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SOME RESULTS ON CSS AND QUARTER-STRATIFIABLE GO-SPACES

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Abstract. We investigate the relation between Harold W. Martin's c-semi-stratifiable (CSS) spaces and T. O. Banakh's quarter-stratifiability among generalized ordered (GO)-spaces. A question in *Compact G_δ sets* (Topology Appl. **153** (2006), no. 12, 2169–2181) is partially answered. Furthermore, we study CSS ordered extensions of GO-spaces and give some surprising results. We show that the properties (X is CSS, X has a G^*_δ-diagonal, X has a G_δ-diagonal, X has a quasi-G^*_δ-diagonal, X has a quasi-G_δ-diagonal, and X is a $\sigma^\sharp$-space) in the class of GO-spaces with a σ-closed-discrete dense subset are equivalent. In addition, we prove a " local implies global" theorem for perfectly normal GO-spaces that are locally CSS. We deduce that four special types of bases (weakly uniform bases, σ-disjoint bases, point-countable bases, and weak monotone ortho-bases) in the class of quarter-stratifiable GO-spaces are equivalent.

1. Introduction and Preliminaries

The following are some basic concepts needed in the following; for other non-explicitly stated elementary notions, please refer to [18], [19], and [20].

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Let $\mathcal{C}$ be a collection of subsets of a topological space X. We say that members of $\mathcal{C}$ are uniformly G_δ-sets if for each $C \in \mathcal{C}$ there are open sets $G(n, C)$ in X such that

(i) $\bigcap\{G(n, C) : n \geq 1\} = C$;
(ii) $G(n+1, C) \subseteq G(n, C)$ for each $n \geq 1$; and
(iii) if $C \subseteq D$ are members of $\mathcal{C}$, then $G(n, C) \subseteq G(n, D)$ for each n.

In case $\mathcal{C}$ is the collection of all closed subsets of X, one obtains the well-known class of semi-stratifiable spaces introduced by Geoffrey D. Creede [13]. In case $\mathcal{C}$ is the collection of all compact subsets of X, one has the class of all c-semi-stratifiable (CSS) spaces introduced by Harold W. Martin [26]. It is easily seen that a semi-stratifiable space is CSS.

According to T. O. Banakh [1], a topological space (X, τ) is quarter-stratifiable if there is a function g (called a quarter-stratification of X) from $\{1, 2, 3, \cdots\} \times X$ into τ such that

(a) for each $n \geq 1$, the collection $\{g(n, x) : x \in X\}$ covers X;
(b) if $y \in g(n, x_n)$ for each n, then the sequence $\langle x_n \rangle$ converges to y.

In this paper we investigate the relation between CSS and quarter-stratifiability among the class of GO-spaces. A question in [2] is partially answered. Furthermore, we show that any quarter-stratifiable GO-space is CSS and give an example that there is a CSS LOTS that is not quarter-stratifiable. We show that, among GO-spaces, X with either $R(X)$ or $L(X)$ is dense σ-closed-discrete, being CSS and quarter-stratifiable are equivalent properties. We study the CSS ordered extensions of GO-spaces and give some surprising results. We show that the properties (X is CSS, X has a G^*_δ-diagonal, X has a G_δ-diagonal, X has a quasi-G^*_δ-diagonal, X has a quasi-G_δ-diagonal, and X is a $\sigma^\sharp$-space) in the class of GO-spaces with a σ-closed-discrete dense subset are equivalent. In addition, we prove a " local implies global" theorem for perfectly normal GO-spaces that are locally CSS. In the paper's final section, we deduce that four special types of bases (weakly uniform bases, σ-disjoint bases, point-countable bases, and weak monotone ortho-bases) in the class of quarter-stratifiable GO-spaces are equivalent.

Recall that a *generalized ordered space* (GO-space) is a triple $(X, <, \tau)$ where (X, τ) is a Hausdorff space that has a base of order-convex sets. If τ is the usual open-interval topology of the order $<$, then X is a *linearly ordered topological space* (LOTS). It is clear that the class of GO-spaces is strictly larger than the class of LOTS, and it is known that GO-spaces are exactly those topological spaces that embed (topologically) in some LOTS.

We reserve the symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$, and $\mathbb{Z}$ for the usual sets of real, rational, and irrational numbers, and for the set of all integers, respectively.

2. The Relation Between CSS and Quarter-Stratifiable GO-Spaces

Here, we characterize those GO-spaces that are quarter-stratifiable in terms of certain special subsets of any GO-space: $R(X)$, $E(X)$, $I(X)$, and $L(X)$. For any GO-space $(X, \tau, <)$, let $I(X)$ be the set of all isolated points of (X, τ). Define $R(X) = \{x \in X - I(X) : [x, \rightarrow) \in \tau\}$ and $L(X) = \{x \in X - I(X) : (\leftarrow, x] \in \tau\}$. Let $E(X) = X - (I(X) \cup R(X) \cup L(X))$. Thus, $R(X)$, $E(X)$, $I(X)$, and $L(X)$ are pairwise disjoint.

Lemma 2.1 (see [10])**.** *Let X be a quarter-stratifiable GO-space. Then*
(a) *X has a G_δ-diagonal;*
(b) *X has a σ-closed-discrete dense subset and therefore is perfect;*
(c) *X is first-countable and hereditarily paracompact.*

Lemma 2.2 (see [2])**.** *Let X be a GO-space. If X has a quasi-G_δ-diagonal, then X is CSS.*

Proposition 2.3. *Any quarter-stratifiable GO-space is CSS.*

Proof. Let X be a quarter-stratifiable GO-space. By Lemma 2.1(a), X has a G_δ-diagonal, and therefore, X has a quasi-G_δ-diagonal. In light of Lemma 2.2, X is CSS. □

Example 2.4. The familiar Sorgenfrey line $\mathbb{S}$ is quarter-stratifiable (using the function $g(n, x) = (x - \frac{1}{n}, x - \frac{1}{2n})$ for each rational x and $g(n, x) = \emptyset$ for each irrational x) and therefore is CSS (by Proposition 2.3) but is not semi-stratifiable (see [1] or [25]).

Example 2.5. *There is a CSS LOTS that is not quarter-stratifiable.*

Proof. Let $M^* = (\mathbb{R} \times \{0\}) \bigcup (\mathbb{P} \times \mathbb{Z})$, with the lexicographic order and the associated open-interval topology. In the light of Example 4.6 in [2], the space M^* is CSS and cannot have a G_δ-diagonal. By Lemma 2.1(a), M^* is not quarter-stratifiable. □

Example 4.6 in [2] gives three CSS LOTS. But none of the three spaces is perfect. By Lemma 2.1(b), none of the three spaces is quarter-stratifiable. This is no accident because, for a large class of perfect GO-spaces, being CSS is equivalent to being quarter-stratifiable, as our next proposition shows. Recall that any GO-space having a σ-closed-discrete dense subspace is perfect [3] and that there is no known ZFC example of a perfect GO-space that does not have a σ-closed-discrete dense subset. (See [9] for related material.) We begin with a lemma.

Lemma 2.6 (see [2]). *Suppose $(X, \tau, <)$ is a GO-space with a σ-closed-discrete dense subset. Then X is CSS if and only if X has a G_δ-diagonal.*

Lemma 2.7 (see [10]). *Suppose that X is a perfect GO-space with a G_δ-diagonal. If either $R(X)$ or $L(X)$ is σ-closed-discrete, then X is quarter-stratifiable.*

Proposition 2.8. *Suppose that X is a GO-space. If either $R(X)$ or $L(X)$ is dense σ-closed-discrete, then X is CSS if and only if X is quarter-stratifiable.*

Proof. Half of the proof follows from Proposition 2.3.

For the converse, suppose X is CSS. In light of Lemma 2.7, it is enough to prove that X is perfect and X has a G_δ-diagonal. To see that X is perfect, we have only to note that X is a GO-space with a σ-closed-discrete dense subset. To see that X has a G_δ-diagonal, we may apply Lemma 2.6, because X is CSS. $\square$

Even though Proposition 2.8 does not characterize quarter-stratifiable GO-spaces, it does allow us to put quarter-stratifiable GO-spaces into a more familiar context. Recall M. J. Faber's metrization theorem for GO-spaces [16]: A perfect GO-space with a G_δ-diagonal is metrizable provided both $R(X)$ and $L(X)$ are σ-closed-discrete in X. If one, but not both, of $R(X)$ and $L(X)$ is σ-closed-discrete, then X is quarter-stratifiable, but not metrizable. Therefore, we have the following corollary.

Corollary 2.9. *Suppose that X is a GO-space. If both $R(X)$ and $L(X)$ are σ-closed-discrete and either of them is dense, then X is CSS if and only if X is metrizable.*

3. The Relation of Various Kinds of Diagonals in CSS GO-Spaces

Let $\mathcal{G} = \{\mathcal{G}(n) : n \geq 1\}$ be a countable family of collections of open subsets of a space X. Consider the following conditions on $\mathcal{G}$.

(a) For each $x \in X$, $\bigcap\{St(x, \mathcal{G}(n)) : n \geq 1\} = \{x\}$.
(b) For each $x \in X$, $\bigcap\{cl(St(x, \mathcal{G}(n))) : n \geq 1\} = \{x\}$.
(c) For any distinct $x, y \in X$, there exists some n such that $x \in St(x, \mathcal{G}(n)) \subseteq X - \{y\}$.
(d) For any distinct $x, y \in X$, there exists some n such that $x \in cl(St(x, \mathcal{G}(n))) \subseteq X - \{y\}$.

Definition 3.1 (see [27]). X has a quasi-G_δ^*-diagonal if there exists a family $\mathcal{G}$ satisfying (d). Recall that X has a G_δ-diagonal (a G_δ^*-diagonal, a quasi-G_δ-diagonal) if there exists a family $\mathcal{G}$ satisfying (a), ((b), and (c), respectively).

From the definition it is clear that if the space X has a quasi-G_δ^*-diagonal (a G_δ^*-diagonal), then X has a quasi-G_δ-diagonal (a G_δ-diagonal).

Lemma 3.2 (see [22]). *Every regular sub-metacompact (= θ-refinable) space with a G_δ-diagonal has a G_δ^*-diagonal.*

Proposition 3.3. *Suppose that X is a CSS GO-space. Then X has a G_δ-diagonal if and only if X has a G_δ^*-diagonal.*

Proof. That any space with a G_δ^*-diagonal has a G_δ-diagonal. For the converse, suppose X has a G_δ-diagonal. To see that X has a G_δ^*-diagonal, apply Lemma 3.2, because any CSS GO-space is paracompact (see [15]), and therefore is sub-metacompact. □

Let us note that Lemma 2.6 and Proposition 3.3 yield the following.

Corollary 3.4. *Suppose X is a GO-space with a σ-closed-discrete dense subset. Then X is CSS if and only if X has a G_δ^*-diagonal.*

Note the fact that any G_δ^*-diagonal is quasi-G_δ^*-diagonal, therefore quasi-G_δ-diagonal, so we obtain the following corollary by Lemma 2.2 and Proposition 3.3.

Corollary 3.5. *Suppose X is a GO-space with a σ-closed-discrete dense subset. Then X is CSS if and only if X has a quasi-G_δ^*-diagonal (or X has a quasi-G_δ-diagonal).*

From Lemma 2.1(b), Proposition 2.3, and Corollary 3.4, we obtain the following.

Corollary 3.6. *Any quarter-stratifiable GO-space has a G_δ^*-diagonal.*

Lemma 3.7 (see [22]). *Every sub-paracompact space with a G_δ-diagonal is a $\sigma^\sharp$-space.*

Lemma 3.8 (see [2]). *Let X be a Hausdorff space. If X is a $\sigma^\sharp$-space, (i.e., X has a σ-closure-preserving collection $\mathcal{C}$ of closed sets with the property that if $x \neq y$ are points of X, then some $C \in \mathcal{C}$ has $x \in C$ and $y \notin C$), then X is CSS.*

Let us observe that Lemma 2.6, Lemma 3.7, and Lemma 3.8 immediately yield the following corollary.

Corollary 3.9. *Suppose X is a GO-space with a σ-closed-discrete dense subset. Then X is CSS if and only if X is a $\sigma^\sharp$-space.*

Corollary 3.9, along with Lemma 2.1(b) and Proposition 2.3, yields the following corollary.

Corollary 3.10. *Any quarter-stratifiable GO-space is a $\sigma^\sharp$-space.*

In Question 4.9 of [2], the authors ask whether there is a GO-space that is CSS but not a $\sigma^\sharp$-space. In the category of GO-spaces with a σ-closed-discrete dense subset, that question has a negative answer, as we show in the following by virtue of Corollary 3.9.

Proposition 3.11. *If X is a GO-space and if its ordered extension X^* has a σ-closed-discrete dense subset, then X is a $\sigma^\sharp$-space provided X is CSS.*

From Lemma 2.6, Corollary 3.4, Corollary 3.9, and Corollary 3.5, we obtain the following.

Proposition 3.12. *Suppose X is a GO-space with a σ-closed-discrete dense subset, then the following are equivalent:*
(a) *X is CSS;*
(b) *X has a G^*_δ-diagonal;*
(c) *X has a G_δ-diagonal;*
(d) *X has a quasi-G^*_δ-diagonal;*
(e) *X has a quasi-G_δ-diagonal;*
(f) *X is $\sigma^\sharp$-space.*

Definition 3.13 (see [15])**.** Let σ be the set of all finite sequences of positive integers and let Σ be the set of all infinite sequences of positive integers. For a point $n = (n_1, n_2, ...)$ of Σ and for $k \geq 1$, write $n|k = (n_1, ..., n_k)$. A space X is said to have a $\mathscr{G}$-Souslin diagonal if for each $p \in \sigma$ there is an open subset $G(p)$ of $X \times X$ such that $\{(x,x) : x \in X\} = \bigcup\{\bigcap\limits_{k=1}^{\infty} G(n|k) : n \in \Sigma\}$.

Lemma 3.14 (see [15])**.** *If a GO-space has a $\mathscr{G}$-Souslin diagonal, then it is a $\sigma^\sharp$-space, and therefore is CSS.*

Following the proof of Lemma 3.14 in [15], we obtain the following corollary.

Corollary 3.15. *If a GO-space has a $\mathscr{G}$-Souslin diagonal, then it has a quasi-G_δ-diagonal.*

Corollary 3.16. *If a GO-space has a quasi-G_δ-diagonal, then it is a $\sigma^\sharp$-space.*

Proposition 3.12 and Lemma 3.14 yield the following corollary.

Corollary 3.17. *Suppose X is a GO-space with a σ-closed-discrete dense subset. If X has a $\mathscr{G}$-Souslin diagonal, then X has a G^*_δ-diagonal.*

Example 3.18. *There is a non-metrizable LOTS having a $\mathscr{G}$-Souslin diagonal; hence, a LOTS can have a $\mathscr{G}$-Souslin diagonal without having a G^*_δ-diagonal.*

Proof. Let $X = \{(x, n) \in \mathbb{R} \times \mathbb{Z}$: if x is rational, then $n = 0\}$ and order X lexicographically. The open-interval topology of that order has a σ-disjoint base and is, therefore, quasi-metrizable. Then X is not metrizable, but X has a $\mathscr{G}$-Souslin diagonal (see [15]). Clearly, X does not have a G_δ^*-diagonal because any LOTS with a G_δ-diagonal is metrizable (see [25]). □

Lemma 3.19 (see [15])**.** *Suppose that each point of a space X is a G_δ in X and that X has at most countably many non-isolated points. Then X has a $\mathscr{G}$-Souslin diagonal.*

Lemma 3.19, along with Definition 3.1, yields the following corollary.

Corollary 3.20. *Suppose that a space X has at most countably many non-isolated points. If X has a G_δ-diagonal, then X has a $\mathscr{G}$-Souslin diagonal.*

Definition 3.21 (see [22])**.** A topological space (X, τ) is an α-space if there is a function g (called an α-function for X) from $\{1, 2, 3, \cdots\} \times X$ into τ such that

(a) $\{x\} = \bigcap\limits_{n=1}^{\infty} g(n, x)$ for each x in X;

(b) if $y \in g(n, x)$, then $g(n, x) \subseteq g(n, y)$ for each n.

Lemma 3.22 (see [22])**.** *Every $\sigma^\sharp$-space is an α-space.*

Proposition 3.23. *Let X be a LOTS. If X is CSS, then X is an α-space.*

Proof. Lemma 3.22, along with the equivalence of a $\sigma^\sharp$-space and CSS in any LOTS [15] and Corollary 3.9, yields this result. □

Corollary 3.24. *If X is a GO-space with a σ-closed-discrete dense set and if X is CSS, then X is an α-space.*

Corollary 3.25. *Any quarter-stratifiable GO-space is an α-space.*

4. CSS Ordered Extensions of GO-Spaces

Lemma 4.1 (see [10])**.** *Suppose that X is a perfect GO-space and $X = Y \cup M$, where, in their relative topologies, Y has a G_δ-diagonal and M is metrizable. Then X has a G_δ-diagonal.*

Lemma 4.1 yields a CSS sum theorem for GO-spaces with a σ-closed-discrete dense subset that have a large metrizable part. (Our next result is a variant of Proposition 4.5 in [10].)

Proposition 4.2. *Let X be a GO-space with a σ-closed-discrete dense subset. If $X = Y \cup M$, where Y is CSS and M is metrizable, then X has a G_δ-diagonal and therefore X is CSS.*

Proof. Suppose X is a GO-space with a σ-closed-discrete dense subset, then X is perfect and the subspace Y also has a σ-closed-discrete dense subset (see [12]). In light of Y being CSS and Lemma 2.6, Y has a G_δ-diagonal. Applying Lemma 4.1, we see that X has a G_δ-diagonal. Therefore, Lemma 2.2 or Lemma 2.6 completes the proof. $\square$

Lemma 4.1 and Proposition 4.2 both have a surprising corollary. Recall that for any GO-space $(X, \tau, <)$, there is a canonical LOTS $(X^*, \tau^*, <^*)$ that contains X as a closed subspace, where X^* is obtained by adding a certain collection of isolated points to X and where $<^*$ is a natural lexicographic extension of $<$ (see [25]). It is often of interest to know which topological properties of X are passed on to X^*. Example 4.13 in [2] shows that being a CSS space is not a property of that type. But we can obtain the following.

Corollary 4.3. *For any GO-space X, the following are equivalent:*
(a) *X^* is metrizable;*
(b) *X is metrizable;*
(c) *X has a G_δ-diagonal and X^* is perfect;*
(d) *X^* has a G_δ-diagonal.*

Proof. That (d) $\Rightarrow$ (a) follows from the fact that any LOTS with a G_δ-diagonal is metrizable.

Clearly, (a) $\Rightarrow$ (b) and it is known (see [25]) that if X is metrizable, then so is X^*; thus, X^* is perfect. Hence, (b) $\Rightarrow$ (c).

To see that (c) $\Rightarrow$ (d), suppose X has a G_δ-diagonal. Note that X^* is perfect and $X^* = X \cup M$, where M is a certain set of isolated points. Then applying Lemma 4.1, we see that X^* satisfies (d). $\square$

The following corollary is a consequence of Corollary 4.3 and the fact that any G_δ^*-diagonal is a G_δ-diagonal.

Corollary 4.4. *For any GO-space X, the following are equivalent:*
(a) *X^* is metrizable;*
(b) *X is metrizable;*
(c) *X has a G_δ^*-diagonal and X^* is perfect;*
(d) *X^* has a G_δ^*-diagonal.*

Lemma 2.1(a) and Corollary 4.3 allow us to obtain an analogous result for quarter-stratifiability in GO-spaces.

Corollary 4.5 (see [10]). *For any GO-space X, the following are equivalent:*
(a) *X^* is metrizable;*
(b) *X is metrizable;*

(c) *X is quarter-stratifiable and X^* is perfect;*
(d) *X^* is quarter-stratifiable.*

Lemma 2.1(a), Lemma 2.3, and Lemma 3.3 yield the following corollary.

Corollary 4.6. *Any quarter-stratifiable GO-space has a G^*_δ-diagonal.*

Remark 4.7. By contrast with the best known result for quarter- stratifiability in GO-spaces (see Corollary 4.7 in [2]) and the result for GO-spaces with a G^*_δ-diagonal above, we may see that the result for GO-spaces with a G_δ-diagonal is best.

Proposition 4.2 also has a surprising corollary for the CSS property in GO-spaces.

Corollary 4.8. *For any GO-space X, if X^* has a σ-closed-discrete dense subset, then the following are equivalent:*
(a) *X^* is CSS;*
(b) *X is CSS ;*
(c) *X^* is metrizable;*
(d) *X is metrizable.*

Proof. That (a) $\Rightarrow$ (b) follows from the fact that the CSS property is hereditary.

Clearly, (c) $\Rightarrow$ (d) and it is known (see [25]) that if X is metrizable, then so is X^*; thus, X^* is CSS. Hence, (d) $\Rightarrow$ (a).

To complete the proof, we must show that (b) $\Rightarrow$ (c), so suppose X is CSS. Because $X^* = X \cup M$, where M is a certain set of isolated points, assertion Proposition 4.2 yields that X^* is CSS, and hence (see Lemma 2.6) has a G_δ-diagonal. But any LOTS with a G_δ-diagonal is metrizable (see [25]). Hence, X^* satisfies (c). □

A similar result for GO-spaces with a quasi-G_δ-diagonal (a quasi-G^*_δ-diagonal, a $\sigma^\sharp$-space) is listed in the following corollary, which is a consequence of Lemma 2.2, Corollary 3.9, and Corollary 4.8.

Corollary 4.9. *For any GO-space X, if X^* has a σ-closed-discrete dense subset, then the following are equivalent:*
(a) *X^* is metrizable;*
(b) *X is metrizable;*
(c) *X^* has a quasi-G^*_δ-diagonal;*
(d) *X has a quasi-G^*_δ-diagonal;*
(e) *X^* has a quasi-G_δ-diagonal;*
(f) *X has a quasi-G_δ-diagonal;*
(g) *X^* is a $\sigma^\sharp$-space;*
(h) *X is a $\sigma^\sharp$-space.*

We obtain the following corollary from Lemma 3.19.

Corollary 4.10. *Suppose that each point of a GO-space X is a G_δ in X and that X has at most countably many non-isolated points. Then X^* has a $\mathscr{G}$-Souslin diagonal, and therefore is a $\sigma^\sharp$-space.*

Proof. Because $X^* = X \cup M$, where M is a certain set of isolated points, assertion Lemma 3.19 yields that X^* has a $\mathscr{G}$-Souslin diagonal. The other is a consequence of Lemma 3.14. □

Corollary 4.11. *For any GO-space X, if X^* has a σ-closed-discrete dense subset, then the following are equivalent:*

(a) *X has a $\mathscr{G}$-Souslin diagonal;*
(b) *X^* has a $\mathscr{G}$-Souslin diagonal;*
(c) *X^* is metrizable;*
(d) *X is metrizable.*

Proof. Clearly, (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

That (b) $\Rightarrow$ (c) is a consequence of Corollary 3.17 and Corollary 4.4.

To complete the proof, we must show that (a) $\Rightarrow$ (b), so suppose X has a $\mathscr{G}$-Souslin diagonal. Clearly, X has a σ-closed-discrete dense subset. Using Corollary 3.17, we obtain that X has a G^*_δ-diagonal. According to Corollary 4.4, X^* is metrizable; thus, X^* has a $\mathscr{G}$-Souslin diagonal. Hence, X^* satisfies (b). □

Lemma 4.12 (see [2]). *Suppose that X is normal and sub-metacompact ($=\theta$-refinable). If X is locally CSS, then X is CSS.*

The following corollary is an immediate consequence of Lemma 4.12 and the fact that any perfectly normal GO-space is paracompact [25].

Corollary 4.13. *Suppose that X is a perfectly normal GO-space. If X is locally CSS, then X is CSS.*

We close this section with some results on the role of the CSS and quarter-stratifiable properties in metrization theory among LOTS (or GO-spaces). First, recall a classical metrization theorem of Arhangel'skii.

Lemma 4.14 (see [8]). *Suppose X is a paracompact p-space in the sense of Arhangel'skii. Then the following are equivalent:*

(a) *X is metrizable;*
(b) *X has a G_δ-diagonal.*

Proposition 4.15. *Let X be a LOTS. Then X is metrizable if and only if X is CSS with a σ-closed-discrete dense subset.*

Proof. It is enough to show that every CSS LOTS with a σ-closed-discrete dense subset is metrizable. To that end, we invoke the corollary to Theorem 2.1.6 in [30] to see that any LOTS with a σ-closed-discrete dense subset must be a paracompact p-space in the sense of Arhangel'skii, and then use Lemma 2.6 and Lemma 4.14 to conclude that X is metrizable. $\square$

Remark 4.16. In the light of Proposition 4.15, each quarter-stratifiable LOTS is metrizable. However, as the Sorgenfrey line shows, a GO-space can be quarter-stratifiable and nonmetrizable.

Results in [3] allow us to extend Proposition 4.15 to a perfect CSS LOTS that has certain other properties.

Proposition 4.17. *Suppose X is a perfect CSS LOTS. Then the following are equivalent:*

(a) *X is metrizable;*

(b) *X is the union of countably many metrizable subspaces;*

(c) *X can be mapped by a continuous s-mapping onto some topological space with a G_δ-diagonal.*

Lemma 4.18 (see [2])**.** *Let X be a completely regular space. Then X is metrizable if and only if X is a paracompact p-space in the sense of Arhangel'skii and is CSS.*

Proposition 4.19. *Let X be a GO-space. Then X is metrizable if and only if X is CSS and a p-space.*

Proof. That X is a completely regular space follows from the fact that any GO-space is collectionwise normal [25]. Half of the proof follows from Lemma 4.18.

For the converse, suppose X is CSS and a p-space. To get the result, we can apply Lemma 4.18, because any CSS GO-space is paracompact [15]. $\square$

Definition 4.20 (see [22])**.** A topological space (X, τ) is a β-space if there is a function g (called a β-function for X) from $\{1, 2, 3, \cdots\} \times X$ into τ such that

(a) for all $x \in X$ and each $n \geq 1$, $x \in g(n, x)$;

(b) if $x \in g(n, x_n)$ for each n, then the sequence $\langle x_n \rangle$ has a cluster point.

Definition 4.21 (see [23])**.** A topological space (X, τ) is a γ-space if there is a function g (called a γ-function for X) from $\{1, 2, 3, \cdots\} \times X$ into τ such that

(a) $x \in \bigcap\limits_{n=1}^{\infty} g(n, x)$ for each x in X;

(b) if $y_n \in g(n, x_n)$ and $x_n \in g(n, y_n)$ for each n, then the sequence $\langle x_n \rangle$ has a cluster point. It is easily seen that a γ-space is CSS.

From Corollary 3.24 and Definition 4.21 we have the following.

Corollary 4.22. *Let X be a GO-space with a σ-closed-discrete dense subset. Then every γ-space is an α-space.*

Lemma 4.23 (see [23])**.** *Every T_1 space which is a β-space and a γ-space is developable.*

Lemma 4.23, along with the equivalence of CSS and a γ-space in any LOTS (see [15]), yields the following.

Proposition 4.24. *Let X be a LOTS. Then X is metrizable if and only if X is CSS and a β-space.*

Lemma 4.25 (see [22])**.** *Let X be a regular space. Then X is semi-stratifiable if and only if X is a β-space with a G_δ^*-diagonal.*

Lemma 4.26. *Let X be a GO-space with a G_δ-diagonal. Then X has a G_δ^*-diagonal.*

Proof. Suppose X is a GO-space with a G_δ-diagonal. To get the result, apply the fact that any GO-space with a G_δ-diagonal is paracompact [25] and any paracompact space with a G_δ-diagonal has a G_δ^*-diagonal. $\square$

The following proposition is a consequence of lemmas 4.25 and 4.26.

Proposition 4.27. *Let X be a GO-space. Then X is metrizable if and only if X is a β-space with a G_δ-diagonal.*

Lemma 2.6 and Proposition 4.27 yield the following corollary.

Corollary 4.28. *Let X be a GO-space. Then X is metrizable if and only if X is CSS and a β-space with a σ-closed-discrete dense subset.*

Corollary 4.29. *Suppose X is a perfect CSS GO-space that is also a β-space. Then the following are equivalent:*

(a) *X is metrizable;*
(b) *X is the union of countably many metrizable subspaces;*
(c) *X can be mapped by a continuous s-mapping onto some topological space with a G_δ-diagonal.*

Corollary 4.30. *Let X be a GO-space. Then X is metrizable if and only if X is quarter-stratifiable and a β-space.*

5. Special Bases in Quarter-Stratifiable GO-Spaces

Lemma 5.1 (see [7]). *Let X be a GO-space. Then the following assertions are equivalent:*

(a) *X has a σ-disjoint base (equivalently, is quasi-developable);*
(b) *X has a point-countable base and a quasi-G_δ-diagonal.*

Lemma 2.1(a) and Lemma 5.1 give the following corollary.

Corollary 5.2. *Any GO-space with a σ-disjoint base is CSS.*

Example 5.3. There is a GO-space with a point-countable base which is not CSS (see Example 5.8).

Lemma 5.4 (see [5]). *A GO-space X has a weakly uniform base if and only if X has a σ-disjoint base (equivalently, is quasi-developable) and has a G_δ-diagonal.*

We obtain the following corollary from Lemma 5.1 and Lemma 5.4.

Corollary 5.5. *A GO-space X has a weakly uniform base if and only if X has a point-countable base and a G_δ-diagonal.*

Lemma 5.6 (see [3]). *Let X be a GO-space. Then the following assertions are equivalent:*

(a) *X is metrizable;*
(b) *X has a point-countable base and a σ-closed-discrete dense subset;*
(c) *X has a weak monotone ortho-base and a σ-closed-discrete dense subset.*

The following proposition is an immediate consequence of Lemma 2.1, Corollary 5.5, and Lemma 5.6.

Proposition 5.7. *Let X be a GO-space. Then the following assertions are equivalent:*

(a) *X is metrizable;*
(b) *X is quarter-stratifiable and has a point-countable base;*
(c) *X is quarter-stratifiable and has a weakly uniform base;*
(d) *X is quarter-stratifiable and has a weak monotone ortho-base.*

Example 5.8. In [17], Gary Gruenhage constructed a LOTS with a point-countable base that is not quasi-metrizable. In light of Proposition 4.7 in [2], that space cannot be CSS. Therefore, the space is not quarter-stratifiable.

Example 5.9. *There is a GO-space with a weakly uniform base that is CSS but not quarter-stratifiable.*

Proof. The Michael line M, which is a non-metrizable CSS GO-space, was shown by R. W. Heath and W. F. Lindgren to have a weakly uniform base [21]. Therefore, by Proposition 5.7, M is not quarter-stratifiable. $\square$

Example 5.10. *There is a LOTS with a weak monotone ortho-base that is not quarter-stratifiable.*

Proof. Using the extended Big Bush appearing in section 3 of [6], we can obtain a LOTS X that has a weak monotone ortho-base but not a point-countable base. Then, by Proposition 5.7, X is not quarter-stratifiable. $\square$

We close this section with a proposition which is a consequence of Lemma 5.1 and Proposition 5.7.

Proposition 5.11. *Let X be a quarter-stratifiable GO-space. Then the following assertions are equivalent:*

(a) *X has a point-countable base;*
(b) *X has a σ-disjoint base;*
(c) *X has a weakly uniform base;*
(d) *X has a weak monotone ortho-base.*

Recall that a space X is called *weak-σ* if and only if there exists a σ-disjoint network $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_k$ such that, for each $k \in \mathbb{N}$, $\mathcal{M}_k$ is discrete with respect to $\mathcal{M}_k^*$ [28].

Theorem 5.12. *A GO-space is metrizable if and only if it is a weak-σ-space and a quarter-stratifiable space.*

Proof. Apply [4, Theorem 7.1], which states that a GO-space is quasi-developable if and only if it is a weak-σ-space with a quasi-G_δ-diagonal, and Lemma 2.1. Since a space X is developable if and only if X is quasi-developable and perfect, and every developable GO-space is metrizable, the proof is complete. $\square$

Lemma 5.13 (see [11])**.** *A GO-space is metrizable if and only if it is a quasi-developable β-space.*

Proposition 5.14. *A GO-space is metrizable if and only if it is a weak-σ-space and a β-space with a quasi-G_δ-diagonal.*

Proof. Apply [4, Theorem 7.1], (as in the proof of Theorem 5.12) and Lemma 5.13. $\square$

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