
TOPOLOGY PROCEEDINGS



Volume 42, 2013

Pages 327–340

<http://topology.auburn.edu/tp/>

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Electronically published on February 9, 2013

Topology Proceedings

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ISSN: 0146-4124

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CONCERNING CHAINABILITY OF INVERSE LIMITS ON $[0, 1]$ WITH SET-VALUED FUNCTIONS

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ABSTRACT. Suppose that X_1, X_2, X_3, \dots is a sequence of continua and $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous set-valued function for each positive integer i ; let $G'_n = \{\mathbf{x} \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$. We show that if G'_n is a chainable continuum for each $n \in \mathbb{N}$, then the inverse limit $\varprojlim \mathbf{f}$ is chainable. We use this to show that two well-studied examples of inverse limits on $[0, 1]$ with set-valued functions, along with two companion examples (the inverse limit with the inverses of those functions), are chainable. In the process we prove a union theorem for chainable continua, specifically, if A and B are chainable continua such that $A \cap B$ is a terminal C-set in each of A and B , then $A \cup B$ is chainable.

1. INTRODUCTION

In two recent papers (see [5] and [6]), the author considered conditions under which an inverse limit with a single set-valued bonding function on $[0, 1]$ is tree-like. These results suggest posing the following problem.

Problem 1.1. *Find sufficient conditions on an upper semi-continuous function $f : [0, 1] \rightarrow 2^{[0,1]}$ so that $\varprojlim \mathbf{f}$ is a chainable continuum.*

Pertinent to this problem is a result of M. M. Marsh [9, Corollary 6] that if $f : X \rightarrow 2^X$ is an upper semi-continuous function on a continuum X and there is a mapping $g : X \rightarrow X$ such that $g^{-1} \subseteq G(f)$, then $\varprojlim \mathbf{f}$ contains a homeomorphic copy of $G(f)$. Thus, a function having a graph that contains the inverse of a mapping cannot contain a triod or a simple closed curve in its graph and produce a chainable inverse limit.

2010 *Mathematics Subject Classification.* Primary 54F15, 54H20.

Key words and phrases. atriodic, chainable, inverse limit, set-valued function.

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On the other hand, there are examples of functions having a graph that is an arc but for which the inverse limit contains a triod [4, Example 2.3 and Example 3.11]. The functions in both of these examples have graphs containing the inverse of a mapping of $[0, 1]$. Examples of arcs that are inverse limits with a single set-valued function that is not a mapping include Example 2.2 and Example 2.6 of [4]. However, the author knows of no previously published example of a chainable continuum other than an arc that can be obtained as an inverse limit with a single set-valued function that is not a mapping. In section 5 of this paper, we provide such examples thus showing that Problem 1.1 is a reasonable problem to consider. Constructing our proof that there are such examples led us to what we believe to be a new theorem about the chainability of unions of two chainable continua (see Theorem 3.2).

2. DEFINITIONS AND SOME BACKGROUND THEOREMS

By a *compactum* we mean a compact metric space. If X is a compactum we denote the collection of closed subsets of X by 2^X ; $C(X)$ denotes the connected elements of 2^X . If each of X and Y is a compactum, a function $f : X \rightarrow 2^Y$ is said to be *upper semi-continuous at the point x of X* provided that if V is an open subset of Y that contains $f(x)$, then there is an open subset U of X containing x such that if t is a point of U , then $f(t) \subseteq V$. A function $f : X \rightarrow 2^Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of X . If $f : X \rightarrow 2^Y$ is a set-valued function, by the *graph* of f , denoted $G(f)$, we mean $\{(x, y) \in X \times Y \mid y \in f(x)\}$. It is known that if M is a subset of $X \times Y$ such that X is the projection of M to its set of first coordinates, then M is closed if and only if M is the graph of an upper semi-continuous function [7, Theorem 2.1]. In the case that f is upper semi-continuous and single-valued, i.e., $f(t)$ is degenerate for each $t \in X$, f is a continuous function. We call a continuous function a *mapping*; if $f : X \rightarrow Y$ is a surjective mapping (i.e., for each $y \in Y$ there is a point $x \in X$ such that $y = f(x)$), we denote that f is surjective by $f : X \twoheadrightarrow Y$. If X and Y are compacta, $Y \subseteq X$, and $r : X \rightarrow Y$ is a mapping, then r is called a *retraction* provided $f(x) = x$ for each $x \in Y$.

We denote by \mathbb{N} the set of positive integers. If $\mathbf{s} = s_1, s_2, s_3, \dots$ is a sequence, we normally denote the sequence in boldface type and its terms in italics. Suppose \mathbf{X} is a sequence of compacta each having diameter bounded by 1 and $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function for each $n \in \mathbb{N}$. By the *inverse limit* of \mathbf{f} , denoted $\varprojlim \mathbf{f}$, we mean $\{\mathbf{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1}) \text{ for each positive integer } i\}$; we call the pair $\{\mathbf{X}, \mathbf{f}\}$ an inverse limit sequence. Inverse limits on compacta with

upper semi-continuous bonding functions are nonempty and compact [7, Theorem 3.2]; they are metric spaces being subsets of the metric space $\prod_{i>0} X_i$. Because every metric space has an equivalent metric that is bounded by 1, we assume throughout that all of our spaces have metrics bounded by 1. This allows us to use the metric d on this product given by $d(\mathbf{x}, \mathbf{y}) = \sum_{i>0} d_i(x_i, y_i)/2^i$. In the case that each f_n is a mapping, the definition of the inverse limit reduces to the usual definition of an inverse limit on compacta with mappings. If $A \subseteq \mathbb{N}$, we denote by p_A the projection of $\prod_{n>0} X_n$ onto $\prod_{n \in A} X_n$ given by $p_A(\mathbf{x}) = \mathbf{y}$ provided $y_i = x_i$ for each $i \in A$. If $A = \{n\}$, $p_{\{n\}}$ is normally denoted p_n . In the case that $A \subseteq B \subseteq \mathbb{N}$, we normally also denote the restriction of p_A to $\prod_{n \in B} X_n$ by p_A , inferring by context that we are using this restriction. We denote the projection from the inverse limit into the i th factor space by π_i , and, more generally, for $A \subseteq \mathbb{N}$, we denote by π_A the restriction of p_A to the inverse limit. If $\{\mathbf{X}, \mathbf{f}\}$ is an inverse limit sequence and $M = \varprojlim \mathbf{f}$, we say that M has the *full projection property* provided that if H is a subcontinuum of M such that $\pi_i(H) = X_i$ for infinitely many integers i , then $H = M$.

A set traditionally used in the proof that $\varprojlim \mathbf{f}$ is nonempty and compact is $\{\mathbf{x} \in \prod_{k>0} X_k \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$. Because this set was originally denoted G_n , we adopt and use throughout this article the notation $G'_n = \{\mathbf{x} \in \prod_{k=1}^{n+1} X_k \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$. Note that for $A = \{1, 2, \dots, n+1\}$, $G'_n = \pi_A(G_n)$.

A *continuum* is a compact, connected subset of a metric space. A continuum M is said to be *chainable* provided that, for each $\varepsilon > 0$, there is a finite collection $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ of open sets covering M such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$ and $\text{diam}(C_i) < \varepsilon$ for $1 \leq i \leq n$. A *triod* is a continuum M containing a subcontinuum K (called the *core* of the triod) such that $M - K$ has at least three components. A continuum is *atriodic* provided it does not contain a triod. A subset K of a continuum M is a *C-set* in M provided it is true that if H is a subcontinuum of M containing a point of K and a point of $M - K$, then $K \subseteq H$. For additional information on C-sets, see [2]. A subcontinuum C of a continuum M is said to be *terminal* in M provided if H and K are subcontinua of M each intersecting C , then $H \subseteq K \cup C$ or $K \subseteq H \cup C$. If A is a set, $\text{Cl}(A)$ denotes the closure of A .

If Y_1, Y_2, Y_3, \dots is a sequence of compact subsets of a compact metric space Y and, for each $i \in \mathbb{N}$, $\psi_i : Y \rightarrow Y_i$ is a mapping of Y onto Y_i , then $\psi_1, \psi_2, \psi_3, \dots$ is said to *converge uniformly* to the mapping $\psi : Y \rightarrow Y$ provided it is true that if $\varepsilon > 0$, then there is a positive integer N such that $d(\psi(x), \psi_n(x)) < \varepsilon$ for each $n \geq N$ and $x \in Y$. In a couple of our

examples we make use of the following theorem of Fort and Segal as found in [10, p. 30].

Theorem 2.1. *Suppose Y is a compact metric space and Y_1, Y_2, Y_3, \dots is a sequence of compact subsets of Y such that, for each $i \in \mathbb{N}$, there exist surjective mappings $g_i : Y_{i+1} \twoheadrightarrow Y_i$ and $\psi_i : Y \twoheadrightarrow Y_i$ such that $\psi_i = g_i \circ \psi_{i+1}$. If ψ converges uniformly to the identity on Y , then Y is homeomorphic to $\varprojlim g$.*

Proof. It is not difficult to show that $h : Y \rightarrow \varprojlim g$ given by $h(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \dots)$ is a surjective homeomorphism. We leave the details to the reader (as does Sam B. Nadler, Jr., in [10, p. 30]). \square

If $f : X \rightarrow Y$ is a mapping and $\varepsilon > 0$, then f is called an ε -map provided that, for each point $x \in Y$, $\text{diam}(f^{-1}(x)) < \varepsilon$. There are two well-known characterizations of chainability. A proof of Theorem 2.2 may be found in [8, Theorem 96 and Theorem 102]; the second theorem is a consequence of Theorem 175 and Theorem 176 of [8].

Theorem 2.2. *A continuum is chainable if and only if it is homeomorphic to an inverse limit on arcs with mappings.*

Theorem 2.3. *A continuum M is chainable if and only if, for each $\varepsilon > 0$, there is an ε -mapping f of M onto an arc.*

3. ATRIODICITY AND UNIONS OF CHAINABLE CONTINUA

In [1] it is shown that if A and B are chainable continua with a point in common, then $A \cup B$ is chainable if and only if $A \cup B$ is atriodic and $A \cap B$ is connected. In this section we show that a continuum is chainable if it is the union of two chainable continua whose intersection is a terminal continuum and a C-set in each of them. We begin with examples that show that even if A and B are arcs, neither $A \cap B$ being terminal nor $A \cap B$ being a C-set alone is sufficient for the union to be atriodic.

Let T be the unit triod in the plane. Specifically, let A_1 be the arc on the x -axis in the plane having endpoints $(0, 0)$ and $(1, 0)$; let A_2 be the arc on the y -axis having endpoints $(0, 0)$ and $(0, 1)$; let A_3 be the arc on the x -axis having endpoints $(0, 0)$ and $(-1, 0)$; let $T = A_1 \cup A_2 \cup A_3$. Although T is a triod, the set $\{(0, 0)\}$ is a C-set in A_1 and in the arc $A_2 \cup A_3$; the arc A_1 is terminal in $A_1 \cup A_2$ and in $A_1 \cup A_3$. Thus, the union of two atriodic continua can fail to be atriodic when their common part is either a C-set in both or a terminal subcontinuum of both. However, when the common part of two atriodic continua is a continuum that is both terminal and a C-set in each, their union is atriodic. This is our first theorem of this section.

Theorem 3.1. *If A and B are atriodic continua and $A \cap B$ is a continuum that is terminal and a C -set in both A and B , then $A \cup B$ is atriodic.*

Proof. Suppose $J_1 \cup J_2 \cup J_3$ is a triod with core K lying in $A \cup B$. We first show that K is not a subset of $A - (A \cap B)$. If $K \subseteq A - (A \cap B)$, then letting L_i denote the closure of the component of $J_i \cap (A - (A \cap B))$ containing K for $i = 1, 2, 3$, we see that each L_i contains a point not in L_j for $i \neq j$. Thus, $L_1 \cup L_2 \cup L_3$ is a triod lying in A . Similarly, K is not a subset of $B - (A \cap B)$ so $K \cap (A \cap B) \neq \emptyset$.

Because $J_1 \cup J_2 \cup J_3$ is not a subset of either A or B , it follows that one of J_1, J_2 , and J_3 contains a point of $A - (A \cap B)$ or one of them contains a point of $B - (A \cap B)$. Suppose J_1 contains a point x of $A - (A \cap B)$. Denote by L the component of $J_1 \cap (A - (A \cap B))$ containing x . Then L has a limit point in $A \cap B$ [8, Theorem 276]. Because $A \cap B$ is a C -set in A , $A \cap B \subseteq \text{Cl}(L)$. Thus, $A \cap B \subseteq J_1$, so neither J_2 nor J_3 is a subset of $A \cap B$. It now follows in a similar manner that $A \cap B \subseteq J_2$ and $A \cap B \subseteq J_3$. Therefore, $A \cap B \subseteq K$.

Let p_i be a point of $J_i - K$ for $i = 1, 2, 3$. If $\{p_1, p_2, p_3\} \subseteq A$, then, by letting L_i denote the closure of the component of $J_i \cap (A - (A \cap B))$ that contains p_i for $i = 1, 2, 3$, we have that L_1, L_2, L_3 are three subcontinua of A with a common point ($A \cap B$ is a subset of each L_i) such that no one of them is a subset of the union of the other two. By a theorem of Sorgenfrey [11, Theorem 1.8], $L_1 \cup L_2 \cup L_3$ contains a triod. However, this involves a contradiction because $L_1 \cup L_2 \cup L_3 \subseteq A$. Similarly, $\{p_1, p_2, p_3\}$ is not a subset of B . Suppose $p_1 \in A$ and $p_2 \in B$. If $p_3 \in A$, then, because $A \cap B$ is terminal in A , $L_1 \subseteq L_3$ or $L_3 \subseteq L_1$. If $L_1 \subseteq L_3$, then $p_1 \in J_3$ while if $L_3 \subseteq L_1$, then $p_3 \in J_1$. Either possibility involves a contradiction. A similar contradiction follows if $p_3 \in B$. \square

Theorem 3.2. *If A and B are chainable continua and $A \cap B$ is a continuum that is terminal and a C -set in both A and B , then $A \cup B$ is chainable. Moreover, $A \cap B$ is a C -set in $A \cup B$.*

Proof. By Theorem 3.1, $A \cup B$ is atriodic. By [1, Theorem 2, p. 196], $A \cup B$ is chainable.

Suppose H is a subcontinuum of $A \cup B$ that contains a point of $A \cap B$ and a point not in $A \cap B$. Assume H contains a point of $A - (A \cap B)$. Then $H \cap A$ is a subcontinuum of A that contains a point of $A \cap B$ and a point not in $A \cap B$. It follows that H contains $A \cap B$. If H contains a point of $B - (A \cap B)$, it follows similarly that H contains $A \cap B$. \square

A C -set in a subcontinuum of a continuum M need not be a C -set in M . For example, if one attaches an arc to the limit bar of a $\sin(1/x)$ -curve

such that the arc intersects the $\sin(1/x)$ -curve in only one point, then, although the limit bar is a C-set in the $\sin(1/x)$ -curve, it is not a C-set in the union. However, the following theorem gives a condition under which a C-set in a subcontinuum is a C-set in the continuum.

Theorem 3.3. *Suppose H is a subcontinuum of the continuum M and C is a C-set in H . If there is an open subset U of M such that $C \subseteq U \subseteq H$, then C is a C-set in M .*

Proof. Suppose K is a subcontinuum of M that contains a point of C and a point not in C . We may assume that K is not a subset of H . If L is a component of $K \cap U$ that intersects C , then L has a limit point in the boundary of U [8, Theorem 276]. Thus, $J = \text{Cl}(L)$ is a subcontinuum of H that contains a point of C and a point not in C . It follows that $C \subseteq J$, but $J \subseteq K$, so $C \subseteq K$. \square

4. PROPERTIES OF THE INVERSE LIMIT DERIVING FROM THE SETS G'_n

Suppose \mathbf{X} is a sequence of compacta and, for each positive integer n , $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function. Because the inverse limit is an intersection of the nested sequence of approximations $G_n = \{\mathbf{x} \in \prod_{i>0} X_i \mid x_j \in f_j(x_{j+1}) \text{ for } 1 \leq j \leq n\}$, many of its properties derive from properties of these sets. For example, an inverse limit on compacta is compact if and only if G_n is compact for each $n \in \mathbb{N}$; an inverse limit on continua is a continuum if and only if G_n is a continuum for each $n \in \mathbb{N}$ [8]. Studies of these approximations can normally be reduced to finite dimensional sets $G'_n = \pi_A(G_n)$ for $A = \{1, 2, \dots, n+1\}$. These sets have also proved useful in proofs of other properties of inverse limits, e.g., if $\dim(G'_n) \leq m$ for each positive integer n , then the dimension of the inverse limit cannot exceed m [8]. Here we show that chainability (tree-likeness, respectively) of the inverse limit derives from the chainability (tree-likeness, respectively) of each of the sets G'_n (Corollary 4.3 (4.4, respectively) below). Lemmas 4.1 and 4.2 have these consequences along with other ramifications. A problem with using these lemmas and other theorems that rely on properties of either G'_n or of G_n often lies in dealing with the dimension of the spaces in which these sets reside. Even in the case that every factor space is the interval $[0, 1]$, G'_n is a subset of $[0, 1]^{n+1}$ while G_n is an infinite dimensional subset of the Hilbert cube. In the next section, we make use of the consequences of these lemmas to obtain the chainability of certain inverse limits on $[0, 1]$ heretofore not shown to be chainable. However, we do not provide conditions on the bonding functions that ensure chainability, as asked in Problem 1.1.

Because chainability and tree-likeness are defined in terms of covers and can be characterized in terms of ε -maps, Corollary 4.3 and Corollary 4.4 follow directly from either Lemma 4.1 or Lemma 4.2 below.

Lemma 4.1. *Suppose \mathbf{X} is a sequence of compacta and $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function for each positive integer n . Then, for $A = \{1, 2, \dots, n+1\}$, π_A is a $1/2^{n+1}$ -map of $\varprojlim \mathbf{f}$ into G'_n .*

Lemma 4.2. *Suppose \mathbf{X} is a sequence of compacta and $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function for each positive integer n . Suppose further that \mathcal{U} is a collection of open sets in $\prod_{i=1}^{n+1} X_i$ covering G'_n such that if $u \in \mathcal{U}$ then $\text{diam}(u) < 1/2^{n+1}$. Then there is a collection \mathcal{V} of open subsets of $\prod_{i>0} X_i$ covering $\varprojlim \mathbf{f}$ and a 1-1 function $\sigma : \mathcal{U} \rightarrow \mathcal{V}$ such that (1) if $v \in \mathcal{V}$, then $\text{diam}(v) < 1/2^n$ and (2) for u_1 and u_2 in \mathcal{U} , $u_1 \cap u_2 \neq \emptyset$ if and only if $\sigma(u_1) \cap \sigma(u_2) \neq \emptyset$.*

Proof. For $u \in \mathcal{U}$, define $\sigma(u)$ to be $u \times (\prod_{i>n+1} X_i)$ and let $\mathcal{V} = \sigma(\mathcal{U})$. The two conclusions of the lemma follow almost immediately. \square

Corollaries 4.3 and 4.4 are immediate consequences of either of these two lemmas. Although we do not make use of Corollary 4.5 elsewhere in this article, we believe it to be of interest. Its proof follows from Lemma 4.2 using techniques already in the literature [8, Section 2.12].

Corollary 4.3. *Suppose \mathbf{X} is a sequence of continua and $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function for each positive integer n . If G'_n is a chainable continuum for each positive integer n , then $\varprojlim \mathbf{f}$ is a chainable continuum.*

Corollary 4.4. *Suppose \mathbf{X} is a sequence of continua and $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function for each positive integer n . If G'_n is a tree-like continuum for each positive integer n , then $\varprojlim \mathbf{f}$ is a tree-like continuum.*

Corollary 4.5. *Suppose \mathbf{X} is a sequence of compacta and $f_n : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function for each positive integer n . If $m \in \mathbb{N}$ and $\dim(G'_n) \leq m$ for each positive integer n , then $\dim(\varprojlim \mathbf{f}) \leq m$.*

In the case that the graph of an upper semi-continuous function $f : [0, 1] \rightarrow 2^{[0,1]}$ contains the inverse of a mapping of $[0, 1]$ into itself and each G'_n is chainable (tree-like, respectively), a theorem of Marsh [9, Corollary 7] can be used to construct an “inverse limit” proof that $\varprojlim \mathbf{f}$ is chainable (tree-like, respectively). In fact, our first proof that Example 5.1 and Example 5.4 in the next section are chainable continua was accomplished in this manner. We do not know of such a proof that Example 5.3 and

Example 5.5 below are chainable. (In each of those examples the graph of the bonding function does not contain the inverse of a mapping of $[0, 1]$.) Corollaries 4.3 and 4.4 do not impose restrictions on the nature of an upper semi-continuous bonding function.

5. EXAMPLES

Our main results on inverse limits with set-valued functions in this paper are in the form of examples. Example 5.1 and Example 5.4 below have been shown to be indecomposable continua [4, Example 3.7 and Example 3.9]. Moreover, both continua have recently been shown to be tree-like [5]. Our contribution here is in showing that both inverse limits are chainable continua. While discussing our results on these two examples with Van Nall, he asked whether the inverses of the functions in these two examples also produce chainable continua. It so happens that both do so. Our other two examples here involve the inverse limits with these inverse functions. We appreciate his asking this question.

Example 5.1. Let $f : [0, 1] \rightarrow C([0, 1])$ be the upper semi-continuous function whose graph consists of three straight line intervals lying in $[0, 1]^2$: one from $(0, 0)$ to $(1/2, 1)$, one from $(1/2, 1)$ to $(1/2, 0)$, and one from $(1/2, 0)$ to $(1, 1)$. Then $\varprojlim f$ is an indecomposable chainable continuum such that every nondegenerate proper subcontinuum is an arc (see the graph on the left in Figure 1 for the graph of f).

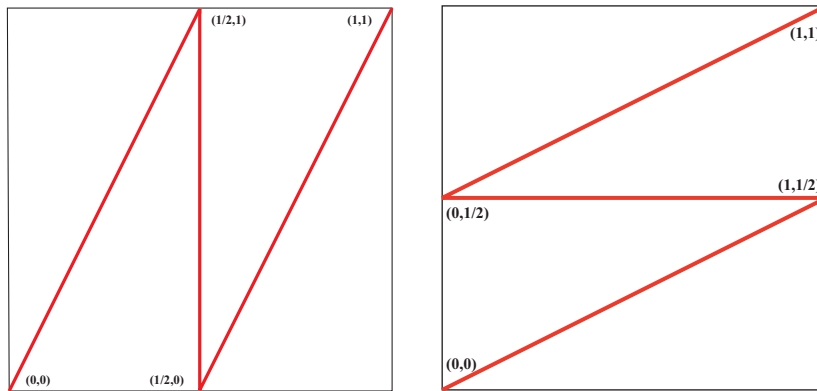


FIGURE 1. The graphs of the bonding functions in Example 5.1 (left) and Example 5.3 (right)

Proof. Let $M = \varprojlim f$. It is well known that an upper semi-continuous function on $[0, 1]$ having connected values produces a continuum in the inverse limit. That M is indecomposable is shown in [4, Example 3.9] (original source is [3]). In view of Corollary 4.3, to show that M is chainable it is sufficient to show that G'_n is chainable for each $n \in \mathbb{N}$. In [4, Example 3.6], it is shown that each G'_n is an arc. Briefly, this is accomplished by letting g_1, g_2 , and g_3 be the three mappings of $[0, 1]$ into $[0, 1]$ whose union is $G(f^{-1})$, i.e., let $g_1(t) = t/2$, $g_2(t) = 1/2$, and $g_3(t) = (t + 1)/2$ for $t \in [0, 1]$. Observe that $G'_1 = G(f^{-1})$ is an arc. Proceeding by induction, suppose k is an integer such that G'_k is an arc. Let $A_j = \psi_j(G'_k)$ for $j \in \{1, 2, 3\}$ where $\psi_j : G'_k \rightarrow G'_{k+1}$ is the homeomorphism given by $\psi_j(\mathbf{x}) = (x_1, \dots, x_k, g_j(x_k))$. Then A_1, A_2 , and A_3 are arcs whose union is G'_{k+1} . Note that $A_1 \cap A_2 = \{(1, 1, \dots, 1, 1/2)\}$ and $(1, 1, \dots, 1, 1/2)$ is a common endpoint of A_1 and A_2 , while $A_2 \cap A_3 = \{(0, 0, \dots, 0, 1/2)\}$ and $(0, 0, \dots, 0, 1/2)$ is a common endpoint of A_2 and A_3 . Because $A_1 \cap A_3 = \emptyset$, it follows that G'_{k+1} is an arc.

Suppose H is a nondegenerate proper subcontinuum of M . To see that H is an arc, we use the fact that if K is a proper subcontinuum of M , then there is a positive integer n such that if $j \geq n$, then $\pi_j(K) \neq [0, 1]$ (i.e., M has the full projection property, established in both [3] and [4]). If $k \in \mathbb{N}$ and $\pi_k(H) = \{1/2\}$, then $\pi_j(H)$ is a singleton for each $j \geq k$ and H is homeomorphic to G'_{k-1} under the homeomorphisms $\varphi : H \rightarrow G'_{k-1}$ given by $\varphi(\mathbf{x}) = (x_1, x_2, \dots, x_k)$. Thus, in this case, H is an arc. Suppose $\pi_i(H) \neq \{1/2\}$ for each $i \in \mathbb{N}$. Let n be a positive integer such that if $j \geq n$, then $\pi_j(H) \neq [0, 1]$. Suppose that $1/2 \in \pi_i(H)$ for infinitely many integers $i \geq n$. Observe that if $0 \in \pi_i(H)$ ($1 \in \pi_i(H)$, respectively) for infinitely many integers i , then $0 \in \pi_i(H)$ ($1 \in \pi_i(H)$, respectively) for all i . Suppose $k \geq n + 1$ and $1/2 \in \pi_k(H)$. There is an integer $j > k$ such that $1/2 \in \pi_j(H)$. Then, because $\pi_j(H) \neq \{1/2\}$, there is a point $s \in \pi_j(H)$ such that $s \neq 1/2$. If $s < 1/2$, then $1 \in \pi_{j-1}(H)$, and thus, $[1/2, 1] \subseteq \pi_k(H)$. It follows that $\pi_{k-1}(H) = [0, 1]$; this involves a contradiction. A similar contradiction occurs in the case that $s > 1/2$. Therefore, $1/2 \notin \pi_k(H)$ for $k \geq n + 1$ and we obtain that $f|(\pi_k(H))$ is a homeomorphism for $k > n + 1$. Define $h : H \rightarrow G'_{n+1}$ by $h(\mathbf{x}) = (x_1, x_2, \dots, x_{n+2})$. Suppose $\mathbf{x}, \mathbf{y} \in H$ and $\mathbf{x} \neq \mathbf{y}$. There is a positive integer j such that $x_j \neq y_j$. If $j \leq n + 1$, then $h(\mathbf{x}) \neq h(\mathbf{y})$. If $j > n + 1$, then $x_{n+1} \neq y_{n+1}$, and again $h(\mathbf{x}) \neq h(\mathbf{y})$. Thus, h is a homeomorphism of H onto a subcontinuum of G'_{n+1} , so H is an arc, being homeomorphic to a subcontinuum of the arc G'_{n+1} . \square

Remark 5.2. Because the continuum in Example 5.1 has two endpoints, $(0, 0, 0, \dots)$ and $(1, 1, 1, \dots)$, and every proper subcontinuum is an arc,

its properties are reminiscent of a two endpoint Knaster continuum, $\varprojlim g$ where g is the piecewise linear mapping whose graph passes through $(0, 0), (1/3, 1), (2/3, 0)$, and $(1, 1)$. It would be interesting to know whether the two continua are homeomorphic.

Our next example is the inverse limit with the inverse of the function in the previous example. This proof is similar to the proof for Example 5.1 so we offer only an outline of the proof that highlights the differences between the proofs.

Example 5.3. *Let f be the inverse of the function in Example 5.1. Then $\varprojlim f$ is a chainable continuum (see the graph on the right in Figure 1 for the graph of f).*

Proof. Let $M = \varprojlim f$. That M is a continuum follows because $f^{-1} : [0, 1] \rightarrow C([0, 1])$ [4, Theorem 2.8]. (This is also a consequence of the fact shown below that G'_n is an arc for each $n \in \mathbb{N}$.) As in the previous example, we show that G'_n is an arc for each $n \in \mathbb{N}$ and appeal to Corollary 4.3. Clearly, G'_1 is an arc. Inductively, suppose k is a positive integer such that G'_k is an arc. Note that the three mappings g_1, g_2 , and g_3 from the previous proof have the property that $G(f) = g_1 \cup g_2 \cup g_3$. Define $\Phi_i : G'_k \rightarrow G'_{k+1}$ by $\Phi(\mathbf{x}) = (g_i(x_1), x_1, x_2, \dots, x_{k+1})$ for $i = 1, 2, 3$. For $i = 1, 2, 3$, Φ_i is a homeomorphism; let $L_i = \Phi_i(G'_k)$. Each L_i is an arc, $L_1 \cap L_3 = \emptyset$, and $L_1 \cap L_2 = \{(1/2, 1, 1, \dots, 1)\}$, while $L_2 \cap L_3 = \{(1/2, 0, 0, \dots, 0)\}$. It follows that G'_{k+1} is an arc. \square

Our next example was first studied by Scott Varagona [12] where he showed that the inverse limit is an indecomposable continuum. It is also included as Example 3.5 in [4]. We now show that it is a chainable continuum. This proof motivated much of this paper.

Example 5.4. *Let $f : [0, 1] \rightarrow C([0, 1])$ be the function whose graph is the union of straight line intervals joining $(1/2^n, 0)$ and $(1/2^{n-1}, 1)$ for all odd positive integers, straight line intervals joining $(1/2^{n-1}, 0)$ and $(1/2^n, 1)$ for all even positive integers, and the straight line interval joining $(0, 0)$ and $(0, 1)$ (a function whose graph is homeomorphic to a $\sin(1/x)$ -curve). Then $\varprojlim f$ is an indecomposable chainable continuum (see the graph on the left in Figure 2 for the graph of f).*

Proof. Let $M = \varprojlim f$. As we remarked earlier, it is known that M is an indecomposable continuum. Here we show that it is a chainable continuum.

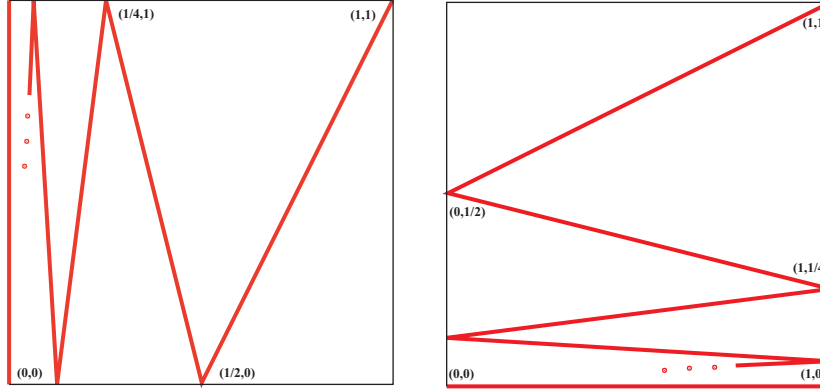


FIGURE 2. The graphs of the bonding functions in Example 5.4 (left) and Example 5.5 (right)

First, we introduce some notation. Let $\varphi_0 : [0, 1] \rightarrow [0, 1]$ be the mapping given by $\varphi_0(t) = 0$ for each $t \in [0, 1]$ and, for each $i \in \mathbb{N}$, let $\varphi_i : [0, 1] \rightarrow [1/2^i, 1/2^{i-1}]$ be the homeomorphism given by $\varphi_i(t) = (t + 1)/2^i$ if i is odd and $\varphi_i(t) = (2 - t)/2^i$ if i is even. Note that $G(f^{-1}) = \varphi_0 \cup \varphi_1 \cup \varphi_2 \cup \dots$.

We now show that, for each positive integer n , G'_n is a chainable continuum such that $\{\mathbf{x} \in G'_n \mid x_{n+1} = t\}$ is a C-set in G'_n for each $t \in [0, 1]$, while $\{\mathbf{x} \in G'_n \mid x_{n+1} = 0\}$ and $\{\mathbf{x} \in G'_n \mid x_{n+1} = 1\}$ are terminal subcontinua of G'_n . We proceed inductively.

The statement is clearly true for $n = 1$. Assume it holds for $n = k$. For each nonnegative integer i , define $\Phi_i : G'_k \rightarrow G'_{k+1}$ by $\Phi_i(\mathbf{x}) = (x_1, x_2, \dots, x_{k+1}, \varphi_i(x_{k+1}))$; let $L_i = \Phi_i(G'_k)$. Observe that Φ_i is a homeomorphism for each $i \geq 0$ so each L_i is a chainable continuum. Moreover, $G'_{k+1} = \bigcup_{i \geq 0} L_i$. To complete the inductive step, we must show that

- (1) $\{\mathbf{x} \in G'_{k+1} \mid x_{k+2} = t\}$ is a C-set in G'_{k+1} for each $t \in [0, 1]$,
- (2) $\{\mathbf{x} \in G'_{k+1} \mid x_{k+2} = 0\}$ and $\{\mathbf{x} \in G'_{k+1} \mid x_{k+2} = 1\}$ are terminal in G'_{k+1} , and
- (3) G_{k+1} is chainable.

Let $C_t = \{\mathbf{x} \in G'_{k+1} \mid x_{k+2} = t\}$. We begin by showing (1) holds. We consider cases $t = 1$, $0 < t < 1$, and $t = 0$. For $t = 1$, $C_t = \{(1, 1, \dots, 1)\}$, a C-set in G'_{k+1} because it is degenerate. Choose t , $0 < t < 1$; let j be a positive integer such that $1/2^j \leq t < 1/2^{j-1}$. Because $j > 0$, φ_j is a homeomorphism; thus, $\varphi_j^{-1}(t)$ is a singleton. Observe that $C_t = \{\mathbf{x} \in G'_{k+1} \mid x_{k+1} = \varphi_j^{-1}(t)\} = \Phi_j(\{\mathbf{x} \in G'_k \mid x_{k+1} = \varphi_j^{-1}(t)\})$. Because Φ_j

is a homeomorphism and $\{\mathbf{x} \in G'_k \mid x_{k+1} = \varphi_j^{-1}(t)\}$ is a C-set in G'_k , it follows that C_t is a C-set in L_j . If $t \neq 1/2^j$, let J be the open interval $(1/2^j, 1/2^{j-1})$. Then $U = G'_{k+1} \cap p_{k+2}^{-1}(J)$ is an open subset of G'_{k+1} containing C_t and lying in L_j , so C_t is a C-set in G'_{k+1} by Theorem 3.3. For $t = 1/2^j$, $\varphi_j^{-1}(t) \in \{0, 1\}$, so C_t is a terminal C-set in L_j and in L_{j+1} . Because $C_t = L_j \cap L_{j+1}$, by Theorem 3.2, $L_j \cup L_{j+1}$ is chainable and C_t is a C-set in $L_j \cup L_{j+1}$. Letting J denote the open interval $(1/2^{j+1}, 1/2^{j-1})$, $U = p_{k+2}^{-1}(J) \cap G'_{k+1}$ is an open set in G'_{k+1} containing C_t and lying in $L_j \cup L_{j+1}$ so C_t is a C-set in G'_{k+1} by Theorem 3.3. For $t = 0$, consider a subcontinuum H of G'_{k+1} that contains a point of C_t and a point not in C_t . There is a positive integer N such that if $i \geq N$, then $L_i \subseteq H$. It follows that H contains C_t , so C_t is a C-set in G'_{k+1} .

To show that (2) holds, first consider the case $t = 1$. To see that $\{(1, 1, \dots, 1)\}$ is terminal in G'_{k+1} , suppose H and K are subcontinua of G'_{k+1} containing $(1, 1, \dots, 1)$. There exist $a, b \in [0, 1]$ such that $p_{k+2}(H) = [a, 1]$ and $p_{k+2}(K) = [b, 1]$. We may assume that $a \leq b$ and $b < 1$; let s be a number such that $b \leq s < 1$. Then H and K are subcontinua of G'_{k+1} each containing a point of $C_s = \{\mathbf{x} \in G'_{k+1} \mid x_{k+2} = s\}$ and a point not in C_s . However, because C_s is a C-set in G'_{k+1} , it is a subset of $H \cap K$. It follows that $K \subseteq H$. Thus, $\{(1, 1, \dots, 1)\}$ is terminal in G'_{k+1} .

To complete showing that (2) holds, consider the case $t = 0$. Suppose H and K are subcontinua of G'_{k+1} and each contains a point of $C_0 = \{\mathbf{x} \in G'_{k+1} \mid x_{k+2} = 0\}$ and a point not in C_0 . Because C_0 is a C-set in G'_{k+1} , $C_0 \subseteq H \cap K$. There exist $a, b \in [0, 1]$ such that $p_{k+2}(H) = [0, a]$ and $p_{k+2}(K) = [0, b]$. We may assume that $a \leq b$ and $0 < a$; let s be a number such that $0 < s \leq a$. Then H and K are subcontinua of G'_{k+1} each containing a point of C_s and a point not in C_s . However, because C_s is a C-set in G'_{k+1} , it is a subset of $H \cap K$. It follows that $H \subseteq K$. Thus, C_0 is terminal in G'_{k+1} .

To see that G'_{k+1} is chainable, recall that we have already shown that for each $i \in \mathbb{N}$, $L_i \cup L_{i+1}$ is chainable. Because $L_i \cap L_{i+2} = \emptyset$ for each positive integer i , it follows that $Y_n = L_1 \cup L_2 \cup \dots \cup L_n$ is chainable for each positive integer n . Define $\psi_n : G'_{k+1} \rightarrow Y_n$ by $\psi_n(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in Y_n$ and $\psi_n(\mathbf{x}) = (x_1, x_2, \dots, x_{k+1}, \varphi_n(x_{k+1}))$ otherwise. Let $g_n : Y_{n+1} \rightarrow Y_n$ be the restriction of ψ_n to Y_{n+1} . Then the sequence ψ converges uniformly to the identity on G'_{k+1} and $\psi_n = g_n \circ \psi_{n+1}$. Thus, by Theorem 2.1, G'_{k+1} is homeomorphic to $\varprojlim g$, a chainable continuum being the inverse limit of chainable continua [8, Theorem 174]. This completes the induction.

Because G'_n is chainable for each $n \in \mathbb{N}$, it follows from Corollary 4.3 that $\varprojlim f$ is chainable. □

Example 5.5. Let f be the inverse of the function in Example 5.4. Then $\varprojlim \mathbf{f}$ is a chainable continuum (see the graph on the right in Figure 2 for the graph of f).

Proof. Let $M = \varprojlim \mathbf{f}$. That M is a continuum is a consequence of the fact that $f^{-1} : [0, 1] \rightarrow C([0, 1])$, [4, Theorem 2.8]. (This also follows from the fact shown below that G'_n is a continuum for each n .) This proof is similar to the proof for Example 5.4. We present an outline of the proof featuring the differences and leave the details to the interested reader. Note that the sequence $\varphi_0, \varphi_1, \varphi_2, \dots$ of mappings of $[0, 1]$ into $[0, 1]$ from the previous proof has the property that $G(f) = \varphi_0 \cup \varphi_1 \cup \varphi_2 \cup \dots$. As in the previous proof, it is sufficient to show that G'_n is chainable for each n and appeal to Corollary 4.3.

Note that G'_1 is chainable, $\{\mathbf{x} \in G'_1 \mid x_1 = t\}$ is a C-set in G'_1 for each $t \in [0, 1]$, and both $\{\mathbf{x} \in G'_1 \mid x_1 = 0\}$ and $\{\mathbf{x} \in G'_1 \mid x_1 = 1\}$ are terminal in G'_1 .

Suppose that k is a positive integer such that G'_k is chainable, $\{\mathbf{x} \in G'_k \mid x_1 = t\}$ is a C-set in G'_k for each $t \in [0, 1]$, and both $\{\mathbf{x} \in G'_k \mid x_1 = 0\}$ and $\{\mathbf{x} \in G'_k \mid x_1 = 1\}$ are terminal in G'_k . For each nonnegative integer i , let $\Phi_i : G'_k \rightarrow G'_{k+1}$ be the homeomorphism given by $\Phi_i(\mathbf{x}) = (\varphi_i(x_1), x_1, x_2, \dots, x_{k+1})$ and let $L_i = \Phi_i(G'_k)$. Each L_i is chainable and $G'_{k+1} = \bigcup_{i \geq 0} L_i$. For each odd positive integer i , $L_i \cap L_{i+1} = \Phi_i(\{\mathbf{x} \in G'_k \mid x_1 = 0\})$; for each even positive integer i , $L_i \cap L_{i+1} = \Phi_i(\{\mathbf{x} \in G'_k \mid x_1 = 1\})$. Proofs that $C_t = \{\mathbf{x} \in G'_{k+1} \mid x_1 = t\}$ is a C-set for each $t \in [0, 1]$ and that C_0 and C_1 are also terminal are much like those in the previous proof. Proof that $L_1 \cup L_2 \cup \dots \cup L_n$ is chainable is also much like that of the previous proof. This allows us to use Theorem 2.1 in much the same way as before to obtain the chainability of G'_{k+1} . \square

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