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THE CAPTURING OPERATION IN TOPOLOGICAL DYNAMICS

JOSEPH AUSLANDER

Abstract. We use the capturing operation to develop a partial classification of minimal flows with the same Ellis group.

1. Introduction

A fundamental problem in topological dynamics is the classification of minimal flows. A partial classification (up to "proximal equivalence") is provided by the *Ellis groups* which are closed subgroups of the automorphism group of the universal minimal flow. A remaining problem is to distinguish minimal flows in the same "proximal class" (equivalently with the same Ellis group). This issue is addressed in the present paper. The main tool is the capturing operation, which is a kind of reverse orbit closure.

2. Some Definitions and Known Results

We begin by reviewing some dynamical notions (see [1] and [3]).

A *flow* (X, T) is a jointly continuous action $(x, t) \mapsto xt$ of a topological group T on a compact Hausdorff space X. A *minimal set* is a non-empty, closed T invariant set, which is minimal with respect to these properties. Equivalently, a non-empty subset K of X is minimal if it is the orbit closure of each of its points $\overline{xT} = K$ for all $x \in K$. It follows from Zorn's Lemma that minimal sets always exist for a flow on a compact space.

If $\overline{xT}$ is minimal, x is said to be an *almost periodic point.* If (X, T) is minimal, (so $\overline{xT} = X$ for all $x \in X$) we say (X, T) is a *minimal flow.*

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If (X,T) and (Y,T) are flows the *product flow* $(X \times Y, T)$ is defined by coordinate action $(x,y)t = (xt, yt)$ for $x \in X$, $y \in Y$, and $t \in T$.

If (X,T) and (Y,T) are flows, a *homomorphism* is a continuous equivariant surjective map $\pi : X \to Y$ such that $\pi(xt) = \pi(x)t$ for $x \in X$ and $t \in T$. We say that Y is a *factor* of X, and X is an *extension* of Y.

If (X,T) is minimal, a homomorphism is determined by its action on a single point. That is, if π and ψ are homomorphisms and $\pi(x_0) = \psi(x_0)$ for some $x_0 \in X$, then $\pi = \psi$.

If (X,T) is a flow, then x and x' in X are said to be *proximal*, if there is a net $\{t_i\}$ in T such that $(xt_i, x't_i) \to (y,y)$ for some $y \in X$. Equivalently, x and x' are proximal if for every neighborhood W of the diagonal Δ there is a $t \in T$ such that $(x,x')t \in W$.

We write $P(X)$ (or just P) for the proximal relation on X. P is reflexive, symmetric, and T-invariant ($(x,x') \in P$ implies $(xt, x't) \in P$ for $t \in T$) but not in general transitive or closed.

If x and y are not proximal, they are said to be *distal*. The flow (X,T) is distal if $P(X) = \Delta$; so if x, $y \in X$ with $x \neq y$, then $(x,y) \notin P$.

If (X,T) and (Y,T) are flows and $\pi : X \to Y$ is a homomorphism, π is said to be a *proximal extension* if whenever $\pi(x) = \pi(x')$, then $(x,x') \in P(X)$.

If T is a group, there is a unique universal minimal flow (M,T) whose defining property is that every minimal flow (X,T) is a factor of (M,T). Equivalently, there is a closed T invariant equivalence relation R on M such that the quotient flow $(M/R, T)$ is isomorphic with (X,T). If $\pi : M \to M/R$ is the canonical map, then $\pi(m) = \pi(m')$ if and only if $(m,m') \in R$.

The latter is the point of view of this paper. That is, we will obtain information about a minimal flow by considering the closed invariant equivalence relation which defines it.

3. The Capturing Operation

Let G be the automorphism group of the universal minimal flow (M,T). If $m \in M$ and $\alpha \in G$, then clearly $(m, \alpha(m))$ is an almost periodic point of the product flow $(M \times M, T)$. In fact, the converse holds as well – if (m,n) is an almost periodic point of $(M \times M, T)$, then there is a (necessarily unique) $\alpha \in G$ such that $n = \alpha(m)$.

If $\alpha \in G$, the graph of α, $\Gamma_\alpha = \{(m, \alpha(m)) | m \in M\}$, and if $A \subset G$, $\Gamma_A = \cup\{\Gamma_\alpha | \alpha \in A\}$.

Let (X,T) be a minimal flow and let $\pi : M \to X$ be a homomorphism. The Ellis group of (X,T) is the subgroup of G defined by $\mathcal{G}(X) = \{\alpha \in$

$G|\pi\alpha = \pi\}$. (Another choice of a homomorphism yields a conjugate subgroup of G.)

If the minimal flow (X, T) is represented as $X = M/R$, then $\alpha \in \mathcal{G}(X)$ if and only if $\Gamma_\alpha \subset R$.

There is a compact, T_1 (but not Hausdorff) topology on G such that $\mathcal{G}(X)$ is closed. (This is the topology of graphs: $\alpha_n \to \alpha$ in $\alpha_n(x_n) \to \alpha(x)$ for some net $\{x_n\}$ in M with $x_n \to x$.) Moreover, every closed subgroup of G is the Ellis group of some minimal flow.

The Ellis groups are "proximal invariants" of minimal flows. That is, two minimal flows (X, T) and (Y, T) have the same Ellis group if and only if there is a minimal flow (Z, T) and proximal homomorphisms $\pi : Z \to X$ and $\psi : Z \to Y$.

If (X, T) is a flow and $K \subset X$. The *capturing set* of K is the set $C(K) = \{x \in X|\overline{xT} \cap K \neq \emptyset\}$, and the *strong capturing set* $C^*(K) = \{x \in X|\overline{xT} \subset C(K)\}$. Clearly, $C^*(K) \subset C(K)$ and both are T invariant. In fact, if $x \in C^*(K)$, then $\overline{xT} \subset C^*(K)$.

We will be mostly concerned with capturing and strongly capturing sets as applied to the product flow $(X \times X, T)$. Note that $C(\Delta) = P$, the proximal relation.

The capturing operation was defined by Eli Glasner and the author [2] to study the distal structure relation of a minimal flow.

Lemma 3.1. *Let A be a closed subgroup of G.*

(i) $(x, y) \in C(\Gamma_A)$ *if and only if* $\Gamma_\alpha \subset \overline{(x, y)T}$ *for some* $\alpha \in A$.

(ii) *If* (x, y) *is an almost periodic point of* $C(\Gamma_A)$, *then* $(x, y) \in \Gamma_A$.

(iii) $(x, y) \in C^*(\Gamma_A)$ *if and only if whenever* $\beta \in G$ *satisfies* $\Gamma_\beta \subset \overline{(x, y)T}$, *then* $\beta \in A$.

(iv) $P(M) \subset C(\Gamma_A)$.

(v) *If* (X, T) *is a minimal flow with* $\mathcal{G}(X) = A$ *and* $\pi : M \to X$ *a homomorphism, then* $\pi^{-1}(P(X)) = C(\Gamma_A)$.

The following are equivalent:

(vi) $(x, y) \in C^*(\Gamma_A)$.

(vii) $\overline{(x, y)T} \subset C^*(\Gamma_A)$.

(viii) *All almost periodic points in* $\overline{(x, y)T}$ *are in* Γ_A.

Proof. (i) and (ii) follow immediately from the definitions, and they imply (iii). Since the identity automorphism *id* is in A, (iv) holds.

As for (v), suppose $(m, m') \in C(\Gamma_A)$, and $(x, x') = \pi(m, m')$. Let $\{t_i\}$ be a net in T such that $(m, m')t_i \to (n, \alpha(n))$ where $\alpha \in A$. Since $\mathcal{G}(X) = A$, $(x, x')t_i \to \Delta_X$ so $(x, x') \in P(X)$.

Suppose $\pi(m, m') = (x, x') \in P(X)$. Let $(x, x')t_i \to (y.y)$, so (a subnet of) $(m, m')t_i \to (n, n')$ and $\pi(n, n') = (y, y)$. Now let $\{s_i\}$ be a net in T

with $s_i(n,n') \to (k,k')$ an almost periodic point. Then $(k,k') = (k,\beta(k))$ for $\beta \in G$, and since $\pi(k.\beta(k)) \in \Delta_X$, we have $\beta \in A$. Since $(k,\beta(k)) \in \overline{(m,m')T}$, $(m,m') \in C(\Gamma_A)$.

The equivalence of (vi) and (viii) follows from (iii)

Obviously, (vii) $\implies$ (viii). Suppose (viii) holds, and let $(x',y') \in \overline{(x,y)T}$. Let $(x_0,y_0) \in \overline{(x',y')T}$ with (x_0,y_0) almost periodic. Then $(x_0,y_0) \in \overline{(x,y)T}$ so $(x_0,y_0) \in \Gamma_A$. Therefore, $(x',y') \in C^*(\Gamma_A)$. □

Theorem 3.2. *Let R be a closed T-invariant equivalence relation on M. Then $\mathcal{G}(M/R) = A$ if and only if $\overline{\Gamma_A} \subset R \subset C^*(\Gamma_A)$.*

Proof. Suppose $\overline{\Gamma_A} \subset R \subset C^*(\Gamma_A)$. Since $\Gamma_A \subset R$, $A \subset \mathcal{G}(M/R)$. If $\beta \in \mathcal{G}(M/R)$, then $(m,\beta(m)) \in R \subset C^*(\Gamma_A)$, and therefore $\beta \in A$.

Conversely, suppose $A = \mathcal{G}(M/R)$. Let $X = M/R$ and let $\pi : M \to X$. Clearly, $\Gamma_A \subset R$, so $\overline{\Gamma_A} \subset R$. For $R \subset C^*(\Gamma_A)$, it is sufficient by Lemma 3.1 to show that if $(m,n) \in R$, all almost periodic points in $\overline{(m,n)T}$ are in Γ_A. If (m',n') is such an almost periodic point, then $(m',n') \in R$, so $n' = \alpha(m')$ for some $\alpha \in A$, and $(m',n') \in \Gamma_A$. □

The next proof makes use of quasifactors of M. If (X,T) is a flow, a *quasifactor* of X is a subflow of the “hyperspace” $(2^X,T)$, where 2^X denotes the collection of closed subsets of X provided with the Hausdorff topology. The action of T on 2^X is defined by $Kt = \{xt | x \in K\}$ for $K \in 2^X$ and $t \in T$. The action of the Stone-Čech compactification βT is denoted by the “circle” operation: If $K \in 2^X$ and $\eta \in \beta T$, we write $K \circ \eta$ for the action of η on K. $y \in K \circ \eta$ if and only if there are x_i in K and t_i in T with $t_i \to \eta$ and $x_i t_i \to y$. In general, $K\eta = \{x\eta | x \in K\}$ is a proper subset of $K \circ \eta$.

If K is a (non-empty, but not necessarily closed) subset of X and I is a minimal right ideal in βT, then $\{K \circ p | p \in I\}$ is a minimal set in $(2^X,T)$.

The relation between factors and quasifactors is still not completly understood.

Theorem 3.3. *Let A be a closed subgroup of G. Then $\overline{\Gamma_A}$ and $C^*(\Gamma_A)$ are equivalence relations.*

Proof. To show that $\overline{\Gamma_A}$ is an equivalence relation, we find a minimal flow (X,T) with $X = M/R$ where $R = \overline{\Gamma_A}$.

Let $\mathcal{M}_A$ be the quasifactor $\{Am \circ p | p \in I\}$ where $Am = \{\alpha(m) | \alpha \in A\}$ and I is a minimal right ideal in βT. Let $\pi : M \to \mathcal{M}_A$ be given by $\pi(mp) = Am \circ p$ for $p \in I$. If we write $\mathcal{M}_A = M/R$, we are to show that $R = \overline{\Gamma_A}$ (so the latter is an equivalence relation). Suppose $(mp.mq) \in R$, so $Am \circ p = \pi(mp) = \pi(mq) = Am \circ q$. Then $mq \in Am \circ q = Am \circ p$. Let

$\{s_i\}$ be a net in T with $s_i \to p$ so $\alpha_i(m)s_i \to mq$ for some $\alpha_i \in A$. Then $(m, \alpha_i(m))s_i \to (mp, mq)$. Now $(m, \alpha_i(m))s_i \in \Gamma_A$ so $(mp, mq) \in \overline{\Gamma_A}$.

Let $(x, y), (y, z) \in C^*(\Gamma_A)$, and let (x', z') be an almost periodic point in $\overline{(x, z)T}$. Then there is a minimal right ideal I in $E(M)$, the enveloping semigroup of (M, T), and a $p \in M$ such that $(x', z') = (xp, zp)$. Now since (x, y) and (y, z) are in $C^*(\Gamma_A)$, (xp, yp) and (yp, zp) are in Γ_A, so $yp = \alpha(xp)$ and $zp = \beta(yp)$ where $\alpha, \beta \in A$, and therefore $(x', z') = (xp, zp) \in \Gamma_A$ so $(x, z) \in C^*(\Gamma_A)$. □

The flow $(M/\overline{\Gamma_A}, T)$ is the "maximal" minimal flow with group A. That is, it has every minimal flow with group A as a factor. When $C^*(\Gamma_A)$ is closed, $M/C^*(\Gamma_A)$ is a factor of all minimal flows with group A. A case where this holds is when A is a distal group, defined below.

The next result shows that the closed invariant equivalence relation R can be recovered from A and $P \cap R$.

Theorem 3.4. *Let R be a closed T invariant equivalence relation on M and let $A = \mathcal{G}(M/R)$. Then $R = \cup\{(x, \alpha(y)) | \alpha \in A, (x, y) \in P(M) \cap R\}$.*

Proof. Since (x, y) and $(y, \alpha(y))$ are in R, the right side is contained in R. Now let $(x, y) \in R$. Let u and v be idempotents in a minimal right ideal I such that $xu = x$ and $yv = y$. Then $(xv, y) = (xv, yv) \in R$ and $(x, xv) \in P \cap R$. Also $y = yv = \alpha(xv)$ with $\alpha \in A$ (since (xv, y) is an almost periodic point of R). Therefore, $(x, y) = (x, \alpha(xv))$ where $(x, xv) \in P \cap R$ and $\alpha \in A$. □

Theorem 3.5. *Let A be a closed subgroup of G. Then the following are equivalent:*

(i) *$C(\Gamma_A)$ is an equivalence relation.*
(ii) *If $(x, y) \in C(\Gamma_A)$, then $\overline{(x, y)T} \subset C(\Gamma_A)$.*
(iii) *$C(\Gamma_A) = C^*(\Gamma_A)$.*
(iv) *Proximal is an equivalence relation for some flow with group A.*
(v) *Proximal is an equivalence relation for all flows with group A.*

Proof. (i) $\implies$ (ii) and (iii): Suppose $C(\Gamma_A)$ is an equivalence relation, and let $(x, y) \in C(\Gamma_A)$. For (ii), it is sufficient to show that any minimal subset N of $\overline{(x, y)T}$ has a point in Γ_A. Let $(x', y') \in N$ with $((x, y), (x', y')) \in P$. Then $(x, x') \in P \subset C(\Gamma_A)$ and similarly $(y, y') \in C(\Gamma_A)$. Since $C(\Gamma_A)$ is an equivalence relation $(x', y') \in C(\Gamma_A)$ and since (x', y') is almost periodic, it is in Γ_A. This proves (ii). Note also that it follows that the minimal set $N \subset \Gamma_A$, so (iii) holds.

(ii) $\implies$ (i): Suppose (x, y) and (y, z) are in $C(\Gamma_A)$, and consider the point $(x, y, z) \in M \times M \times M$. Then there is a point $(x', y', z') \in \overline{(x, y, z)T}$ with $(x', y') \in \Gamma_A$. By (ii), $(y', z') \in C(\Gamma_A)$, so there is a point

$(x'', y'', z'') \in \overline{(x', y', z')T}$ with $(y'', z'') \in \Gamma_A$. Note that also $(x'', y'') \in \Gamma_A$, so $y'' = \alpha(x'')$, $z'' = \beta(y'')$ with $\alpha, \beta \in A$, and $z'' = \beta\alpha(x'') \in \Gamma_A$. Since $(x'', z'') \in \overline{(x, z)T}$, we have $(x, z) \in C(\Gamma_A)$.

(iii) $\implies$ (ii) is an immediate consequence of Lemma 3.1.

Statements (iv) and (v) are equivalent, since "proximal is an equivalence relation" is a proximal invariant hence an Ellis group invariant. (This, in turn, follows from the well-known and easily proved assertion that if $\pi : X \to Y$ is a proximal homomorphism of minimal flows, then $\pi^{-1}(P(Y)) = P(X)$.)

If (X, T) is a minimal flow with $\mathcal{G}(X) = A$ and $\pi : M \to X$, then $\pi^{-1}(P(X)) = C(\Gamma_A)$ (Lemma 3.1). It follows that $P(X)$ is an equivalence relation if and only if $C(\Gamma_A)$ is such, so (iv) and (v) imply (i). □

Corollary 3.6. *If $C(\Gamma_A)$ is closed, it is an equivalence relation, and $C(\Gamma_A) = C^*(\Gamma_A)$.*

Proof. $C(\Gamma_A)$ is T invariant, so if it is closed, then $(x, y) \in C(\Gamma_A)$ implies $\overline{(x, y)T} \subset C(\Gamma_A)$. Now apply Theorem 3.5. □

Let A be a closed subgroup of G. We say that A is a *distal group* if there is a distal minimal flow X with $\mathcal{G}(X) = A$.

Theorem 3.7. *Let A be a closed subgroup of G. Then the following are equivalent:*

(i) *A is a distal group.*
(ii) *$C(\Gamma_A)$ is closed.*
(iii) *$C^*(\Gamma_A)$ is closed and $C(\Gamma_A)$ is an equivalence relation.*

In this case $M/C(\Gamma_A)$ is the (necessarily unique) distal minimal flow with group A.

Proof. Suppose $C(\Gamma_A)$ is closed. By Corollary 3.6, $C(\Gamma_A)$ is a closed invariant equivalence relation. Since $P \subset C(\Gamma_A)$, it follows that $M/C(\Gamma_A)$ is distal. Moreover, by Theorem 3.4, $C(\Gamma_A) = C^*(\Gamma_A)$, so $\mathcal{G}(M/C(\Gamma_A)) = A$ (Theorem 3.2).

Now suppose A is a distal group. Then there is a closed invariant equivalence relation R with $\overline{\Gamma_A} \subset R \subset C^*(\Gamma_A)$ such that M/R is distal. We show that $R = C^*(\Gamma_A)$ (so $C^*(\Gamma_A)$ is closed) and then that $C(\Gamma_A) = C^*(\Gamma_A)$. Let $\pi : M \to M/R$, $\rho : M \to M/\overline{\Gamma_A}$, and $\delta : M/\overline{\Gamma_A} \to M/R$, so $\pi = \delta\rho$. Let $(x, y) \in C^*(\Gamma_A)$. Then there is a minimal right ideal I in $E(M)$ such that if $p \in I$, $(xp, yp) \in \Gamma_A$, so $\rho(xp) = \rho(yp)$. Therefore, $\rho(x)$ and $\rho(y)$ are proximal, and since M/R is distal, $\delta\rho(x) = \delta\rho(y)$. That is, $\pi(x) = \pi(y)$, or equivalently, $(x, y) \in R$. Thus, $C^*(\Gamma_A) = R$, and consequently, $C^*(\Gamma_A)$ is closed and $M/C^*(\Gamma_A)$ is distal. Finally, we show that $C(\Gamma_A) = C^*(\Gamma_A)$, so $C(\Gamma_A)$ is closed. Let $(x, y) \in C(\Gamma_A)$.

Then $(xp, yp) \in \Gamma_A$ for $p \in I$, a minimal right ideal in $E(M)$. Since $\Gamma_A \subset C^*(\Gamma_A)$, $\pi(xp) = \pi(yp)$ so $\pi(x)$ and $\pi(y)$ are proximal, and since $M/C^*(\Gamma_A)$ is distal, we have $\pi(x) = \pi(y)$ or, what is the same thing, $(x, y) \in R = C^*(\Gamma_A)$.

If A is a distal group, then $C(\Gamma_A)$ is closed, so it is an equivalence relation (Corollary 3.6), and, by Theorem 3.5, $C^*(\Gamma_A) = C(\Gamma_A)$ so $C^*(\Gamma_A)$ is closed. Therefore, (i) implies (iii).

Suppose (iii) holds. By Theorem 3.5, $C(\Gamma_A) = C^*(\Gamma_A)$ so $C(\Gamma_A)$ is closed, and therefore A is a distal group. □

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