TOPOLOGY PROCEEDINGS
Volume 42, 2013
Pages 259–273

http://topology.auburn.edu/tp/

Span of Subcontinua

by

Nadeeka De Silva

Electronically published on January 14, 2013

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 42 (2013)
Pages 259-273
E-Published on January 14, 2013

SPAN OF SUBCONTINUA

NADEEKA DE SILVA

ABSTRACT. Let Y be a continuum consisting of a ray limiting to a continuum X. We prove that $\sigma(Y) \leq \max\{\sigma(X), \sigma_0^*(X)\}$. When $\sigma(X) = 0$ or when X is a simple closed curve, we have that $\sigma(Y) = \sigma(X)$. Using this, we construct for each closed subset G of $[0, 1]$ with $0 \in G$ a one-dimensional continuum Y_G such that the set of values of the span of subcontinua of Y_G is the set G. Some other results related to this are also presented.

1. INTRODUCTION

The concept of the span of a metric space was introduced by A. Lelek [4]. In [5], he defined three other versions of span called surjective span, semispan, and surjective semispan. A *continuum* is a nonempty, compact, connected metric space, and a *ray* is a homeomorphic copy of $(0, 1]$. Our focus is directed on studying the span of a continuum Y, consisting of a ray limiting to continuum X, and on the set of values of the span of subcontinua of a continuum.

Definition 1.1 ($S(A)$)**.** Let X be a metric space with distance d. Let A be a nonempty subset of $X \times X$. $S(A)$ is defined by

$$S(A) = \inf\{d(x, y) : (x, y) \in A\}.$$

Definition 1.2 (Span)**.** Let X be a continuum with distance d. For each nonempty subset A of $X \times X$, let $S(A)$ be as in the above definition. The

2010 *Mathematics Subject Classification.* Primary 54F50; Secondary 54F15.
Key words and phrases. ray, span, surjective semispan.
This paper represents part of the author's Ph.D. dissertation at Texas Tech University, August 2012, under the direction of Professor Wayne Lewis. Some of the results included here were presented in a talk in the Special Session on Continuum Theory at the Spring Topology and Dynamics Conference in Mexico City, March 2012.

span of continuum X is defined by

$$span(X) = \sigma(X) =$$

$$\sup\{S(A) : A \subset X \times X, \text{ A is connected, and } \pi_1(A) = \pi_2(A)\}$$

where π_1 and π_2 are the projection maps. The set A in this definition can be considered to be closed.

The *surjective span* $\sigma^*(X)$ of X is defined similarly. The only difference is that we impose one additional condition $\pi_1(A) = \pi_2(A) = X$ on sets A. To obtain the definition of the *semispan* $\sigma_0(X)$ of X, we replace the condition $\pi_1(A) = \pi_2(A)$ in the definition of $\sigma(X)$ by the inclusion $\pi_1(A) \subset \pi_2(A)$, and to define the *surjective semispan* $\sigma_0^*(X)$ of X we additionally require that $\pi_2(A) = X$.

It follows directly from the definitions that the following inequalities hold:

(1.1) $$0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam}(X)$$

(1.2) $$0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq \text{diam}(X)$$

(1.3) $$\sigma(H) \leq \sigma(X), \sigma_0(H) \leq \sigma_0(X) \text{ for } H \subset X.$$

Consider a continuum Y consisting of a ray limiting to continuum X. For arcs and arc-like continua, all versions of span are zero. Thus, it is interesting to study the span of Y. Does $\sigma(X) = \sigma(Y)$? We obtain a partial result in answering this question: $\sigma(Y) \leq \max(\sigma(X), \sigma_0^*(X))$. In [5], Lelek gives an example of a 4-od such that the span is less than the surjective semispan. In [3], L. C. Hoehn and A. Karasev also give an example of such a triod. Thus, it remains an open question whether $\sigma(Y) = \sigma(X)$ in general. This inequality can be improved to $\sigma(Y) = \sigma(X)$ under any of the following conditions:

- $\sigma(X) = 0$.
- X is a simple closed curve.
- $\sigma_0^*(X) \leq \sigma(X)$.

If the second condition is satisfied, that is, if X is a simple closed curve, then $\sigma_0^*(X) = \sigma(X)$. So the second condition implies the third condition. We also explore how these results can be applied to other versions of span.

We use the above results to construct continua such that the set of values of the span of subcontinua is equal to specific sets. There are other uses of these results. In private communication, we learned that Tina Sovič has recently given an independent proof of the span zero case and she is using it to construct more examples of nonchainable continua with span zero.

Furthermore, if Y is a continuum consisting of a ray limiting to continuum X and Z is a continuum formed by a ray limiting to Y, then

$\sigma(Z) \leq \max(\sigma(X), \sigma_0^*(X))$. Thus, if we start from a continuum and keep adding limiting rays in this fashion, the span of the resulting continuum is bounded by the maximum of the span and the surjective semispan of the starting continuum, and equality is obtained under any of the above conditions.

For a given continuum X, our focus is directed on the set of values of the span of subcontinua of X.

Definition 1.3 ($B(X)$)**.** Let X be a continuum. $B(X)$ is defined by:

$$B(X) = \{\sigma(H) : H \text{ is a subcontinuum of X }\}.$$

In most of the cases, we shall assume that the span of X is 1. The following facts are straightforward.

- If X is a simple closed curve with $\sigma(X) = 1$, then $B(X) = \{0, 1\}$.
- If X is a simple triod with $\sigma(X) = 1$, then $B(X) = [0, 1]$.
- $0 \in B(X)$.

Due to the above observations, we study the possibilities of $B(X)$ between the two extreme cases of $\{0, 1\}$ and $[0, 1]$. This leads us to the following question.

Question 1.4. Given that $M \subset [0, 1]$ with $0, 1 \in M$, does there exist a continuum X_M such that $B(X_M) = M$?

In answering this question, we obtain the following results.

- For each closed subset M of $[0, 1]$ with $0, 1 \in M$, there exists a one-dimensional continuum Y_M such that the set of values of the span of subcontinua of Y_M is the set M. There are uncountably many incomparable continua for each closed set M.
- Except for the case where M has an infinite number of nondegenerate components, the examples constructed are planar.

2. Span of a Continuum Consisting of a Ray Limiting to a Continuum

In this section we obtain bounds on the span, semispan, surjective span, and surjective semispan of continuum Y consisting of a ray limiting to continuum X.

Definition 2.0.1 (ray limiting to X)**.** Let X be a continuum and R be a ray. R is a *ray limiting to X* if

(1) $X \cap R = \emptyset$,
(2) $X \cup R = \overline{R}$, and
(3) $\overline{R}$ is compact.

2.1. Span.

First, we provide the following propositions which will be used later.

Proposition 2.1.1. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. There exists a continuous surjection $f : Y \to [0, 1]$ such that*

(1) *$y \in X$ if and only if $f(y) = 0$ and*
(2) *f is one-to-one on R.*

Proposition 2.1.2. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$ and let A be a subcontinuum of $Y \times Y$ such that $S(A) > 0$ and $\pi_1(A) \subseteq \pi_2(A)$. Then $A \cap (X \times X) \neq \emptyset$.*

Proof. Note that $\pi_i(A) \not\subset R$ for $i \in 1, 2$. (Every subcontinuum of R has span zero.) Let f be a continuous function as in Proposition 2.1.1. Define the functions $g_i : A \to [0, 1]$ by $g_i = f \circ \pi_i$ for $i = 1, 2$. The functions g_1 and g_2 satisfy the hypotheses of the coincidence point theorem. Therefore, there exists a point (x, y) in A such that $g_1(x, y) = g_2(x, y)$, which implies $f(x) = f(y)$. Since $f(x) = f(y)$, $(x, y) \in (X \times X) \cup (R \times R)$. If $(x, y) \in R \times R$, then, since f is one-to-one on R, $x = y$ and $S(A) = 0$, which is a contradiction. Therefore, $(x, y) \in X \times X$ and $A \cap (X \times X) \neq \emptyset$. □

Proposition 2.1.3. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$ and D be a subcontinuum of $Y \times Y$. If $\pi_i(D) \cap X \neq \emptyset$ and $\pi_i(D) \cap R \neq \emptyset$, where $i \in \{1, 2\}$, then $X \subset \pi_i(D)$.*

Let us prove the main result for this section.

Theorem 2.1.4. *Let X be a continuum and R be a ray limiting to it. Let $Y = X \cup R$. Then $\sigma(Y) \leq \max(\sigma(X), \sigma_0^*(X))$.*

Proof. If $\sigma(Y) = 0$, then, since $X \subset Y$, we have that $\sigma(Y) = \sigma(X) = 0$. Therefore, we can assume that $\sigma(Y) > 0$.

From the definition of span, there exists a subcontinuum A of $Y \times Y$ such that $S(A) = \sigma(Y)$ and $\pi_1(A) = \pi_2(A)$. From Proposition 2.1.2, $A \cap (X \times X) \neq \emptyset$.

Case 1: $A \subset X \times X$.

In this case, $\sigma(Y) = S(A) \leq \sigma(X) \leq \sigma(Y)$, so $\sigma(Y) = \sigma(X)$.

Case 2: $A \not\subset X \times X$.

Let C be any component of $A \cap (X \times X)$. For each $\epsilon > 0$, let $N(C, \epsilon) = \{t \in A : d(t, C) < \epsilon\}$. Let C_ϵ be the component of $N(C, \epsilon)$ that contains C. From the boundary bumper theorem, C_ϵ has a limit point on the boundary of $N(C, \epsilon)$. Therefore, C_ϵ is a connected subset of A that properly contains C. So C_ϵ contains a point in the complement of $X \times X$, i.e., $\pi_1(C_\epsilon) \cap R \neq \emptyset$

or $\pi_2(C_\epsilon) \cap R \neq \emptyset$. Consequently, using Proposition 2.1.3, it follows that, for each $\epsilon > 0$, $X \subset \pi_1(C_\epsilon)$ or $X \subset \pi_2(C_\epsilon)$. Without loss of generality, there exists a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ converging to 0 such that, for all ϵ_n, $X \subset \pi_1(C_{\epsilon_n})$. Hence, $X \subset \pi_1(C)$. Note that C is a connected subset of $X \times X$ and that C satisfies the condition $\pi_2(C) \subset \pi_1(C) = X$. Therefore, in this case, we conclude that $\sigma(Y) \leq$S(C)$\leq \sigma_0^*(X)$.

In general, we have that $\sigma(Y) \leq \sigma_0^*(X)$ or $\sigma(Y) \leq \sigma(X)$, and hence $\sigma(Y) \leq \max\{\sigma_0^*(X), \sigma(X)\}$. □

As mentioned in the introduction, due to examples by Lelek and by Hoehn and Karasev, it remains an open question whether $\sigma(Y) = \sigma(X)$ in general. However, equality can be obtained in the following special cases. The proofs of the following corollaries are straightforward and hence omitted.

Corollary 2.1.5. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. If $\sigma_0^*(X) \leq \sigma(X)$, then $\sigma(Y) = \sigma(X)$.*

Corollary 2.1.6. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. If X is a simple closed curve, then $\sigma(Y) = \sigma(X)$.*

In [1], James Francis Davis proved that $\sigma(X) = 0$ if and only if $\sigma_0(X) = 0$. Using this result, we have the following corollary.

Corollary 2.1.7. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. If $\sigma(X) = 0$, then $\sigma(Y) = \sigma(X)$.*

Sovič has recently obtained independently and by a different argument the result in Corollary 2.1.7.

2.2. Other versions of span.

Here, similar results to the one in Theorem 2.1.4 are given for the other versions of span. The proofs are similar to the proof of Theorem 2.1.4, and we use inequalities (1.1), (1.2), and (1.3) to improve the results.

Theorem 2.2.1. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. Then $\sigma_0(Y) = \sigma_0(X)$.*

Theorem 2.2.2. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. Then $\sigma_0^*(Y) \leq \sigma_0^*(X)$.*

Corollary 2.2.3. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. Then $\sigma^*(Y) \leq \sigma_0^*(X)$.*

Corollary 2.2.4. *Let X be a continuum and R be a ray limiting to X. Let $Y = X \cup R$. If $\sigma(X) = 0$, then all versions of the span values of Y are 0.*

Theorem 2.2.5. *Let Y be a continuum consisting of a ray limiting to continuum X. Let R be a ray limiting to Y. Let $Z = Y \cup R$. Then $\sigma(Z) \leq \max(\sigma(X), \sigma_0^*(X))$.*

Proof. By applying Theorem 2.1.4 to Z,

$$\sigma(Z) \leq \max(\sigma(Y), \sigma_0^*(Y)). \tag{2.1}$$

By applying Theorem 2.1.4 to Y,

$$\sigma(Y) \leq \max(\sigma(X), \sigma_0^*(X)). \tag{2.2}$$

By applying Theorem 2.2.2 to Y,

$$\sigma_0^*(Y) \leq \sigma_0^*(X). \tag{2.3}$$

From inequalities (2.1), (2.2), and (2.3), $\sigma(Z) \leq \max(\sigma(X), \sigma_0^*(X))$. □

3. Span of Subcontinua

In this section, we construct, for each closed subset G of $[0,1]$ with $0 \in G$, a one-dimensional continuum Y_G such that the $B(Y_G) = G$. Except for the case where G has an infinite number of nondegenerate components, all of the examples are planar. First, we construct the planar examples.

3.1. Construction of a continuum Y such that $B(Y)$ is equal to a given special closed set.

First, we construct planar, one-dimensional continua such that $B(X)$ is

(1) a finite set,
(2) a set which is the union of $\{0\}$ and a closed interval that does not contain 0, or
(3) a Cantor set.

Let F be any finite subset of $[0,1]$ that contains 0 and 1. We construct a continuum Y_F such that $B(Y_F) = F$. In constructing this continuum. we use an extended version of Theorem 2.1.4. The following propositions will be helpful.

Proposition 3.1.1. *Let $C_n, n = 1,2,3,\dots,m$, be disjoint continua. Let $R_n, n = 1,2,3,\dots,m-1$, be pairwise disjoint copies of $(0,1)$ such that the end of R_n corresponding to 0 limits to C_n and the other end limits to C_{n+1} and the R_n's are disjoint from the C_n's. Let $Y_n = \overline{R_n} = C_n \cup R_n \cup C_{n+1}$ and $Y = \bigcup_{n=1}^{m-1} Y_n$. There exists a continuous surjection $f : Y \to [0,1]$ such that*

(1) *$y \in C_n$ if and only if $f(y) = r_n$, for $n = 1, 2, 3, \dots m$ and*
(2) *$y \in R_n$ if and only if $f(y) = h_n(y)$, for $n = 1, 2, 3, \dots, m-1$,*

where $0 = r_1 < r_2 < r_3 < ... < r_m = 1$ and for each $n = 1, 2, 3, 4, \dots m-1$, $h_n : R_n \to (r_n, r_n + 1)$ is a homeomorphism.

Proposition 3.1.2. *Let Y be a continuum as in Proposition 3.1.1. If A is a subcontinuum of $Y \times Y$ such that $S(A) > 0$ and $\pi_1(A) = \pi_2(A)$, then $A \cap \left(\bigcup_{n=1}^{m} (C_n \times C_n) \right) \neq \emptyset$.*

Proof. Note that $\pi_i(A) \not\subset \bigcup_{n=1}^{m-1} R_n$. Let f be a continuous function as in Proposition 3.1.1. Define the functions $g_i : A \to [0, 1]$ by $g_i = f \circ \pi_i$ for $i = 1, 2$. The functions g_1 and g_2 satisfy the hypotheses of the coincidence point theorem. Therefore, there exists a point (x, y) in A such that $g_1(x, y) = g_2(x, y)$, which implies $f(x) = f(y)$. Since $f(x) = f(y)$, $(x, y) \in \left(\bigcup_{n=1}^{m} (C_n \times C_n) \right) \cup \left(\bigcup_{n=1}^{m-1} (R_n \times R_n) \right)$. Assume that $(x, y) \in \bigcup_{n=1}^{m-1} (R_n \times R_n)$. Then $x = y$, and hence $S(A) = 0$, which is a contradiction. Therefore, $(x, y) \in \bigcup_{n=1}^{m} (C_n \times C_n)$, and hence $A \cap \left(\bigcup_{n=1}^{m} (C_n \times C_n) \right) \neq \emptyset$. □

Proposition 3.1.3. *Let Y be a continuum as in Proposition 3.1.1. Let D be a subcontinuum of $Y \times Y$. If there exists an n such that $\pi_i(D) \cap C_n \neq \emptyset$ and $\pi_i(D) \not\subset C_n$ for $i = 1$ or 2, then $C_n \subset \pi_i(D)$.*

Theorem 3.1.4. *Let Y be a continuum as in Proposition 3.1.1. There exists $n_0 \in \{1, 2, 3, \dots, m\}$ such that $\sigma(Y) \leq \max(\sigma(C_{n_0}), \sigma_0^*(C_{n_0}))$.*

Proof. If $\sigma(Y) = 0$, then, since for each n, $C_n \subset Y$, we have that $\sigma(Y) = \sigma(C_n) = 0$. Therefore, we can assume that $\sigma(Y) > 0$.

From the definition of span, there exists a subcontinuum A of $Y \times Y$ such that $S(A) = \sigma(Y)$ and $\pi_1(A) = \pi_2(A)$. Note that from Proposition 3.1.2 there exists n_0 such that $A \cap (C_{n_0} \times C_{n_0}) \neq \emptyset$. Consider the two cases.

Case 1: $A \subset C_{n_0} \times C_{n_0}$.

In this case, $\sigma(Y) = S(A) \leq \sigma(C_{n_0}) \leq \sigma(Y)$, so $\sigma(Y) = \sigma(C_{n_0})$.

Case 2: $A \not\subset C_{n_0} \times C_{n_0}$.

Let C be any component of $A \cap (C_{n_0} \times C_{n_0})$. For each $\epsilon > 0$, let $N(C, \epsilon) = \{t \in A : d(t, C) < \epsilon\}$. Let C_ϵ be the component of $N(C, \epsilon)$ that contains C. From the boundary bumper theorem, C_ϵ has a limit point on the boundary of $N(C,\epsilon)$. Therefore, C_ϵ is a connected subset of A that properly contains C. So C_ϵ contains a point in the complement of $C_{n_0} \times C_{n_0}$, i.e., $\pi_1(C_\epsilon) \not\subset C_{n_0}$ or $\pi_2(C_\epsilon) \not\subset C_{n_0}$. Consequently, using Proposition 3.1.3, we obtain that, for each $\epsilon > 0$, $C_{n_0} \subset \pi_1(C_\epsilon)$ or $C_{n_0} \subset \pi_2(C_\epsilon)$. Without loss of generality, there exists a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ converging to 0 such that, for all ϵ_n, $C_{n_0} \subset \pi_1(C_{\epsilon_n})$. Hence, $C_{n_0} \subset \pi_1(C)$. Note that C is a connected subset of $C_{n_0} \times C_{n_0}$ and that C satisfies the condition $\pi_2(C) \subset \pi_1(C) = C_{n_0}$. Therefore, in this case, we have that $\sigma(Y) \leq S(C) \leq \sigma_0^*(C_{n_0})$.

In general, we have that $\sigma(Y) \leq \sigma_0^*(C_{n_0})$ or $\sigma(Y) \leq \sigma(C_{n_0})$, and hence $\sigma(Y) \leq \max\{\sigma_0^*(C_{n_0}), \sigma(C_{n_0})\}$. □

Corollary 3.1.5. *For $n = 1, 2, 3, \ldots, m$, let C_n be disjoint continua such that $\sigma_0^*(C_n) \leq \sigma(C_n)$. Let Y be a continuum as in Proposition 3.1.1. Then there exists $n_0 \in \{1, 2, 3, \ldots, m\}$ such that $\sigma(Y) = \sigma(C_{n_0})$. Furthermore, $\sigma(C_{n_0}) = \max\{\sigma(C_n) : n = 1, 2, 3, \ldots, m\}$.*

Proof. From Theorem 3.1.4 and from the condition $\sigma_0^*(C_n) \leq \sigma(C_n)$ for each n, it follows that $\sigma(Y) = \sigma(C_{n_0})$ for some $n_0 \in \{1, 2, 3, \ldots, m\}$. But $\sigma(C_n) \leq \sigma(Y) = \sigma(C_{n_0})$ for all n. Hence, $\sigma(C_{n_0}) = \max\{\sigma(C_n) : n = 1, 2, 3, \ldots, m\}$. □

Theorem 3.1.6. *For any finite subset F of $[0,1]$ that contains 0 and 1, there exists a continuum Y_F such that $B(Y_F) = F$.*

Proof. First, we construct the continuum Y_F. Let $F = \{r_1, r_2, r_3 \ldots r_m\}$ where $0 = r_1 < r_2 < r_3 < \ldots < r_m = 1$. For each n, let C_n be a circle in $\mathbb{R}^2$ with center $(0,0)$ and diameter r_n. For $n = 1, 2, 3, \ldots, m-1$, let R_n be disjoint copies of $(0,1)$ that lie in $\mathbb{R}^2$, such that the end corresponding to 0 limits to C_n and the other end limits to C_{n+1}. Let $Y_n = \overline{R_n} = C_n \cup R_n \cup C_{n+1}$ and $Y_F = \bigcup_{n=0}^{m-1} Y_n$. (See Figure 1.)

We claim that $B(Y_F) = \{\sigma(S) : \text{S is a subcontinuum of } Y_F\} = F$. The proof is given below.

Consider any subcontinuum S of Y_F. Let $f : Y_F \to [0,1]$ be a function defined as in Proposition 3.1.1. Then $f(S) \subseteq I$ is compact and connected.

Case 1: $f(S)$ is a singleton and $f(S) \notin F$.
In this case, S is a singleton; thus, $\sigma(S) = 0$.

Case 2: $f(S)$ is a singleton and $f(S) \in F$.
In this case, $S \subseteq C_r$ for some $r \in F$. Hence, $\sigma(S) = \text{diam}\{C_r\}$ or $\sigma(S) = 0$.

Case 3: $f(S)$ is a closed interval and $f(S) \subset F^C$.

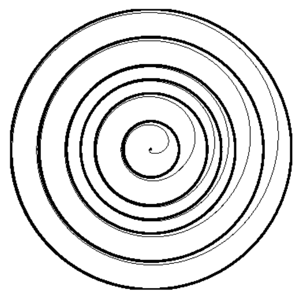

FIGURE 1. Continuum Y_F.

In this case, S is an arc. Hence, $\sigma(S) = 0$.

Case 4: $f(S)$ is a closed interval and $f(S) \cap F \neq \emptyset$.

In this case, S is a continuum consisting of circles and rays limiting to them. Let $L = f(S)$. $f\,|_S$ is a continuous function from S onto L. Therefore, in a similar manner as in the proof of Proposition 3.1.2, we obtain the following result:

If A is a subcontinuum of $S \times S$ such that $S(A) > 0$ and $\pi_1(A) = \pi_2(A)$, then $A \cap \left(\bigcup_{r \in F \cap L} C_r \times C_r \right) \neq \emptyset$.

Using this result, Proposition 3.1.3, and similar arguments as in the proof of Theorem 3.1.4, we conclude that $\sigma(S) = \max\{\text{diam}(C_r) : r \in F \cap L\}$.

In any case, $\sigma(S) \in F$. Hence, $B(Y) \subset F$. For any $m \in F$, there exists a circle $C_m \subset Y$ such that $\text{diam}(C_m) = m$. Therefore, $F \subset B(Y)$, and hence $F = B(Y)$. □

In [7], Z. Waraszkiewicz shows that there is an uncountable family of spirals that limit onto a circle which are pairwise incomparable in the sense that no one admits a continuous surjection to another. So we get the following result.

Corollary 3.1.7. *For any finite subset F of $[0,1]$ that contains 0 and 1, there exists an uncountable family $\mathcal{Y}_F$ of pairwise incomparable continua such that, for each member $Y \in \mathcal{Y}_F$, $B(Y) = F$.*

Before we go on to the general case, we present two more planar examples.

Theorem 3.1.8. *Let $d(x,y)$ be the Euclidean distance in $\mathbb{R}^2$ and let $Y_T = L_1 \cup L_2 \cup L_3 \cup U$, where*

- $L_1 = \{(x,0) : x \in [2,4]\}$
  $\cup \{\big((1+e^{-\theta})\cos(\theta), (1+e^{-\theta})\sin(\theta)\big) : \theta \in [0,\infty)\}$,
- $L_2 = \{(x,y) : y = -\sqrt{3}x : x \in [-2,-1]\}$
  $\cup \left\{\left((1+e^{-\theta})\cos(\theta + \frac{2\pi}{3}), (1+e^{-\theta})\sin(\theta + \frac{2\pi}{3})\right) : \theta \in [0,\infty)\right\}$,
- $L_3 = \{(x,y) : y = \sqrt{3}x : x \in [-2,-1]\}$
  $\cup \left\{\left((1+e^{-\theta})\cos(\theta + \frac{4\pi}{3}), (1+e^{-\theta})\sin(\theta + \frac{4\pi}{3})\right) : \theta \in [0,\infty)\right\}$

and

- *U is the unit circle.*

Then Y_T is a continuum and $B(Y_T) = \{0\} \cup [2, \sigma(Y_T)]$. Furthermore, for any given $0 < t_1 < t_2 < 0$, by adjusting the radius and arm lengths, we can construct a continuum Y such that $B(Y) = \{0\} \cup [t_1, t_2]$. (See Figure 2.)

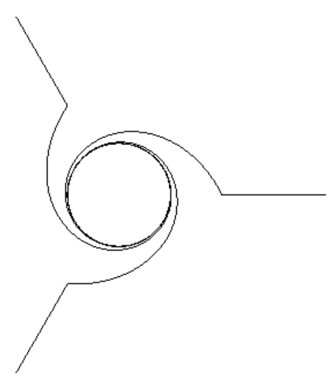

FIGURE 2. Continuum Y_T.

Proof. Let l_1, l_2, and l_3 be the endpoints of L_1, L_2, L_3, respectively. Consider the subcontinuum A of $Y_T \times Y_T$ described by $A = \bigcup_{n=1}^{6} A_n$, where

(1) $A_1 = \{l_1\} \times (L_2 \cup U \cup L_3)$,

(2) $A_2 = (L_1 \cup U \cup L_2) \times \{l_3\}$,
(3) $A_3 = \{l_2\} \times (L_1 \cup U \cup L_3)$,
(4) $A_4 = (L_2 \cup U \cup L_3) \times \{l_1\}$,
(5) $A_5 = \{l_3\} \times (L_1 \cup U \cup L_2)$, and
(6) $A_6 = (L_1 \cup U \cup L_3) \times \{l_2\}$.

It is clear that A is a connected subset and that $\pi_1(A) = \pi_2(A)$ and

$$\inf\{d(x,y) : (x,y) \in A_1\} = \min\{d(l_1, L_2 \cup U \cup L_3)\},$$
$$\inf\{d(x,y) : (x,y) \in A_2\} = \min\{d(l_3, L_1 \cup U \cup L_2)\},$$
$$\inf\{d(x,y) : (x,y) \in A_3\} = \min\{d(l_2, L_1 \cup U \cup L_3)\},$$
$$\inf\{d(x,y) : (x,y) \in A_4\} = \min\{d(l_1, L_2 \cup U \cup L_3)\},$$
$$\inf\{d(x,y) : (x,y) \in A_5\} = \min\{d(l_3, L_1 \cup U \cup L_2)\}, \text{ and}$$
$$\inf\{d(x,y) : (x,y) \in A_6\} = \min\{d(l_2, L_3 \cup U \cup L_1)\}.$$

From the above equations, it follows that

$$\inf\{d(x,y) : (x,y) \in A\} = \min\{d(l_i, L_{i+1} \cup U \cup L_{i+2} : i = 1,2,3\}.$$

Therefore,

$$\sigma(Y_T) \geq \inf\{d(x,y) : (x,y) \in A\} = \min\{d(l_i, L_{i+1} \cup U \cup L_{i+2} : i = 1,2,3\}.$$

The arms are extended far enough such that $\sigma(Y_T) > 2$.

Consider any subcontinuum S of Y_T.

Case 1: There exists n in $\{1,2,3\}$ such that $L_n \cap S = \emptyset$.

If S is a point or an arc, then $\sigma(S) = 0$. Otherwise, from results in previous sections, $\sigma(S) = 2$.

Case 2: $L_i \cap S \neq \emptyset$ for each $i \in \{1,2,3\}$.

In this case, $U \subset S$; therefore, $\sigma(S) \geq 2$. Hence, $B(Y_T) \subset \{0\} \cup [2, \sigma(Y_T)]$.

There is a deformation retract of the entire continuum Y_T which is the identity except on one arm, which it shortens. For any subcontinuum A of $Y_T \times Y_T$, this deformation retract induces a homotopy of A. Under this homotopy, $S(A)$ changes continuously. Thus, $\sigma(Y_T)$ changes continuously as the arm shrinks. Hence, for each value r in $[2, \sigma(Y_T)]$, there exists a subcontinuum S such that $\sigma((S) = r$. Therefore, $B(Y_T) = \{0\} \cup [2, \sigma(Y_T)]$. □

The example for the finite case can be modified to cover the case for the Cantor set.

Proposition 3.1.9. *Let K denote the Cantor set and $\mathcal{K} = \{r/2 : r \in K\}$. For each $r \in \mathcal{K}$, let C_r be the circle in $\mathbb{R}^2$ with center $(0,0)$ and radius r. Consider the disjoint collection of open intervals $\mathcal{G}$ such that*

$\bigcup_{I\in\mathcal{G}} I$ *is the complement of* $\mathcal{K}$ *in* $[0,1/2]$*. For each* $(a,b)\in\mathcal{G}$*, let* R_{ab} *be a copy of* $(0,1)$ *such that the end that corresponds to* 0 *limits to* C_a *and the other end limits to* C_b*, the* R_{ab}*'s are pairwise disjoint, and the* R_{ab}*'s are disjoint from the* C_r*'s. Let* $Y_{ab} = \overline{R_{ab}} = C_a \cup R_{ab} \cup C_b$ *and* $Y_K = \left(\bigcup_{(a,b)\in\mathcal{G}} Y_{ab}\right) \cup \left(\bigcup_{r\in\mathcal{K}} C_r\right)$*. (See Figure 3). Then* $B(Y_K) = \{\sigma(S) : S$ *is a subcontinuum of* $Y_K\} = K$*.*

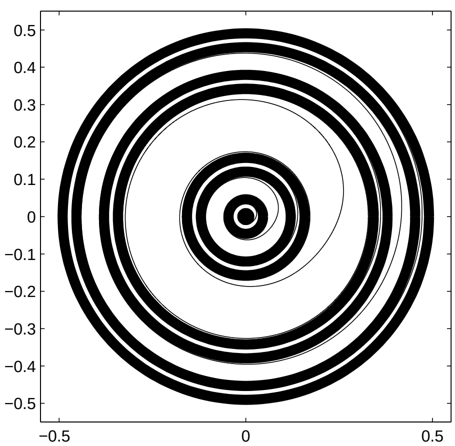

FIGURE 3. Continuum Y_K

Proof. There exists a continuous surjection $f : Y_K \to [0,1]$ such that

(1) $y \in C_r$ if and only if $f(y) = r$, and
(2) $y \in R_{ab}$ if and only if $f(y) = h_{ab}(y)$, where, for each $(a,b)\in\mathcal{G}$, $h_{ab} : R_{ab} \to (a,b)$ is a homeomorphism.

Hence, using similar arguments as in the case of a finite set, we obtain that $K = B(Y_K)$. □

As before, due to a result in [7], we get the following result.

Corollary 3.1.10. *Let* K *denote the Cantor set. There exists an uncountable family* $\mathcal{Y}_K$ *of pairwise incomparable continua such that, for each member* $Y \in \mathcal{Y}_K$*,* $B(Y) = K$*.*

The examples we have discussed up to now are planar, but for the general case we do not have a planar example. First, let us recall a well-known theorem in general topology.

Theorem 3.1.11. *Let M be a nonempty closed subset of $[0,1]$ and let K denote the Cantor set. There exists a continuous surjection $g : K \to M$.*

Using the above theorem, we modify the example for a Cantor set to cover the case of a closed set.

Theorem 3.1.12. *Let K denote the Cantor set and let M be a nonempty closed subset of $[0,1]$ which contains 0 and 1. Let $g : K \to M$ be a continuous surjection such that $g(0) = 0$. For each $r \in K$, let C_r be the circle parallel to the yz plane in $\mathbb{R}^3$ with center $(r,0,0)$ and diameter $g(r)$. Consider the disjoint collection of open intervals $\mathcal{G}$ such that $\bigcup_{I\in\mathcal{G}} I$ is the complement of K in $[0,1]$. For each $(a,b) \in \mathcal{G}$, let R_{ab} be a copy of $(0,1)$ such that the end that corresponds to 0 limits to C_a and the other end limits to C_b, the R_{ab}'s are pairwise disjoint, and the R_{ab}'s are disjoint from the C_r's. Let $Y_{ab} = \overline{R_{ab}} = C_a \cup R_{ab} \cup C_b$ and $Y_M = \left(\bigcup_{(a,b)\in\mathcal{G}} Y_{ab}\right) \cup \left(\bigcup_{r\in K} C_r\right)$. Then $B(Y_M) = \{\sigma(S) : S \text{ is a subcontinuum of } Y_M\} = M$.*

Proof. There exists a continuous surjection $f : Y_M \to [0,1]$ such that

(1) $y \in C_r$ if and only if $f(y) = r$, and
(2) $y \in R_{ab}$ if and only if $f(y) = h_{ab}(y)$, where, for each $(a,b) \in \mathcal{G}$, $h_{ab} : R_{ab} \to (a,b)$ is a homeomorphism.

Using similar arguments as in the case of a finite set, we obtain that $B(Y_M) = M$. □

As before, due to a result in [7], we get the following result.

Corollary 3.1.13. *Let M be a closed subset of $[0,1]$ containing 1 and 0. There exists an uncountable family $\mathcal{Y}_M$ of pairwise incomparable continua, such that, for each member $Y \in \mathcal{Y}_M$, $B(Y) = M$.*

Observe that continuum Y_M is not constructed in the plane.

4. Further Questions

The answers to the following questions could extend the results in this paper in many different directions.

We observed that the inequality $\sigma(Y) \leq \max(\sigma(X), \sigma_0^*(X))$ can be improved to $\sigma(Y) = \sigma(X)$ when $\sigma_0^*(X) \leq \sigma(X)$.

Question 4.1. Let X be a continuum and R be a ray limiting to it. Let $Y = X \cup R$. What conditions on X will guarantee that $\sigma(Y) = \sigma(X)$? One such condition is $\sigma(X) \leq \sigma_0^*(X)$. Is this condition necessary for $\sigma(Y) = \sigma(X)$?

All examples we constructed involve closed sets.

Question 4.2. What conditions on a set $G \subseteq [0, 1]$ will guarantee that there will be a continuum X such that $B(X_G) = G$? Must G be a closed set?

The examples which we constructed contain rays limiting to simple closed curves and hence are not arcwise connected.

Question 4.3. If X is an arcwise connected continuum, what closed sets can $B(X)$ be?

Except for the case of a closed set G with infinitely many nondegenerate components, we have constructed a planar example of a continuum X_G with $B(X_G) = G$.

Question 4.4. Does there exist a planar example for every closed set G?

The following results are straightforward.

- If X is a simple closed curve, $B(X) = \{0, \sigma(X)\}$.
- If X is a simple triod, $B(X) = [0, \sigma(X)]$.
- $0 \in B(X)$.

The above results are independent of the metric on X. When X is a simple closed curve or when X is a triod, if $Z \approx X$, then $B(Z) \approx B(X)$. For some continua X, the set $B(X)$ depends on the metric.

Question 4.5. For which continua X is the homeomorphism type of $B(X)$ a topological invariant of X?

Question 4.6. What properties of $B(X)$ are topological invariants of X?

Acknowledgment. Thanks to Dr. Wayne Lewis for his contribution to this paper and for the help, support, and guidance throughout my graduate studies.

References

[1] James Francis Davis, *The equivalence of zero span and zero semispan*, Proc. Amer. Math. Soc. **90** (1984), no. 1, 133–138.

[2] L. C. Hoehn, *A non-chainable plane continuum with span zero*, Fund. Math. **211** (2011), no. 2, 149–174.

[3] L. C. Hoehn and A. Karassev, *Equivalent metrics and the spans of graphs*, Colloq. Math. **114** (2009), no. 1, 135–153.

[4] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964) 199–214.

[5] ———, *On the surjective span and semispan of connected metric spaces*, Colloq. Math. **37** (1977), no. 1, 35–45.

[6] Andrew Lelek and Thelma West, *Spans of continua and their applications* in Continua. With the Houston Problem Book (Cincinnati, OH, 1994). Ed. Howard Cook, W. T. Ingram, K. T. Kuperberg, Andrew Lelek, and Piotr Minc. Lecture Notes in Pure and Applied Mathematics, 170. New York: Dekker, 1995. 127–132.

[7] Z. Waraszkiewicz, *Une famille indénombrable de continus plans dont aucun n'est l'image continue d'un autre*, Fund. Math. **18** (1932), no. 1, 118–137.

Department of Mathematics & Statistics; Texas Tech University; Lubbock, Texas 79409

E-mail address: nadeeka.de-silva@ttu.edu