}$ is a base and for any point $x \in X$ and for any its neighborhood O, there is another neighborhoods O' of x with the following properties: $O' \subset O$ and at most finitely many members of $\mathcal{P}$ meet both O' and $X \setminus O$ simultaneously.

Lemma 3.1.2. *The family of maximal members π' of any cover $\pi \subset \mathcal{P}$ is a locally finite open cover of X.*

Proof. Let $x \in U$ for some $U \in \pi$. Since the family $\{V \in \pi : U \subset V\}$ is finite by our construction, it contains a maximal member O. Then $x \in O$ and $O \in \pi'$. Since $\mathcal{P}$ is a regular base of Arhangelskiĭ, there is a neighborhood O' of x as above. Since every member of $\pi' \setminus \{O\}$ intersects $X \setminus O$, at most finitely many members of π' intersect O'. □

Moreover, $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ is a π-base and $\mathcal{B}_{i+1} \succ \mathcal{B}_i$ for each $i \in \omega$. So the number $n(U)$ is well defined for every $U \in \mathcal{B}$ as follows:

$$n(U) = i \Leftrightarrow U \in \mathcal{B}_i.$$

We put $\mathcal{B}_t(U) = \{V \in \mathcal{B}_t : V \subset U\}$ for each $t \geq i$. As any $U \in \mathcal{P}$ belongs to unique $\mathcal{P}_{i_0}$ by our construction, we can choose $\mathcal{B}_{i_0}$ so that both of the families $\mathcal{P}_{i_0}$ and $\mathcal{B}_{i_0}$ have one and the same index i_0 and put

$$\hat{U} = \{V \in \mathcal{B}_{i_0} : V \subset U\} = \{V \in \mathcal{B}_{i_0} : V \cap U \neq \emptyset\}.$$

Obviously, $\hat{U}$ is nice in U. For any locally finite cover $\pi \subset \mathcal{P}$ of X, we define $\hat{\pi} = \bigcup\{\hat{U} : U \in \pi\}$. Then

$$\sigma(\pi) = \{V \in \hat{\pi} : V \text{ is maximal in } \hat{\pi}\}$$

belongs to

$$\Sigma = \{\sigma \subset \mathcal{B} : \sigma \text{ is nice}\}.$$

Let $p \in X^*$ be a remote point from now on and $\sigma, \sigma', \sigma'' \in \Sigma$. We may assume $p \notin [U]_{\beta X}$ for any $U \in \mathcal{P}_0$. We denote $\sigma(p) = \{\delta \subset \sigma : p \in [\bigcup \delta]_{\beta X}\} = \{\delta \subset \sigma : p \in \operatorname{Ex} \bigcup \delta\}$.

Definition 3.1.3. A countable collection $\{\sigma(k)\}_{k \in \omega}$ of subfamilies $\sigma(k) \subset \sigma$ is called a *p-partition* of σ if the following hold:

(1) $\{k : \sigma(k) = \emptyset\}$ is finite or empty;
(2) $\sigma(k) \cap \sigma(k') = \emptyset$ if $k \neq k'$;
(3) $\sigma = \bigcup_{k \in \omega} \sigma(k)$;
(4) $p \notin [\bigcup \sigma(k)]_{\beta X}$ for any $k \in \omega$.

We denote $\mathcal{P}art(\sigma, p)$ as all the p-partitions of σ.

Denote

$\sigma \prec_p \sigma'$ if $\delta \prec \sigma'$ for some $\delta \in \sigma(p)$ and
$\sigma =_p \sigma'$ if $p \in \operatorname{Ex} \bigcup \sigma \cap \sigma'$.

Then $\sigma \preceq_p \sigma'$ if either $\sigma \prec_p \sigma'$ or $\sigma =_p \sigma'$. Obviously, $\sigma \prec_p \sigma'$ if and only if $\sigma \prec \delta$ for some $\delta \in \sigma'(p)$.

By the next lemma, cases 1 and 2 in the main construction do cover all possibilities.

Lemma 3.1.4. *The binary relation $\prec_p$ on Σ is a linear order on Σ.*

Proof. For transitivity, let $\sigma \prec_p \sigma'$ and $\sigma' \prec_p \sigma''$. Then $\delta \prec \sigma'$ for some $\delta \in \sigma(p)$ and $\delta' \prec \sigma''$ for some $\delta' \in \sigma'(p)$. For $\varphi = \{U \in \delta : U \supset V \text{ for some } V \in \delta'\}$, we obtain $\varphi \in \sigma(p)$ and $\varphi \prec \sigma''$. Hence, $\sigma \prec_p \sigma''$.

For linearity, σ is a free union of $\delta_1 = \sigma \cap \sigma'$, $\delta_2 = \{U \in \sigma \setminus \delta_1 : U \subset V \text{ for some } V \in \sigma'\}$, and $\delta_3 = \{U \in \sigma \setminus \delta_1 : V \subset U \text{ for some } V \in \sigma'\}$. If $p \in \operatorname{Ex} \bigcup \delta_1$, then $\sigma =_p \sigma'$. If $p \in \operatorname{Ex} \bigcup \delta_2$, then $\sigma \prec_p \sigma'$, and vice versa. □

3.2. Embedding and raising the number maps $\xi : \mathcal{B} \to \mathcal{B}$.

Let Ξ be all the maps $\xi : \mathcal{B} \to \mathcal{B}$, which are "embedding and raising the number," that is, $[\xi(U)] \subset U$ and $n(\xi(U)) > n(U)$ for each $U \in \mathcal{B}$. For every $\xi \in \Xi$, denote

$$\mathcal{B}(U,\xi) = \mathcal{B}_{n(\xi(U))}(U) = \{V \in \mathcal{B} : n(V) = n(\xi(U)) \text{ and } V \subset U\}.$$

Define by induction the following families:

$$\mathcal{D}(U,\xi,0) = \mathcal{B}(U,\xi,0) = \emptyset,$$
$$\mathcal{D}(U,\xi,1) = \{\xi(U)\}, \text{ and}$$
$$\mathcal{B}(U,\xi,1) = \mathcal{B}(U,\xi).$$

If $\mathcal{B}(U,\xi,n)$ has been defined for some $n \in \omega$, then

$$\mathcal{D}(U,\xi,n+1) = \{\xi(V) : V \in \mathcal{B}(U,\xi,n)\} \text{ and}$$
$$\mathcal{B}(U,\xi,n+1) = \bigcup\{\mathcal{B}(V,\xi) : V \in \mathcal{B}(U,\xi,n)\}.$$

As it is easy to see by induction, every $\mathcal{B}(U,\xi,n)$ is cellular and locally finite. Denote

$$\mathcal{D}(U,\xi,\le n) = \bigcup_{i\le n} \mathcal{D}(U,\xi,i) \text{ and } \mathcal{B}(U,\xi,\le n) = \bigcup_{i\le n} \mathcal{B}(U,\xi,i).$$

For any $\delta \subset \mathcal{B}$, we put

$$\mathcal{D}(\delta,\xi,n) = \bigcup_{U\in\delta} \mathcal{D}(U,\xi,n) \text{ and}$$
$$\mathcal{D}(\delta,\xi,\le n) = \bigcup_{U\in\delta} \mathcal{D}(U,\xi,\le n) = \bigcup_{i\le n} \mathcal{D}(\delta,\xi,i).$$

Here, $\mathcal{D}(\emptyset,\xi,n) = \mathcal{D}(\delta,\xi,0) = \emptyset$. If $\mathcal{P}art(\sigma,p) \neq \emptyset$, then the following sets are well defined:

$$\Omega(\sigma,p) = \{\bigcup_{k\in\omega} \mathcal{D}(\sigma(k),\xi,\le k) : \xi \in \Xi \text{ and } \{\sigma(k)\}_{k\in\omega} \in \mathcal{P}art(\sigma,p)\} \text{ and}$$

$$K(\sigma,p) = \bigcap\{[\bigcup \mathcal{D}]_{\beta X} : \mathcal{D} \in \Omega(\sigma,p)\}.$$

By Proposition 3.2.5 below, $K(\sigma,p)$ is non-empty and consists of remote points. Probably, $p \notin K(\sigma,p)$.

Lemma 3.2.1. *Let* $\bigcup_{k\in\omega} \mathcal{D}(\sigma(k),\xi,\le k) \in \Omega(\sigma,p)$. *Then*

$$\bigcup_{k\in\omega} \mathcal{D}(\sigma(2k)\cup\sigma(2k+1),\xi,\le k) \in \Omega(\sigma,p) \text{ and}$$

$$\bigcup_{k>n} \mathcal{D}(\sigma(k),\xi,\le k) \in \Omega(\sigma,p)$$

for every $n \in \omega$.

Proof. If $\sigma(k)' = \sigma(2k) \cup \sigma(2k+1)$ for each $k \in \omega$, then $\{\sigma(k)'\}_{k\in\omega} \in \mathcal{P}art(\sigma, p)$ by our construction. The same is true if $\sigma(0)' = \bigcup_{k\le n} \sigma(k)$, $\sigma(1)' = ... = \sigma(n)' = \emptyset$, and $\sigma(k)' = \sigma(k)$ for all $k > n$. But then the families above coincide with $\bigcup_{k\in\omega} \mathcal{D}(\sigma(k)', \xi, \le k) \in \Omega(\sigma, p)$. □

Lemma 3.2.2. *Let* $\mathcal{P}art(\sigma, p) \neq \emptyset$ *for some* $\sigma \in \Sigma$. *Then* $K(\sigma, p) \subset [\bigcup \delta]_{\beta X}$ *for any* $\delta \in \sigma(p)$.

Proof. Any $\{\sigma(k)\}_{k\in\omega} \in \mathcal{P}art(\sigma, p)$ may be rebuilt as follows: $\sigma(0)' = \sigma(0) \cup (\sigma \setminus \delta)$ and $\sigma(k)' = \sigma(k) \cap \delta$ for each $k > 0$. Here, we can assume all $\sigma(k)'$ to be non-empty, rejecting empty ones, if necessary. Then for any $\xi \in \Xi$, $D(\sigma(0)', \xi, \le 0) = \emptyset$ implies

$$K(\sigma, p) \subset [\bigcup \bigcup_{k\in\omega} D(\sigma(k)', \xi, \le k)]_{\beta X} \subset [\bigcup \delta]_{\beta X}. \qquad \square$$

Lemma 3.2.3. *Let* $\mathcal{P}art(\sigma, p) \neq \emptyset$ *for some* $\sigma \in \Sigma$. *Then for any* $q \in K(\sigma, p)$ *and* $\sigma' \in \Sigma$, *the following hold:*

(1) $\sigma(p) = \sigma(q)$;
(2) $\mathcal{P}art(\sigma, p) = \mathcal{P}art(\sigma, q)$;
(3) $\Omega(\sigma, p) = \Omega(\sigma, q)$;
(4) $K(\sigma, p) = K(\sigma, q)$;
(5) $\sigma \prec_p \sigma' \Leftrightarrow \sigma \prec_q \sigma'$.

Proof. (1) Let $\delta \subset \sigma$. Then $\delta \in \sigma(p)$ implies $q \in K(\sigma, p) \subset [\bigcup \delta]_{\beta X}$ by Lemma 3.2.2. So $\delta \in \sigma(q)$. On the other hand, $\delta \notin \sigma(p)$ implies $p \in [\bigcup(\sigma \setminus \delta)]_{\beta X}$ and $q \in K(\sigma, p) \subset [\bigcup(\sigma \setminus \delta)]_{\beta X}$. So $q \notin [\bigcup \delta]_{\beta X}$ and $\delta \notin \sigma(q)$.

(2) Let a collection $\{\sigma(k)\}_{k\in\omega}$ of subfamilies $\sigma(k) \subset \sigma$ satisfy conditions (1)–(3) of the p-partition definition (Definition 3.1.3).

Then $\{\sigma(k)\}_{k\in\omega} \in \mathcal{P}art(\sigma, p)$ implies $p \notin [\bigcup \sigma(k)]_{\beta X}$ for all $k \in \omega$. That is, $K(\sigma, p) \cap [\bigcup \sigma(k)]_{\beta X} = \emptyset$. But then $q \notin [\bigcup \sigma(k)]_{\beta X}$ implies $\{\sigma(k)\}_{k\in\omega} \in \mathcal{P}art(\sigma, q)$.

On the other hand, $\{\sigma(k)\}_{k\in\omega} \notin \mathcal{P}art(\sigma, p)$ implies $p \in [\bigcup \sigma(k)]_{\beta X}$ for some $k \in \omega$ and $q \in K(\sigma, p) \subset [\bigcup \sigma(k)]_{\beta X}$. So $\{\sigma(k)\}_{k\in\omega} \notin \mathcal{P}art(\sigma, q)$.

Obviously, (3), (4), and (5) are the easy corollaries of (1) and (2). □

Lemma 3.2.4. *Let* $F \subset \beta X$. *Then* F *is* β*-nd if and only if*

$$\forall \sigma \in \Sigma \, \forall n \in \omega \, \exists \xi \in \Xi \, ([\bigcup \mathcal{D}(\sigma, \xi, \le n)]_{\beta X} \cap F = \emptyset).$$

Proof. Let $F \subset \beta X$ be β-nd, $\sigma \in \Sigma$, and $n \in \omega$. Then $\sigma = \{U\}$ is locally finite and cellular. Any family of non-empty open sets $\{V_U : U \in \sigma\}$, where $[V_U] \subset U$, is discrete. There are $W_U \in \mathcal{B}$ such that $W_U \subset V_U$ and $[\bigcup_{U\in\sigma} W_U]_{\beta X} \cap F = \emptyset$ by the definition of β-nd sets. For any $\xi \in \Xi$, we define $\xi^1 \in \Xi$ as follows: $\xi^1(U) = W_U$ if $U \in \sigma$ and $\xi^1(U) = \xi(U)$ if $U \in \mathcal{B} \setminus \sigma$. Then

$$\mathcal{D}(\sigma, \xi^1, \leq 1) = \{W_U : U \in \sigma\}$$

and ξ^1 is as required for $n = 1$.

By our construction, $\sigma_1 = \bigcup_{U \in \sigma} \mathcal{B}(U, \xi^1)$ is also locally finite and cellular. We choose $\{W_U : U \in \sigma_1\}$ as above and define $\xi^2 \in \Xi$ as follows: $\xi^2(U) = W_U$ if $U \in \sigma_1$ and $\xi^2(U) = \xi^1(U)$, otherwise. Then

$$\mathcal{D}(\sigma, \xi^2, \leq 2) = \{W_U : U \in \sigma_1\} \bigcup \mathcal{D}(\sigma, \xi^1, \leq 1)$$

and ξ^2 is as required for $n = 2$.

For $\sigma_2 = \bigcup_{U \in \sigma_1} \mathcal{B}(U, \xi^2)$, we choose $\{W_U : U \in \sigma_2\}$ as above and define $\xi^3 \in \Xi$ as follows: $\xi^3(U) = W_U$ if $U \in \sigma_2$ and $\xi^3(U) = \xi^2(U)$, otherwise. Then

$$\mathcal{D}(\sigma, \xi^3, \leq 3) = \{W_U : U \in \sigma_2\} \bigcup \mathcal{D}(\sigma, \xi^2, \leq 2)$$

and ξ^3 is as required for $n = 3$. In the same way we can construct $\xi = \xi^n$ for any $n \in \omega$.

Conversely, let $\delta = \{U\}$ be a discrete in X family of non-empty open sets. We fix non-empty open $U' \subset X$ with $[U'] \subset U$ and denote by π all maximal members of the cover

$$\{V \in \mathcal{P} : \forall U \in \delta (V \cap U' \neq \emptyset \Rightarrow V \subset U)\}.$$

Then $\sigma = \sigma(\pi)$ belongs to Σ and $[\bigcup \mathcal{D}(\sigma, \xi, \leq 1)]_{\beta X} \cap F = \emptyset$ for some $\xi \in \Xi$ by the lemma assumption. Every U' meets some $W_U \in \sigma$. But then $\xi(W_U) \subset W_U \subset V \subset U$ for some $V \in \pi$ and $\{\xi(W_U) : U \in \sigma\}$ satisfies the definition of β-nd sets. □

Giving a scheme of the proof, for completeness, we shall use the result of Henriksen and Peters [7] in the following form.

Proposition 3.2.5. *For any $O \in \mathcal{B}_0$ and $\xi^0, ..., \xi^n \in \Xi$, the following holds: $\bigcap_{i \leq n} \bigcup \mathcal{D}(O, \xi^i, \leq n+1) \neq \emptyset$.*

Proof. We shall construct by induction an embedded sequence of sets $U_i \in \mathcal{D}(O, \xi^i, \leq n+1)$ as follows:

For $i = 0, ..., n-1$, we may assume that $n(\xi^i(O)) \leq n(\xi^{i+1}(O))$ and put $U_0 = \xi^0(O)$. This allows us to find, step by step, $V_i^0 \in \mathcal{B}(O, \xi^i)$ so that

$$U_0 \supset V_1^0 \supset ... \supset V_n^0.$$

At the second step of induction, a member of every $\mathcal{B}(O, \xi^i, \leq n+1)$ sequence

$$U_0 \supset ... \supset U_t \supset V_{t+1}^t \supset ... \supset V_n^t$$

has been constructed for some $t < n$ so that

(1) $U_i \in \mathcal{D}(O, \xi^i, \leq t+1)$ if $i \leq t$;
(2) $V_i^t \in \mathcal{B}(O, \xi^i, \leq t+1)$ if $i > t$;
(3) $V_i^t \in \mathcal{B}(W_i, \xi^i)$ for some $W_i \in \mathcal{B}(O, \xi^i, \leq t)$ with $W_i \supset U_t \supset V_i^t$.

Then $\mathcal{A} = \{U \in \bigcup_{i>t} \mathcal{D}(O, \xi^i, \leq t+2) : U \subset U_t\}$ contains every $\xi^i(V_i^t)$. There is $U_{i_0} \in \mathcal{A}$ with $n(U_{i_0}) = \min\{n(U) : U \in \mathcal{A}\}$. Let $i > t$ and $i \neq i_0$. By (3), for some $V_i \in \hat{\xi}^i(W_i)$, we have either $W_i \supset U_t \supset U_{i_0} \supset V_i$ or $W_i \supset U_t \supset V_i \supset U_{i_0}$. In the last case, $n(\xi^i(V_i)) \geq n(U_{i_0})$ by the choice of U_{i_0}. So there is $V_i' \in \mathcal{B}(V_i, \xi^i)$ with $V_i \supset U_{i_0} \supset V_i'$. By our construction, all the sets V_i and V_i' may be chosen embedded into each other. Hence, we obtain the sequence

$$U_0 \supset ... \supset U_t \supset U_{i_0} \supset V_{i_{t+2}}^{t+1} \supset ... \supset V_{i_n}^{t+1}$$

with properties similar to those of the above sequence. After the n steps of induction, the proof is complete. □

3.3. Construction.

First step. Let $\sigma_0 = \mathcal{B}_0$. We can choose a collection $\mathcal{P} = \{\sigma_0(k)\}_{k\in\omega}$ of subfamilies $\sigma_0(k) \subset \sigma_0$ so that conditions (1)–(3) of the p-partition definition (Definition 3.1.3) hold. There is a remote point p_0 in $K = \bigcap_{n\in\omega}[\bigcup\bigcup_{k>n} \sigma_0(k)]_{\beta X}$. Then $\mathcal{P} \in \mathcal{P}art(\sigma_0, p_0)$. By Proposition 3.2.5 and Lemma 3.2.2, $K_0 = K(\sigma_0, p_0)$ is a non-empty subset of K consisting of remote points. For any $q \in K_0$, $\mathcal{P}$ is also a q-partition. Hence, $\mathcal{P}art(\sigma, q) \neq \emptyset$ for every $\sigma \in \Sigma$. Probably, $p_0 \notin K_0$. For any $p_1 \in K_0$, we choose $\sigma_1 \in \Sigma$ with $\sigma_0 \prec_{p_1} \sigma_1$ and put $K_1 = K(\sigma_1, p_1)$. For any point p_2 of the set $K_0 \cap K_1$, which is non-empty by Lemma 3.3.1, we choose $\sigma_2 \in \Sigma$ with $\sigma_1 \prec_{p_2} \sigma_2$ and put $K_2 = K(\sigma_2, p_2)$ and so on.

Second step. For some ordinal λ assume that p_α and $\sigma_\alpha \in \Sigma$ have been constructed for all $\alpha < \lambda$ so that for $\mathcal{P}_\alpha = \mathcal{P}art(\sigma_\alpha, p_\alpha)$, $\Omega_\alpha = \Omega(\sigma_\alpha, p_\alpha)$, and $K_\alpha = K(\sigma_\alpha, p_\alpha)$, the following hold:
(1) $p_\alpha \in \bigcap_{\beta<\alpha} K_\beta$;
(2) $\sigma_\beta \prec_{p_\alpha} \sigma_\alpha$ if $\beta < \alpha$;
(3) $\bigcap_{\alpha<\lambda} K_\alpha \neq \emptyset$.
Then we fix any $p_\lambda \in \bigcap_{\alpha<\lambda} K_\alpha$.

Case 1: There is $\sigma \in \Sigma$ with $\sigma_\alpha \prec_{p_\lambda} \sigma$ for all $\alpha < \lambda$. Then we put $\sigma_\lambda = \sigma$.

Notice that if for some $\sigma \in \Sigma$, $\sigma_\alpha \preceq_{p_\lambda} \sigma$ for all $\alpha < \lambda$, then it is easy to find $\sigma' \in \Sigma$ with $\sigma \prec_{p_\lambda} \sigma'$ and, by transitivity, we again obtain Case 1.

Lemma 3.3.1. $\bigcap_{\alpha<\lambda} K_\alpha \cap K(\sigma_\lambda, p_\lambda) \neq \emptyset$.

Proof. Let $n, m \in \omega$, and let $\alpha_0 \leq ... \leq \alpha_n < \lambda$ be any finite sequence of ordinals. It is enough to show that

$$\bigcap_{s\leq n}(\bigcup \mathcal{D}_s) \cap \bigcap_{t\leq m}(\bigcup \mathcal{D}_t) \neq \emptyset$$

for any $\mathcal{D}_s \in \Omega_{\alpha_s}$ and $\mathcal{D}_t \in \Omega(\sigma_\lambda, p_\lambda)$. Our goal is to choose by induction $U_s \in \mathcal{D}_s$ and $U_t \in \mathcal{D}_t$, making up an embedded sequence.

By our construction, $\mathcal{D}_s = \bigcup_{k_s\in\omega} \mathcal{D}(\sigma_{\alpha_s}(k_s), \xi^s, \leq k_s)$ for some $\xi^s \in \Xi$ and $\{\sigma_{\alpha_s}(k_s)\}_{k_s\in\omega} \in \mathcal{P}_{\alpha_s}$. By Lemma 3.2.3, $\mathcal{P}_{\alpha_s} = \mathcal{P}art(\sigma_{\alpha_s}, p_\lambda)$. By Lemma 3.2.1,

$$\mathcal{D}'_s = \bigcup_{k_s>n+m} \mathcal{D}(\sigma_{\alpha_s}(2k_s) \cup \sigma_{\alpha_s}(2k_s+1), \xi^s, \leq k_s)$$

also belongs to Ω_{α_s}. Hence, $p_\lambda \in [\bigcup \mathcal{D}'_s]_{\beta X}$ and $\mathcal{D}(U, \xi^s, \leq n+m) \subset \mathcal{D}_s$ for each $U \in \mathcal{D}'_s$.

Similarly, $\mathcal{D}_t = \bigcup_{k_t\in\omega} \mathcal{D}(\sigma_\lambda(k_t), \xi^t, \leq k_t)$ for some $\xi^t \in \Xi$, and $\{\sigma_\lambda(k_t)\}_{k_t\in\omega} \in \mathcal{P}art(\sigma_\lambda, p_\lambda)$. Let $\delta \in \sigma_\lambda(p_\lambda)$. Since for all $s \leq n$, $\sigma_{\alpha_s} \prec_{p_\lambda} \sigma_\lambda$, we may assume $\sigma_{\alpha_s} \prec \delta$. Since for all $t \leq m$, $p_\lambda \notin [\bigcup \sigma_\lambda(k_t)]_{\beta X}$ for each $k_t \in \omega$, we may assume $\delta \subset \bigcup_{k_t>n+m} \sigma_\lambda(k_t)$. Then $\mathcal{D}(O, \xi^t, \leq n+m) \subset \mathcal{D}_t$ for each $O \in \delta$ by our construction.

By Lemma 3.0.1, we obtain $p_\lambda \in [\bigcap_{s\leq n}(\bigcup \mathcal{D}'_s) \cap (\bigcup \delta)]_{\beta X}$. Then, for some $U_s \in \mathcal{D}'_s$ and $O \in \delta$, we have $\bigcap_{s\leq n} U_s \cap O \neq \emptyset$. We consider only the case when, for some $r < n$, we have the following embedded sequence:

$$U_{s_0} \supset ... \supset U_{s_r} \supset O \supset U_{s_{r+1}} \supset ... \supset U_{s_n}.$$

We leave to the reader the other simple cases. For each $l = r+1, ..., n$, we have $O_{s_l} \supset O \supset U_{s_l}$ for some $O_{s_l} \in \sigma_{\alpha_{s_l}}$ because $\sigma_{\alpha_{s_l}} \prec \delta$. By our construction we can make inside $\mathcal{D}'_{s_l}$ the following descending induction: $U_{s_l} = \xi^{s_l}(U^1_{s_l})$, $U^1_{s_l} \in \mathcal{B}(U^2_{s_l}, \xi^{s_l})$, $U^2_{s_l} \in \mathcal{B}(U^3_{s_l}, \xi^{s_l})$, and so on, until we get $V_{s_l} \in \mathcal{B}(W_{s_l}, \xi^{s_l})$ and $O_{s_l} \supset W_{s_l} \supset O \supset V_{s_l}$, where $V_{s_l} \in \mathcal{D}'_{s_l}$, and either $W_{s_l} \in \mathcal{D}'_{s_l}$ or $W_{s_l} = O_{s_l}$. Then $\mathcal{D}(W_{s_l}, \xi^{s_l}, \leq n+m) \subset \mathcal{D}_{s_l}$ and we obtain, in general, a new embedded sequence:

$$U_{s_0} \supset ... \supset U_{s_r} \supset O \supset V_{s'_{r+1}} \supset ... \supset V_{s'_n}.$$

We can find U_0 in $\mathcal{A} = \{\xi^{s_l}(V_{s'_l})\}^n_{l=r+1} \cup \{\xi^t(O)\}_{t\leq m}$ with $n(U_0) = \min\{n(U) : U \in \mathcal{A}\}$ and proceed with the second step of the proposition. □

After at most $| \Sigma |$ steps of induction, we have the following case.

Case 2: For any $\sigma \in \Sigma$, $\sigma \prec_{p_\lambda} \sigma_\alpha$ for some $\alpha < \lambda$. Then we shall show that $p = p_\lambda$ is super-remote.

3.4. A super-remote point.

Lemma 3.4.1. *The ordinal λ is not countably cofinal.*

Proof. Let $\alpha_i < \lambda$ for all $i \in \omega$ be any ordinals. By our construction, $p \in \bigcap_{i\in\omega} O_i \subset X^*$ for some open $O_i \subset \beta X$ with $[O_{i+1}]_{\beta X} \subset O_i$ for all $i \in \omega$. For any $x \in X$, there is $Ox \subset X$ disjoint from some O_{i_x}. For some $O^{'}x \subset Ox$, $\sigma(x) = \{V \in \bigcup_{i\leq i_x} \sigma_{\alpha_i} : V \cap O^{'}x \neq \emptyset\}$ is finite. If $n_x \in \omega$ is big enough, then $n(V) < n_x$ for each $V \in \sigma(x)$ and $x \in U_x \subset O^{'}x$ for some $U_x \in \mathcal{P}_{n_x}$. Then $\sigma(x) \prec \hat{U}_x$. In other words, if $U_x \cap O_i \neq \emptyset$ for any $i \in \omega$, then $\sigma_{\alpha_i} \prec \hat{U}_x$. Denote by π all maximal members of the cover $\{U_x : x \in X\}$ and $\sigma = \sigma(\pi)$. Then $\{V \in \sigma : V \cap O_i \neq \emptyset\} \subset \bigcup\{\hat{U}_x : U_x \cap O_i \neq \emptyset\}$ implies $\sigma_{\alpha_i} \prec_p \sigma$. By Case 2, $\sigma \prec_p \sigma_\beta$ for some $\sigma_\beta \in \mathcal{F}$. Hence, $\sigma_{\alpha_i} \prec_p \sigma_\beta$ implies $\alpha_i < \beta$. □

Lemma 3.4.2. *Let $F \subset \beta X \setminus \{p\}$ be σ-compact. Then*

$$F \subset \bigcup_{k\in\omega} [\bigcup \sigma_\beta(k)]_{\beta X}$$

for some $\beta < \lambda$ and $\{\sigma_\beta(k)\}_{k\in\omega} \in \mathcal{P}art(\sigma_\beta, p)$.

Proof. Let $F = \bigcup_{k\in\omega} F_k$, where every F_k is compact. Let OF_k and O_kp be disjoint with closure neighborhoods, π_k – all maximal members of the cover $\{U \in \mathcal{P} : U \cap O_kp \neq \emptyset \Rightarrow U \cap OF_k = \emptyset\}$, and $\sigma^k = \sigma(\pi_k)$. By Case 2 and Lemma 3.4.1, $\sigma^k \prec_p \sigma_\beta$ for all $k \in \omega$ and some $\beta < \lambda$, i.e., $\sigma^k \prec \delta_k$ for some $\delta_k \in \sigma_\beta(p)$. For $\delta'_k = \{V \in \delta_k : V \cap O_kp \neq \emptyset\}$, we get

$$p \in [\bigcup \delta'_k]_{\beta X} \subset [\bigcup\{V \in \sigma^k : V \cap O_kp \neq \emptyset\}]_{\beta X}$$

$$\subset [\bigcup\{U \in \pi_k : U \cap O_kp \neq \emptyset\}]_{\beta X} \subset \beta X \setminus F_k.$$

For $\sigma_\beta(0) = \sigma_\beta \setminus \delta'_0$ and $\sigma_\beta(k) = \sigma_\beta \setminus \delta'_k \setminus \bigcup_{i<k} \sigma_\beta(i)$, we obtain

$$F_k \subset [\bigcup(\sigma_\beta \setminus \delta'_k)]_{\beta X} \subset \bigcup_{i\leq k} [\bigcup \sigma_\beta(i)]_{\beta X}$$

for all $k \in \omega$. If necessary, we can add finitely many members of $\bigcap_{i\leq k} \delta'_i$ to $\sigma_\beta(k)$ to provide conditions (1) and (3) of Definition 3.1.3. □

Lemma 3.4.3. *Let $F \subset \beta X \setminus \{p\}$ be σ-compact and β-nd. Then $p \notin [\bigcup F]_{\beta X}$.*

Proof. Let $\{\sigma_\beta(k)\}_{k\in\omega}$ be defined as in the previous lemma. By Lemma 3.2.4 there is $\xi \in \Xi$ with the following property: For any $k \in \omega$, the closure of $\mathcal{D}_k = \mathcal{D}(\sigma_\beta(k), \xi, \leq k)$ does not meet $[F]_{\beta X}$. By our construction there are open $O_k \subset X$ with $[\bigcup \mathcal{D}_k] \subset O_k \subset [O_k] \subset \bigcup \sigma_\beta(k)$. Let $\mathcal{D} = \bigcup_{k\in\omega} \mathcal{D}_k$

and $O = \bigcup_{k\in\omega} O_k$. Then $\mathcal{D} \in \Omega_\beta$ implies $p \in K_\beta \subset [\bigcup \mathcal{D}]_{\beta X} \subset \operatorname{Ex} O$. Moreover,

$$F \cap [O]_{\beta X} \subset \bigcup_{k\in\omega} [\bigcup \sigma_\beta(k)]_{\beta X} \cap [O]_{\beta X} = \bigcup_{k\in\omega} [O_k]_{\beta X} \subset \bigcup_{k\in\omega} \operatorname{Ex} \bigcup \sigma_\beta(k).$$

There are open $U_k \subset \beta X$ with $F \cap [O_k]_{\beta X} \subset U_k \subset [U_k]_{\beta X} \subset \operatorname{Ex} \bigcup \sigma_\beta(k) \setminus [\bigcup \mathcal{D}]_{\beta X}$. Since σ_β is locally finite, $W = \bigcup_{k\in\omega} [U_k]_{\beta X} \cap X$ is closed in X and so completely separated from $\bigcup \mathcal{D}$. Since $F \cap [O]_{\beta X} \subset [W]_{\beta X}$, our proof is complete. □

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