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A GENERALIZATION OF THE NOTION OF A P-SPACE TO PROXIMITY SPACES

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Abstract. In this note, we shall generalize the notion of a P-space to proximity spaces and investigate the basic properties of these proximities. We, therefore, define a $P_{\aleph_1}$-proximity to be a proximity where, if $A_n \prec B$ for all $n \in \mathbb{N}$, then $\bigcup_n A_n \prec B$. It turns out that the category of $P_{\aleph_1}$-proximities is isomorphic to the category of σ-algebras. Furthermore, the $P_{\aleph_1}$-proximity coreflection of a proximity space is the σ-algebra of proximally Baire sets.

1. Introduction

We begin by reviewing basic facts on proximity spaces without proofs. All our preliminary information on proximity spaces can be found in [2]. In this paper, we shall assume all proximity spaces are separated and all topologies are Tychonoff. If δ is a relation, then we shall write $\overline{\delta}$ for the negation of the relation δ. In other words, we have $R\overline{\delta}S$ if and only if we do not have $R\delta S$. The complement of a subset A of a set X will be denoted by A^c.

A *proximity space* is a pair (X, δ) where X is a set and δ is a relation on the power set $P(X)$ that satisfies the following axioms.

1. $A\delta B$ implies $B\delta A$.
2. $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.
3. If $A\delta B$, then $A \neq \emptyset$ and $B \neq \emptyset$.
4. If $A\overline{\delta}B$, then there is a set E such that $A\overline{\delta}E$ and $E^c\overline{\delta}B$.
5. If $A \cap B \neq \emptyset$, then $A\delta B$.

A proximity space is *separated* if and only if $\{x\}\delta\{y\}$ implies $x = y$.

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Intuitively, $A\delta B$ whenever the set A touches the set B in some sense. Therefore, a proximity space is a set with a notion of whether two sets are infinitely close to each other.

If (X,δ) is a proximity space, then let $\prec_\delta$ (or, simply, $\prec$) be the binary relation on $P(X)$ where $A \prec B$ if and only if $A\overline{\delta}B^c$. The relation $\prec$ satisfies the following.

1. $X \prec X$.
2. If $A \prec B$, then $A \subseteq B$.
3. If $A \subseteq B, C \subseteq D, B \prec C$, then $A \prec D$.
4. If $A \prec B_i$ for $i = 1, \ldots, n$, then $A \prec \bigcap_{i=1}^n B_i$.
5. If $A \prec B$, then $B^c \prec A^c$.
6. If $A \prec B$, then there is some C with $A \prec C \prec B$.

If $\prec$ satisfies 1–6 and if we define δ by letting $A\overline{\delta}B$ if and only if $A \prec B^c$, then (X,δ) is a proximity space.

If (X,δ) is a proximity space, then we put a topology τ_δ on X by letting $\overline{A} = \{x|x\delta A\}$. A set $U \subseteq X$ is open if and only if $\{x\} \prec U$ whenever $x \in U$.

If (X,δ) and (Y,ρ) are proximity spaces, then a function $f : X \to Y$ is a *proximity map* if $f(A)\rho f(B)$ whenever $A\delta B$. It can easily be shown that f is a proximity map if and only if $f^{-1}(C)\overline{\delta}f^{-1}(D)$ whenever $C\overline{\rho}D$. Furthermore, f is a proximity map if and only if $f^{-1}(C) \prec_\delta f^{-1}(D)$ whenever $C \prec_\rho D$. Every proximity map is continuous.

Example 1.1. If X is a set, then δ is the discrete proximity if $A\overline{\delta}B$ whenever $A \cap B = \emptyset$.

Example 1.2. Let (X,τ) be a Tychonoff space. Let $A\overline{\delta}B$ if there is a continuous function $f : (X,\tau) \to [0,1]$ with $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$. Then (X,δ) is a proximity space that induces the topology on X (i.e., δ is *compatible* with τ). It is well known that a topology X is induced by some proximity if and only if X is Tychonoff.

Example 1.3. Compact spaces have a unique compatible proximity where $A\delta B$ if and only if $\overline{A} \cap \overline{B} \neq \emptyset$. Furthermore, if (X,δ) is a compact proximity space and (Y,ρ) is a proximity space, then a map $f : X \to Y$ is continuous if and only if f is a proximity map.

If (X,δ) is a proximity space and $Y \subseteq X$, then define a relation δ_Y on $P(Y)$ by letting $A\delta_Y B$ if and only if $A\delta B$. Then δ_Y is a proximity on Y that induces the subspace topology on Y called the induced proximity.

If (X,δ) is a proximity space, then $A\overline{\delta}B$ if and only if there is a proximity map $g : (X,\delta) \to [0,1]$ with $g(A) \subseteq \{0\}, g(B) \subseteq \{1\}$.

If (X,δ) is a proximity space, then there is a unique compactification $\mathcal{C}$ of X where $A\delta B$ if and only if $(\mathrm{cl}_{\mathcal{C}} A) \cap (\mathrm{cl}_{\mathcal{C}} B) \neq \emptyset$. This compactification is called the *Smirnov compactification* of X and the proximity δ is the proximity induced by the unique proximity on the compact space $\mathcal{C}$. If X is a Tychonoff space, then the proximity spaces that induce the topology on X are in a one-to-one correspondence with the compactifications of X. If (X, δ) and (Y, ρ) are proximity spaces and $\mathcal{C}$ is the Smirnov compactification of X and $\mathcal{D}$ is the Smirnov compactification of Y, then a function $f : X \to Y$ is a proximity map if and only if f has a unique extension to a continuous map from $\mathcal{C}$ to $\mathcal{D}$.

An algebra of sets $(X, \mathcal{M})$ is reduced if and only if whenever $x, y \in X$ are distinct, then there is an $R \in \mathcal{M}$ with $x \in R$ and $y \in R^c$. We assume that all algebras of sets are reduced. If X is a topological space, then a *zero set* is a set of the form $f^{-1}(0)$ where $f : X \to \mathbb{R}$ is continuous. The union of finitely many zero sets is a zero set, and the intersection of countably many zero sets is a zero set. A *P-space* is a Tychonoff space where every G_δ-set is open. It is well known and it can easily be shown that a Tychonoff space is a P-space if and only if every zero set is open. If X is a Tychonoff space, then the *P-space coreflection* $(X)_{\aleph_1}$ is the space with underlying set X and where the G_δ-sets in X form a basis for the topology on $(X)_{\aleph_1}$.

2. $P_{\aleph_1}$-Proximities

A separated proximity space (X, δ) is a $P_{\aleph_1}$*-proximity space* if whenever $A_n \subseteq X$ for $n \in \mathbb{N}$ and $B \subseteq X$ and $\bigcup_{n=0}^{\infty} A_n \delta B$, then $A_n \delta B$ for some n. In other words, X is a $P_{\aleph_1}$-proximity space if and only if whenever $A_n \prec B$ for each natural number n, then $\bigcup_n A_n \prec B$. Equivalently, X is a $P_{\aleph_1}$-proximity if and only if whenever $A \prec B_n$ for all n, then $A \prec \bigcap_n B_n$.

A proximity space (X, δ) is said to be *zero-dimensional* if and only if whenever $A \bar{\delta} B$, then there is a $C \subseteq X$ with $A \bar{\delta} C$, $B \bar{\delta} C^c$, and $C \bar{\delta} C^c$. In other words, (X, δ) is zero-dimensional if and only if, whenever $A \prec B$, there is a C with $A \prec C \prec C \prec B$. If (X, δ) is a proximity space, then let $\mathcal{M}_\delta = \{R \subseteq X | R \bar{\delta} R^c\} = \{R \subseteq X | R \prec R\}$. If $(X, \mathcal{M})$ is an algebra of sets, then let $\delta_{\mathcal{M}}$ be the relation on $P(X)$ where $U \overline{\delta_{\mathcal{M}}} V$ if and only if there is some $R \in \mathcal{M}$ with $U \subseteq R, V \subseteq R^c$.

Theorem 2.1. (1) *If (X, δ) is a proximity space, then $(X, \mathcal{M}_\delta)$ is an algebra of sets.*

(2) *If $(X, \mathcal{M})$ is an algebra of sets, then $(X, \delta_{\mathcal{M}})$ is a zero-dimensional proximity space.*

(3) *If δ is a zero-dimensional proximity, then $\delta_{\mathcal{M}_\delta} = \delta$.*

(4) *If $(X, \mathcal{M})$ is an algebra of sets, then $\mathcal{M} = \mathcal{M}_{\delta_{\mathcal{M}}}$.*

(5) *If (X, δ) is a zero-dimensional proximity space, then $\mathcal{M}_\delta$ is a basis for the topology on X.*

Proof. See [1]. □

Theorem 2.2. *Let $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ be algebras of sets. Then a mapping $f:(X,\delta_{\mathcal{M}})\rightarrow(Y,\delta_{\mathcal{N}})$ is a proximity map if and only if $f^{-1}(U)\in\mathcal{M}$ for each $U\in\mathcal{N}$.*

Proof. $(\rightarrow)$ Assume that f is a proximity map. For each $R\in\mathcal{N}$, we have $R\prec R$, so $f^{-1}(R)\prec f^{-1}(R)$, and thus $f^{-1}(R)\in\mathcal{M}$.

$(\leftarrow)$ Let $U,V\subseteq Y$ and let $U\prec V$. Then there is an $R\in\mathcal{N}$ with $U\subseteq R\subseteq V$. Therefore, $f^{-1}(R)\in\mathcal{M}$ and $f^{-1}(U)\subseteq f^{-1}(R)\subseteq f^{-1}(V)$. Therefore, $f^{-1}(U)\prec f^{-1}(V)$, so f is a proximity map. □

Theorem 2.3. *Every $P_{\aleph_1}$-proximity space is a zero-dimensional proximity space.*

Proof. Let X be a $P_{\aleph_1}$-proximity space. Assume $A\prec B$. Then there is a sequence $(C_n)_{n\in\mathbb{N}}$ of subsets of X where $A=C_0\prec C_1\prec ...\prec C_n\prec ...\prec B$. Therefore, let $C=\bigcup_{n\in\mathbb{N}}C_n$. Since X is a $P_{\aleph_1}$-proximity space, we have $A\prec C\prec B$. Furthermore, since $C_n\prec C$ for all n and X is a $P_{\aleph_1}$-proximity space, we have $C=\bigcup_n C_n\prec C$. Therefore, X is a zero-dimensional proximity space. □

Theorem 2.4. *An algebra of sets $(X,\mathcal{M})$ is a σ-algebra if and only if $(X,\delta_{\mathcal{M}})$ is a $P_{\aleph_1}$-proximity space.*

Proof. $(\rightarrow)$ Assume $(X,\mathcal{M})$ is a σ-algebra. Assume that $A_n\prec B$ for all n. Then, for each n, there is a $C_n\in\mathcal{M}$ with $A_n\subseteq C_n\subseteq B$. Therefore, $\bigcup_n A_n\subseteq\bigcup_n C_n\subseteq B$, so $\bigcup_n A_n\prec B$.

$(\leftarrow)$ Assume that $\delta_{\mathcal{M}}$ is a $P_{\aleph_1}$-proximity space and let $R_n\in\mathcal{M}$ for each natural number n. Then $R_n\prec R_n\subseteq\bigcup_n R_n$, so $\bigcup_n R_n\prec\bigcup_n R_n$, so $\bigcup_n R_n\in\mathcal{M}$. Therefore, $(X,\mathcal{M})$ is a σ-algebra. □

Corollary 2.5. (1) *The category of zero-dimensional proximity spaces with proximity maps is isomorphic to the category of reduced algebras of sets with measurable maps (i.e., maps $f:(X,\mathcal{N})\rightarrow(Y,\mathcal{M})$ where if $R\in\mathcal{M}$, then $f^{-1}(R)\in\mathcal{N}$).*

(2) *Their corresponding subcategories of $P_{\aleph_1}$-proximity spaces and reduced σ-algebras are isomorphic as well.*

Now given a proximity space (X,δ), we shall characterize the smallest σ-algebra $(X,\mathcal{M})$ such that the identity function from $(X,\delta_{\mathcal{M}})$ to (X,δ) is a proximity map, but we must first generalize the notion of a zero set to proximity spaces. Let (X,δ) be a proximity space. Then a *proximally zero set* is a set of the form $f^{-1}(0)$ for some proximity map $f:(X,\delta)\rightarrow[0,1]$. If $\mathcal{C}$ is the Smirnov compactification of X, then f has a unique extension to a continuous function $\hat{f}:\mathcal{C}\rightarrow[0,1]$. Hence, $f^{-1}(0)=\hat{f}^{-1}(0)\cap X$.

Therefore, the proximally zero sets on a proximity space are precisely the sets of the form $X \cap Z$ where $Z \subseteq \mathcal{C}$ is a zero set. As a consequence, the intersection of countably many proximally zero sets is a proximally zero set and the union of finitely many proximally zero sets is a proximally zero set.

The σ-algebra $\mathcal{B}^*(X,\delta)$ of *proximally Baire sets* on a proximity space (X,δ) is the smallest σ-algebra containing the proximally zero sets. If (X,δ) is a proximity space with Smirnov compactification $\mathcal{C}$ and $\mathcal{M}$ is the Baire σ-algebra on $\mathcal{C}$, then $\{R \cap X | R \in \mathcal{M}\}$ is the σ-algebra of proximally Baire sets on X.

Remark 2.6. Every proximally zero set is a zero set, but in general there are zero sets that are not proximally zero sets. For example, let A be an uncountable discrete space, and let δ be the proximity induced by the one point compactification $A \cup \{\infty\}$. It is well known that for normal spaces the closed G_δ-sets are precisely the zero sets, so it suffices to characterize the closed G_δ-subsets of $A \cup \{\infty\}$. Let $R \subseteq A \cup \{\infty\}$ be a closed G_δ-set. If $\infty \notin R$, then R is finite. If $\infty \in R$, then $R = \bigcap_n U_n$ for some sequence of open sets $U_n \subseteq A \cup \{\infty\}$, but each U_n is co-finite, so R is co-countable. Therefore, each zero set in $A \cup \{\infty\}$ is either co-finite or co-countable. Hence, every proximally zero set in A is either co-finite or co-countable. We conclude that not every zero set in A is a proximally zero set in A.

Theorem 2.7. *Let (X,δ) be a proximity space. Then a set $Z \subseteq X$ is a proximally zero set if and only if there is a sequence $(Z_n)_{n\in\mathbb{N}}$ with $Z = \bigcap_n Z_n$ and where $\ldots Z_n \prec Z_{n-1} \prec \ldots \prec Z_1$.*

Proof. ($\rightarrow$) If $Z \subseteq X$ is a proximally zero set, then there is a proximity map $f : (X,\delta) \to [0,1]$ with $Z = f^{-1}(0)$. For all $n \geq 1$, we have $[0, \frac{1}{n+1}] \prec [0, \frac{1}{n}]$, so $f^{-1}([0, \frac{1}{n+1}]) \prec f^{-1}([0, \frac{1}{n}])$ and $Z = f^{-1}(\{0\}) = \bigcap_n f^{-1}([0, \frac{1}{n}])$.

($\leftarrow$) Suppose that $(Z_n)_{n\in\mathbb{N}}$ is such a sequence. Then, for all n, we have $Z_{n+1}\bar{\delta}Z_n^c$, so there is a proximity map $f_n : (X,\delta) \to [0,1]$ with $Z_{n+1} \subseteq f_n^{-1}(0)$ and $Z_n^c \subseteq f_n^{-1}(1) \subseteq f_n^{-1}(0)^c$. Thus, $Z_{n+1} \subseteq f_n^{-1}(0) \subseteq Z_n$. Therefore, $\bigcap_n Z_n = \bigcap_n f_n^{-1}(0)$ is a proximally zero set being the intersection of countably many proximally zero sets. □

Corollary 2.8. *A proximity space (X,δ) is a $P_{\aleph_1}$-proximity space if and only if $\mathcal{M}_\delta$ contains each proximally zero set.*

Proof. ($\rightarrow$) Let (X,δ) be a $P_{\aleph_1}$-proximity space. If Z is a proximally zero set, then there is a sequence $(Z_n)_{n\in\mathbb{N}}$ with $Z = \bigcap_{n\in\mathbb{N}} Z_n$ and where $Z_{n+1} \prec Z_n$ for all n. Therefore, $Z \prec Z_n$ for all n, so $Z \prec \bigcap_{n\in\mathbb{N}} Z_n = Z$, so $Z \in \mathcal{M}_\delta$.

($\leftarrow$) Assume $\mathcal{M}_\delta$ contains each proximally zero set. Assume $A \prec B_n$ for all $n \geq 0$. For $n \geq 0$, there is a proximally zero set Z_n with $A \subseteq Z_n \subseteq B_n$, so $\bigcap_n Z_n$ is a proximally zero set, so $A \subseteq \bigcap_n Z_n \prec \bigcap_n Z_n \subseteq \bigcap_n B_n$. $\square$

If X is a topological space, then a *cozero set* is a set of the form $f^{-1}(0,1]$ for some continuous $f : X \to [0,1]$. If (X,δ) is a proximity space, then a *proximally cozero set* is a set of the form $f^{-1}(0,1]$ for some proximity map $f : (X,\delta) \to [0,1]$. In other words, a cozero set is a complement of a zero set and a proximally cozero set is a complement of a proximally zero set.

Theorem 2.9. (1) *Every Lindelöf open subset of a proximity space is a proximally cozero set.*

(2) *In a Lindelöf proximity space, every cozero set is a proximally cozero set.*

Proof. (1) Let (X,δ) be a proximity space and let U be a Lindelöf open subset of X. Then, for each $x \in X$, there is a proximity map $f_x : (X,\delta) \to [0,1]$ such that $f_x(x) = 1$ and $f_x(U^c) \subseteq \{0\}$. If $U_x = f_x^{-1}(0,1]$, then U_x is a proximally cozero set with $x \in U_x \subseteq U$. Since $\{U_x | x \in X\}$ covers U, there is a countable subcover $\{U_{x_n} | n \in \mathbb{N}\}$ of the set U. Therefore, $U = \bigcup_n U_{x_n}$ is a proximally cozero set since U is the countable union of proximally cozero sets.

(2) Assume (X,δ) is a Lindelöf proximity space. If $U \subseteq X$ is a cozero set, then U is an F_σ-set, so the set U is Lindelöf. Therefore, by (1), the set U is a proximally cozero set. $\square$

In particular, in every Lindelöf proximity space, the proximally Baire sets coincide with the Baire sets.

Remark 2.10. We may have $O\delta E$ even when O and E are disjoint proximally zero sets. Furthermore, it is possible that $C\delta C^c$ even though C and C^c are proximally zero sets. Give $\mathbb{Z}$ the proximity induced by the one-point compactification. Let C be the collection of even integers. Then C^c is the collection of all odd integers. Since C and C^c are both closed sets, we have C and C^c being disjoint proximally zero sets. On the other hand, $\mathrm{cl}_{\mathbb{Z}\cup\{\infty\}}(C) \cap \mathrm{cl}_{\mathbb{Z}\cup\{\infty\}}(C^c) = \{\infty\}$, so $C\delta C^c$.

Definition 2.11. If (X,δ) is a proximity space, then let $(X,\delta)_{\aleph_1} = (X,\delta_{\mathcal{B}^*(X,\delta)})$ be the proximity space equivalent to the σ-algebra of proximally Baire sets on X.

Theorem 2.12. *Let $(X,\mathcal{N})$ be a σ-algebra and let (Y,δ) be a proximity space. Then a map $f : (X,\delta_{\mathcal{N}}) \to (Y,\delta)$ is a proximity map if and only if f is a proximity map from $(X,\delta_{\mathcal{N}})$ to $(Y,\delta)_{\aleph_1}$.*

Proof. ($\rightarrow$) Assume that $f : (X, \delta_{\mathcal{N}}) \to (Y, \delta)$ is a proximity map. If $C \subseteq Y$ is a proximally zero set, then there is a proximity map $g : (Y, \delta) \to [0, 1]$ with $C = g^{-1}(0)$. Therefore, $g \circ f$ is a proximity map as well; $f^{-1}(C) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$ is a proximally zero set; and since $(X, \delta_{\mathcal{N}})$ is a $P_{\aleph_1}$-proximity space (by Theorem 2.4), we obtain, using Corollary 2.8 and Theorem 2.1, that $f^{-1}(C) \in \mathcal{N}$ for each proximally zero set C. Therefore, $f^{-1}(R) \in \mathcal{N}$ for each proximally Baire set R.

($\leftarrow$) Now assume that $f : (X, \delta_{\mathcal{N}}) \to (Y, \delta)_{\aleph_1}$ is a proximity map. Then let $C, D \subseteq Y$ be sets with $C\overline{\delta}D$. Then there is a proximally zero set $Z \subseteq Y$ with $C \subseteq Z$ and $D \subseteq Z^c$. Therefore, by Theorem 2.2, $f^{-1}(Z) \in \mathcal{N}$ and $f^{-1}(C) \subseteq f^{-1}(Z), f^{-1}(D) \subseteq f^{-1}(Z^c)$; thus, $f^{-1}(C)\overline{\delta_{\mathcal{N}}}f^{-1}(D)$. □

In particular, if $(X, \mathcal{M})$ is a σ-algebra and δ is any proximity on $\mathbb{R}$ that induces the Euclidean topology, then $f : (X, \mathcal{M}) \to \mathbb{R}$ is measurable if and only if $f : (X, \delta_{\mathcal{M}}) \to \mathbb{R}$ is a proximity map.

Theorem 2.13. *If (X, δ) is a proximity space, then the topology on $(X, \delta)_{\aleph_1}$ is the topology on the P-space coreflection of the topology on X.*

Proof. The proof is left to the reader. □

Lemma 2.14. *Let $f : (X, \delta) \to (Y, \rho)$ be a function. Then f is a proximity map if and only if whenever $g : (Y, \rho) \to [0, 1]$ is a proximity map, then $g \circ f : (X, \delta) \to [0, 1]$ is a proximity map.*

Proof. ($\rightarrow$) If f is a proximity map, then clearly any composition $g \circ f$ must be a proximity map.

($\leftarrow$) Now assume that each composition $g \circ f$ is a proximity map. Assume that $A, B \subseteq Y$ are sets with $A\overline{\rho}B$. Then there is a proximity map $g : (Y, \rho) \to [0, 1]$ with $A \subseteq g^{-1}(0)$ and $B \subseteq g^{-1}(1)$. Therefore, $f^{-1}(A) \subseteq f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$ and $f^{-1}(B) \subseteq f^{-1}(g^{-1}(1)) = (g \circ f)^{-1}(1)$, so $f^{-1}(A)\overline{\delta}f^{-1}(B)$ since $g \circ f$ is a proximity map. □

We shall write χ_A for the characteristic function on A. In other words, $\chi_A(A) = 1$ and $\chi_A(A^c) = 0$.

Theorem 2.15. *Let (X, δ) be a proximity space. Then the following are equivalent.*

(1) *(X, δ) is a $P_{\aleph_1}$-proximity space.*
(2) *If $f_n : (X, \delta) \to [0, 1]$ is a proximity map for each $n \in \mathbb{N}$ and $f_n \to f$ pointwise (here we do not assume f is continuous), then $f : (X, \delta) \to [0, 1]$ is also a proximity map.*

(3) *For each proximity space* (Y,ρ)*, if* $f_n : (X,\delta) \to (Y,\rho)$ *is a proximity map for each* $n \in \mathbb{N}$ *and* $f_n \to f$ *pointwise, then* $f : (X,\delta) \to (Y,\rho)$ *is also a proximity map.*

Proof. ($1 \to 2$) The map f is a proximity map if and only if f is a measurable function on the σ-algebra $(X, \mathcal{M}_\delta)$. The implication follows since measurable functions are closed under pointwise convergence.

($2 \to 3$) Assume that (Y,ρ) is any proximity space and assume that $f_n \to f$ pointwise and each f_n is a proximity map from (X,δ) to (Y,ρ). Then let $g : (Y,\rho) \to [0,1]$ be a proximity map. Then $g \circ f_n \to g \circ f$ pointwise, and, since each $g \circ f_n : (X,\mathcal{M}) \to [0,1]$ is a proximity map, $g \circ f$ is also a proximity map. Therefore, f is a proximity map by Lemma 2.14.

($3 \to 2$) This is trivial.

($2 \to 1$) Assume $Z \subseteq X$ is a proximally zero set. Then there is a proximity map $f : (X,\delta) \to [0,1]$ with $Z = f^{-1}(1)$. On the other hand, we have $f^n \to \chi_Z$ pointwise where χ_Z denotes the characteristic function, so χ_Z is a proximity map. Therefore, $Z\overline{\delta}Z^c$, so $Z \in \mathcal{M}_\delta$. Thus, using Corollary 2.8, we conclude that X is a $P_{\aleph_1}$-proximity space. □

Theorem 2.16. *Let* X *be a Tychonoff space. Then* X *is a* P*-space if and only if whenever* $f_n : X \to [0,1]$ *is continuous for all* n *and* $f_n \to f$ *pointwise, then* f *is continuous.*

Proof. ($\to$) Assume X is a P-space. Then, for each continuous $f_n : X \to [0,1]$ and $x \in X$, there is an open neighborhood U_n of x where $f_n(U_n) = f_n(x)$. Therefore, if $U = \bigcap_n U_n$, then U is an open neighborhood of x. Furthermore, if $f_n \to f$ pointwise, then $f(U) = f(x)$. Therefore, f is continuous at each point $x \in X$.

($\leftarrow$) Assume $Z \subseteq X$ is a zero set. Then let $f : X \to [0,1]$ be a continuous function where $f^{-1}(1) = Z$. Then $f^n \to \chi_Z$ pointwise. Therefore, since χ_Z is continuous, we have that Z is open. Therefore, X is a P-space. □

Corollary 2.17. *If* X *is Tychonoff and* δ *is the proximity induced by the Stone-Čech compactification of* X*, then* X *is a* P*-space if and only if* (X,δ) *is a* $P_{\aleph_1}$*-proximity space.*

Proof. The proximity maps from (X,δ) to $[0,1]$ are precisely the continuous functions from X to $[0,1]$. Therefore, by theorems 2.15 and 2.16, we have that X is a P-space if and only if (X,δ) is a $P_{\aleph_1}$-proximity space. □

3. Conclusions and applications

We conclude this paper by demonstrating that it is sometimes better to consider σ-algebras as $P_{\aleph_1}$-proximities since proximity spaces are often easier to work with than σ-algebras.

If $(X, \mathcal{M})$ is a σ-algebra, then let $L^\infty(X, \mathcal{M})$ denote the collection of all bounded measurable functions from $(X, \mathcal{M})$ to $\mathbb{C}$. Clearly, $L^\infty(X, \mathcal{M})$ is a Banach-algebra and even a C^*-algebra. If Y is a compact space, then let $C(Y)$ be the Banach-algebra which consists of all continuous functions from Y to $\mathbb{C}$. Let $\mathcal{C}$ be the Smirnov compactification of $(X, \delta_{\mathcal{M}})$. Then $L^\infty(X, \mathcal{M})$ is isomorphic as a Banach-algebra to $C(\mathcal{C})$. One can easily show that $\mathcal{C}$ is the collection of all ultrafilters on the Boolean algebra $\mathcal{M}$. The maximal ideal space of $L^\infty(X, \mathcal{M})$ is therefore homeomorphic to the collection of all ultrafilters on $\mathcal{M}$. Furthermore, from these facts, one can easily show that if $(X, \mathcal{M}, \mu)$ is a measure space, then the maximal ideal space of $L^\infty(\mu)$ is homeomorphic to the collection of all ultrafilters on the quotient Boolean algebra $\mathcal{M}/\{R \in \mathcal{M} | \mu(R) = 0\}$.

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