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by

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Abstract. The Hilbert-Smith conjecture asks if a p-adic group can act effectively upon an n-manifold. Recently, John Pardon proved this conjecture for $n = 3$. For non-manifold spaces there are a variety of p-adic actions to be found in the literature (for example A. N. Dranishnikov showed that there are free p-adic actions on Menger manifolds). The goal of this paper is to further explore the more general setting of Peano continua for p-adic actions. The first result is that every effective p-adic action on a Peano continuum admits an equivariant partitioning. The second main result is: any map from a simply connected continuum, such as an arc, can be lifted from the orbit space of such an action to the top space. Examples are constructed of invariant sub-continua.

1. Introduction

The long standing Hilbert-Smith conjecture asks whether or not a p-adic group can act effectively on a manifold. While John Pardon recently proved this result in the case of 3 dimensional manifolds [6], the conjecture remains open for higher dimensions. With this conjecture in mind, we address the consequences of a p-adic group acting on a Peano continuum.

A great deal of study has gone into properties of the orbit space of p-adic actions. If the group of p-adic numbers, A_p, acts effectively on a locally connected Hausdorff space X, then the orbit space (or quotient space) satisfy $\dim_{\mathbb{Z}} X/A_p \leq 3 + \dim_{\mathbb{Z}} X$ [10], where $\dim_{\mathbb{Z}} X$ denotes the integral cohomology dimension. If the space X is compact that bound can be tightened to $\dim_{\mathbb{Z}} X/A_p \leq 2 + \dim_{\mathbb{Z}} X$ [8]. If the space X is a

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manifold, then the quotient space must have integral cohomological dimension $\dim_{\mathbb{Z}} {}^{X}/_{A_p} = 2 + \dim_{\mathbb{Z}} X$ [10]. This equality also holds in the case where X is an ANR when the A_p-action is free [4]. When the quotient map raises integral cohomological dimension, it is known that the quotient space cannot be dimensionally full-valued [8, 4]. It is the goal of this paper to add to these well known results.

Every Peano continuum, or continuous curve, is known to be partitionable [2]. We show that whenever a p-adic group acts effectively on a Peano continuum, the continuous curve can be partitioned into members that finitely permute under the group action.

If X is a Peano continuum upon which the p-adic numbers act effectively, this paper will demonstrate the existence of lifts of arcs from the quotient space of the action. Likewise, lifts exist for any simply connected sub-continua of the orbit space. There is an isomorphism between the higher homotopy groups $\pi_n(X) \cong \pi_n({}^{X}/_{A_p})$ for all $n \geq 2$.

In considering the possibility of a p-adic action on a manifold of higher dimension than 3, the last section of the paper restricts down to the case where the A_p action is a free action. If X is a Peano continuum that cannot be locally separated by any one-dimensional set, then for every point $x \in X$, there are specific invariant subsets of X containing the point x. These subsets include p-adic solenoids, p^k distinct p-adic solenoids for any natural number k, the space $A_p \times S^1$, and the Menger curve μ^1. The sub-action of the Menger curve is similar, but not identical to the non-dimension raising action given in the paper by A. Dranishnikov [4].

2. Notation

Let $\mathbb{Z}$ denote the integers and $\mathbb{N}$ denote the strictly positive integers. Let $\mathbb{Z}_k := {}^{\mathbb{Z}}/_{k\mathbb{Z}}$ denote the integers modulo k. For a given prime number p, let $A_p := \varprojlim\{\mathbb{Z}_{p^n}, \phi_n^{n+1}\}$ denote the group of p-adic numbers, where the maps $\phi_n^{n+1} : \mathbb{Z}_{p^{n+1}} \to \mathbb{Z}_{p^n}$ are the group homomorphisms obtained by taking modulo p^n, thus $|\ker \phi_n^{n+1}| = p$. The p-adic numbers are endowed with the topology of a Cantor set, and can be topologically generated by a single element. When there is need to pick one of these elements that topologically generates A_p, it will be denoted by $\tau \in A_p$. Proper, nontrivial subgroups of A_p can then be written in the form $\tau^{p^k} A_p$ for each choice of $k \in \mathbb{N}$ (these sub-groups are independent of the choice of τ). Denote by $\Delta^k := \tau^{p^k} A_p$ the subgroup of A_p with index p^k.

When A_p acts on a space X then there naturally arises a system of maps $\{p_n : {}^{X}/_{\Delta^n} \to {}^{X}/_{\Delta^{n-1}}\}$:

$$X \cdots \xrightarrow{p_5} {}^{X}/_{\Delta^4} \xrightarrow{p_4} {}^{X}/_{\Delta^3} \xrightarrow{p_3} {}^{X}/_{\Delta^2} \xrightarrow{p_2} {}^{X}/_{\Delta^1} \xrightarrow{p_1} {}^{X}/_{A_p}. \tag{2.1}$$

The maps $p_n : {}^{X}/_{\Delta^n} \to {}^{X}/_{\Delta^{n-1}}$ are induced by the homomorphisms ϕ^n_{n-1}. If the A_p action is free, then each of the maps p_n is a p-to-1 covering map. While if the A_p action is merely effective on X, then some of the maps are only branched covering maps. For each $n \in \mathbb{N}$, define the p^n-to-1 branched covering map $P_n := p_1 \circ p_2 \circ \cdots \circ p_{n-1} \circ p_n$. Define $\pi_0 : X \to {}^{X}/_{A_p}$ to be the quotient map of the A_p action and, for each $n \in \mathbb{N}$, let the map $\pi_n : X \to {}^{X}/_{\Delta^n}$ be the quotient map induced by the Δ^n-subgroup action, meaning the composition $P_n \circ \pi_n = \pi_0$.

3. Equivariant Partitions of Peano continuua

A Peano continuum, X, is defined as a compact, connected, locally connected metric space. Since this criteria is equivalent to saying that the space X is the continuous image of an arc, Peano continua are also referred to as continuous curves. We remind the reader of some concepts of partitions as defined by Bing:

Definition 3.1 (Bing [1])**.**

(1) A partitioning of X is a finite collection of mutually exclusive connected domains whose union is dense in X.
(2) A partitioning μ of X is called a brick partitioning if each of the elements of μ is uniformly locally connected and equal to the interior of its closure while the interior of the closure of the union of two adjacent elements of μ is connected and uniformly locally connected.
(3) If each element of μ is of diameter less than ϵ, μ is an ϵ-partitioning.
(4) The brick partitioning ν is a core refinement of the brick partitioning μ if
   (a) ν is a refinement of μ,
   (b) for each pair of adjacent elements u', u'' of μ there is a pair of adjacent elements v', v'' of ν in u' and u'' respectively such that $\bar{v}' \cup \bar{v}''$ is a subset of the interior of $\bar{u}' \cup \bar{u}''$, and
   (c) for each element u of μ, the elements of ν in u may be ordered $v_0, v_1, \ldots, v_n$ such that $\bar{v}_0$ intersects each $\bar{v}_i$, while $\bar{v}_i$ intersects the boundary of u if and only if $i > 0$.
   We call v_0 a core element and $v_i, i > 0$ border elements.

In 1949, R.H. Bing proved that every continuous curve is partitionable [2]. Moreover every partitionable set admits a sequence of brick $\frac{1}{n}$-partitions $\{\mu_n\}_{n=1}^{\infty}$ such that μ_{k+1} core refines μ_k[1].

If there is an effective p-adic group action on a Peano continuum X, then a natural question is to ask whether or not there are partitions of X that respect the group action. This section demonstrates that, for every $\epsilon > 0$, the space X can be partitioned by partitionable sets of diameter less than ϵ such that the group action merely finitely permutes these sets.

Theorem 3.2 (e.g. Theorem 3.7.2 in Engelking[5])**.** *If $f : X \to Y$ is a perfect mapping, then for every compact subspace $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.*

Lemma 3.3. *Let X and Y be connected, locally connected metric spaces. If $f : X \to Y$ is a perfect, light open map and $U \subset Y$ is open with the property that $\bar{U}$ is compact and locally connected, then $V := f^{-1}(U)$ has finitely many components.*

Proof. Pick a point $y \in U$. Since the mapping f is continuous, the set $V \subset X$ is open. Since X is locally connected, for each point $x \in W := f^{-1}(y) \subset V$, there is a connected open set $O_x \subset V$. Since the mapping f is perfect, the set W is compact. The collection $\{O_x\}$ is an open cover of W, so there are a finite sub-collection of open sets O_{x_i} which cover W. Since each O_{x_i} is connected, there are finitely many components of V containing points of W, which implies that each of those components is both open and closed in V. Since f is an open map, the image of each component is open. Likewise, since f is a perfect mapping, it is a closed map, which implies the image of each component is also closed. Since U is connected, the image of each component is all of U.

Thus, the pre-image V has only finitely many components. □

Theorem 3.4. *Let X and Y be connected, locally connected metric spaces. If $f : X \to Y$ is a perfect, light open map and $U \subset Y$ is open, with the property that $\bar{U}$ is compact and locally connected, then $f^{-1}(\bar{U})$ is also compact and locally connected.*

Proof. From Theorem 3.2, the pre-image $V := f^{-1}(\bar{U})$ is compact. Thus, it suffices to show that V has property S. By Bing's partitioning theorems [2], this condition is equivalent to V being partitionable.

Since $\bar{U}$ is compact and locally connected, $\bar{U}$ has property S, and is thus partitionable. Let $G := \{g_i\}$ be a partitioning of $\bar{U}$. From Lemma 3.3, the finite collection $\{h \subset V|\ h$ is a component of $f^{-1}(g_i)$, for some $g_i \in G\}$ forms a partitioning, H, of V. It suffices to show that the size of the elements of partitions of V can be controlled.

Choose a point $y \in \bar{U}$. Since f is a light mapping, the set $f^{-1}(y)$ is 0-dimensional, thus totally disconnected. Given $\epsilon > 0$; cover $f^{-1}(y)$ by a finite number of disjoint open sets of diameter less than $\frac{\epsilon}{3}$. Denote this cover by $\{J_i\}$.

Since f is an open map, consider the open neighborhood $K_y := \bigcap f(J_i)$ of the point y. The set $f^{-1}(K_y) \subset \bigcup J_i$ is open, contains the points $f^{-1}(y)$, and the components of $f^{-1}(K_y)$ have diameter less than $\frac{\epsilon}{3}$.

Since the point y was an arbitrarily chosen point in $\bar{U}$, form an open cover $\{K_y\}_{y \in \bar{U}}$ of $\bar{U}$. Since $\bar{U}$ is compact, take a finite sub-cover of $\{K_y\}$, and denote it by $\{K_i\}$. Let $\delta > 0$ be the minimum distance between non-intersecting members of the finite closed cover $\{\bar{K}_i\}$.

Let G be a δ-partition of $\bar{U}$, and let H be the associated partition of V under the mapping f. Since any member, $g \in G$, will lie in the star of the finite sub-cover, $\{\bar{K}_i\}$, the collection H is at most an ϵ partition of V.

Since ϵ was arbitrary, the set V is partitionable thus V has property S, which implies that it is locally connected as desired. □

In the specific case of a p-adic group acting on a Peano continuum, Theorem 3.4 and Lemma 3.3 provide a means to construct equivariant partitions. Just as the group of p-adic numbers can be seen as an inverse limit of finite p-power groups, we demonstrate that a p-adic action on a Peano continuum can be approximated by finite permutations. We see from Corollary 3.5 below that every effective p-adic action on a Peano continuum (including any possible counter-example to the Hilbert-Smith conjecture) admits a refining sequence of partitions each with a corresponding finite action on its members.

Corollary 3.5 (Equivariant Partitions)**.** *If the group of p-adic numbers, A_p acts on a Peano continuum X, then for each $\epsilon > 0$, there is an ϵ-partition of X, where the group A_p acts on the members of the partition by finite permutation.*

Proof. Since the quotient map $\pi_0 : X \to {}^{X}\!/_{A_p}$ is continuous and X is a Peano continuum, the quotient space $Y := {}^{X}\!/_{A_p}$ is also a Peano continuum. Moreover, the map π_0 is a perfect, light open map, and both X and Y are connected, locally connected, compact metric spaces, so the criteria for Theorem 3.4 and Lemma 3.3 are satisfied.

Since Y is a Peano continuum, by Bing's partitioning theorems [2], the space Y can be partitioned. Let $G = \{g_i\}$ be a partition of Y. From Lemma 3.3, the finite collection $H := \{h \subset X |\ h$ is a component of $\pi_0^{-1}(g_i)$, for some $g_i \in G\}$ forms a partitioning of X. For a given $g_i \in G$, the full pre-image $h_i := \pi_0^{-1}(g_i) \subset X$ is fixed by the group A_p. Since, by Lemma 3.3, each h_i has only finitely many components, the group A_p finitely permutes the components of h_i, and thus the members of the partition H.

Just as in the proof of Theorem 3.4, the size of elements of these induced partitions of X can be controlled, and the result follows. □

It is easy to generalize this result (and those from the rest of the paper) to more arbitrary zero-dimensional compact groups. Pontryagin showed that every zero-dimensional compact group is the inverse limit of finite groups [7]. It is easy to see that for a given zero-dimensional compact group, C, either C is finite or C has the topology of a Cantor set. When C is not finite, the group C is called a *Cantor Group*. In either case, write $C = \varprojlim\{C_i, \psi_i^{i+1}\}$ where

(1) each C_i is a finite group,
(2) $C_0 = \{e\}$ is the trivial group,
(3) $C_i \leq C_{i+1}$ with equality if and only if $C_i = C$, and
(4) for each $i \geq 0$, the map $\psi_i^{i+1} : C_{i+1} \to C_i$ is an onto homomorphism.

The group of p-adic numbers is a Cantor group: $A_p = \varprojlim\{\mathbb{Z}_{p^n}, \phi_n^{n+1}\}$.

Corollary 3.6. *If $C = \varprojlim\{C_i, \psi_i^{i+1}\}$ is a zero-dimensional compact group acting on a Peano continuum X, then for each $\epsilon > 0$, there is an ϵ-partition of X, where the group C acts on the members of the partition by finite permutation.*

4. Lifting Arcs and Homotopies

While some preliminary work in this section is done in more abstraction, the motivation is still the setting where a p-adic group, A_p, acts on a space X. While the results will also hold for any zero-dimensional compact group, the focus will be when the group is a p-adic group.

Although all of the maps $p_n : {}^{X}/_{\Delta^n} \to {}^{X}/_{\Delta_{n-1}}$ are covering or branched covering maps, the projection $\pi_0 : X \to {}^{X}/_{A_p}$ is not, in general, a covering map. Nevertheless, some standard results on covering spaces can still be obtained. It is always possible to lift an arc from the quotient space ${}^{X}/_{A_p}$ to an arc in X. There is an isomorphism between the higher homotopy groups- that is $\pi_n(X) \cong \pi({}^{X}/_{A_p})$ for all $n \geq 2$.

Theorem 4.1 (Whyburn [9]). *Let $T(A) = B$ be a light interior transformation, where A is compact. Then if pq is any simple arc in B and p_0 is any point in $T^{-1}(p)$, there exists a simple arc p_0q_0 in A such that $T(p_0q_0) = pq$ and T is topological on p_0q_0.*

In other words, if $p : X \to Y$ is a light open perfect map, $A \subset Y$ is an arc, and $a \in A$, then given a point $\alpha \in p^{-1}(a)$ there is an arc $\mathcal{A} \subset X$ such that $\alpha \in \mathcal{A}$, $p(\mathcal{A}) = A$, and $p|_{\mathcal{A}}$ is an embedding. This shows us that arcs can be lifted from the quotient space.

Corollary 4.2. *If a p-adic group, A_p, acts on a space X, the set $A \subset {}^{X}/_{A_p}$ is an arc, the point $a \in A$, and the map $\pi_0 : X \to {}^{X}/_{A_p}$ is the quotient map induced by the group action, then given any $\alpha \in \pi_0^{-1}(a)$ there is an arc $\mathcal{A} \subset X$ such that $\pi_0(\mathcal{A}) = A$, $\pi_0(\alpha) = a$, and $\pi_0|_{\mathcal{A}}$ is an embedding.*

In the case of a free action, a stronger statement is possible.

Theorem 4.3. *If A_p acts freely on X and if $\pi_0 : X \to {}^{X}/_{A_p}$ is the quotient map, then for any arc $A \subset {}^{X}/_{A_p}$, the pre-image $\pi_0^{-1}(A) \cong A_p \times [0,1]$.*

Proof. Let $h : [0,1] \hookrightarrow X/A_p$ be a parameterization of the arc $A := h([0,1]) \subset X/A_p$.

For each point $a \in h^{-1}(0)$, there is an arc $\mathcal{A}_a \subset X$, by Corollary 4.2. Define the map $h_a : [0,1] \hookrightarrow X$ so that $h_a(t) \in \pi_0^{-1}(h(t)) \cap \mathcal{A}_a$. Since $\pi_0|_{\mathcal{A}_a}$ is an embedding, the map h_a is a well-defined embedding.

Fix the point $a \in h^{-1}(0)$, and let $\tilde{A} := \cup_{g\in A_p} g(\mathcal{A}_a)$. Since $\mathcal{A}_a \subset \tilde{A}$, the set $\pi_0(\tilde{A}) = A$. If a point $x \in \pi_0^{-1}(A)$, then $\pi_0(x) \in A$ thus there is a parameter $t_x \in [0,1]$ such that $\pi_0(x) = h(t_x)$. Let $y = h_a(t_x) \in \mathcal{A}_a$. Since $\pi_0(y) = \pi_0(h_a(t_x)) = h(t_x) = \pi_0(x)$, there is a $g_x \in A_p$ such that $g_x(y) = x$. Since $g_x(\mathcal{A}_a) \subset \tilde{A}$, the point $x \in \tilde{A}$ and thus $\tilde{A} = \pi_0^{-1}(A)$.

Suppose, for sake of contradiction, there is a nontrivial element $g \in A_p$ such that there is a point $x \in \mathcal{A}_a \cap g(\mathcal{A}_a)$. Let $t_a \in [0,1]$ such that $h_a(t_a) = x$, and hence such that $h(t_a) = \pi_0(x)$. Since g is a homeomorphism, there is a parameterization $h_g : [0,1] \to g(\mathcal{A}_a)$ such that $g(h_a(t)) = h_g(t)$. Since $\pi_0(g(z)) = p(z)$ for all $z \in X$, it follows that $\pi_0(h_a(t)) = \pi_0(g(h_a(t)) = \pi_0(h_g(t))$ for all $t \in [0,1]$. Let $t_b \in [0,1]$ such that $x = h_g(t_b)$. Since h is an embedding and $h(t_a) = \pi_0(x) = \pi_0(h_g(t_b)) = \pi_0(g(h_a(t_b))) = \pi_0(h_a(t_b)) = h(t_b)$, then the parameter $t_a = t_b$. Since $x = h_g(t_b) = h_g(t_a) = g(h_a(t_a)) = g(x)$ and g is non-trivial, this is a contradiction with A_p acting freely. Since the intersection $\mathcal{A}_a \cap g(\mathcal{A}_a) = \emptyset$ for every $g \in A_p \setminus \{e\}$, the pre-image $\pi_0^{-1}(A) \cong A_p \times [0,1]$. □

While the quotient map $\pi_0 : X \to X/A_p$ is not a covering map (in general), the maps $p_n : X/\Delta^n \to X/\Delta^{n-1}$ are p-fold branched covers. Given a mapping h into the quotient space X/A_p, a sequence of lifts of h to the quotient spaces X/Δ^n can be constructed in many cases in order to obtain a lifting of h to X.

Theorem 4.4. *If a p-adic group, A_p, acts on a space X, the map $\pi_0 : X \to X/A_p$ is the quotient map induced by the group action, and the map $h : K \to X/A_p$ is a continuous map from a simply connected compact space K, then there is a lift of this map $\hat{h} : K \to X$ such that $\pi_0 \circ \hat{h} = h$.*

Proof. Since the map h is continuous, the space K is compact, and the map π_0 is perfect, the set $Y := \pi_0^{-1}(h(K))$ is compact. Let τ be a generator of the group A_p. The restriction $\tau_{|Y}$ is uniformly continuous, and $\tau_{|Y}^{p^n} \xrightarrow{n\to\infty} 1_Y$ uniformly.

Pick a basepoint $k_0 \in K$ and then fix a basepoint $x_\omega \in \pi_0^{-1}(h(k_0)) \subset Y \subset X$. From the inverse sequence of p-fold covering maps on the quotient spaces X/Δ^n (labeled (2.1) on page 2) there is, for our basepoint $x_\omega \in X$, a corresponding sequence $\{x_n\}_{n=0}^\infty$ such that $x_0 := h(k_0)$, $\pi_n(x_\omega) = x_n$, and $p_n(x_n) = x_{n-1}$ for all $n \in \mathbb{N}$. Likewise with this choice fixed, since K is simply connected there is a sequence of maps $\{\mathcal{H}_n : K \to X/\Delta^n\}_{n=0}^\infty$ such that $\mathcal{H}_n(k_0) = x_n \in X/\Delta^n$ and $P_n \circ \mathcal{H}_n = h$.

Define $\hat{h} : K \to Y \subset X$ by $\hat{h}(k) \in \bigcap \pi_n^{-1}\mathcal{H}_n(k)$, for any $k \in K$. The map $\hat{h}$ is well-defined, since

- $\lim_{n\to\infty} \operatorname{diam}(\pi_n^{-1}\mathcal{H}_n(k)) = 0$,
- the set $\pi_{n+1}^{-1}\mathcal{H}_{n+1}(k) \subset \pi_n^{-1}\mathcal{H}_n(k)$, for each $n \in \mathbb{N}$, and
- the set $\pi_n^{-1}\mathcal{H}_n(k)$ is compact, for each $n \in \mathbb{N}$.

The map $\hat{h}$ is continuous because the map $\mathcal{H}_N$ is continuous and that for any $\epsilon > 0$, there is an $N > 0$ such that $\|\tau^{p^N}_{|Y}\|_\infty < \frac{\epsilon}{3}$. □

Corollary 4.5. *If a p-adic group, A_p, acts on a space X, the map $\pi_0 : X \to {}^X/_{A_p}$ is the quotient map induced by the group action, the map $h : [0,1] \to {}^X/_{A_p}$ is a path in the quotient space, and the map $H : [0,1] \times [0,1] \to {}^X/_{A_p}$ is a homotopy in the quotient space where $h(t) = H(t,0)$ for each $t \in [0,1]$, then there is a path $\hat{h} : [0,1] \to X$ such that $\pi_0 \circ \hat{h} = h$ and a homotopy $\hat{H} : [0,1] \times [0,1] \to X$ such that $\hat{h}(t) = \hat{H}(t,0)$ for each $t \in [0,1]$ and $\pi_0 \circ \hat{H} = H$.*

Proof. Both the spaces $[0,1]$ and $[0,1] \times [0,1]$ are compact simply connected spaces. □

Corollary 4.6. *If a p-adic group, A_p, acts on a space X, then for any $n \geq 2$ there is an isomorphism of the higher homotopy groups $\pi_n(X) \cong \pi_n({}^X/_{A_p})$.*

Proof. For any $n \geq 2$, the space S^n is compact and simply connected. □

5. Invariant Sets

The motivation for this section, like the entirety of the paper, is the Hilbert-Smith conjecture. While the results of this section extend to more arbitrary Cantor groups, the attention is restricted to p-adic groups. Should a counter-example exist to the conjecture, then it would perforce have certain properties. The aim here is to add to those known properties.

In this section, the action is assumed to be a free action. This assumption could be lessened to an action that is free except on a set that nowhere locally separates the space without significantly lessening the results. While the requirement in the theorems below that a finite number of arcs nowhere locally separate the quotient space might seem like an imposing restriction, in the context of trying to find a counter-example for the Hilbert-Smith conjecture, it is a trivial requirement. It is already known that such a counter-example would have to be a manifold of dimension four or higher, and as such cannot be locally separated by any one dimensional subset.

The pair of lemmas below serve as the workhorse for this section. They show that, between any two points, x and y, one can construct an arc J that can be made to not only avoid the remainder of the orbits of its end points, but also to avoid its own orbit (excluding the possibility that x is in the orbit of y). The remainder of the section demonstrates what can be made from such constructions by looking at the full orbit of these constructed arcs. The section concludes with a construction of a p-adic invariant Menger curve from these arcs.

Lemma 5.1. *If a p-adic group, A_p acts freely on a space X, then for any $x, y \in X$ such that there is a connected property S sub-space $Y \subset X$ with $x, y \in Y$ whose quotient ${}^{Y}\!/_{A_p}$ cannot be locally separated by a finite number of arcs there is, for every subgroup $\Delta^h \leq A_p$, an arc $J \hookrightarrow X$ from x to some $z \in \Delta^h y$ with $J \cap (A_p x \cup A_p y) = \{x, z\}$ and $g(J) \cap J \subset \{x, z\}$ for every $g \in A_p$. Moreover, if $L \subset {}^{X}\!/_{A_p}$ is a finite union of arcs, then the arc J can be chosen such that $\pi_0(J) \cap L \subset \{\pi_0(x), \pi_0(y)\}$.*

Proof. Given $x, y \in Y \subset X$ and $\Delta^h \leq A_p$.

Let $\pi_h : A_p Y \to {}^{Y}\!/_{\Delta^h}$ and $\pi_0 : A_p Y \to {}^{Y}\!/_{A_p}$ be the quotient maps induced by the action of A_p. Since ${}^{Y}\!/_{G} = {}^{GY}\!/_{G}$ for any group G acting on X, they may be used interchangeably in this proof. Since Y is connected with property S, the quotient spaces ${}^{Y}\!/_{A_p}$ and ${}^{Y}\!/_{\Delta^h}$ are as well. There is a p^h-to-1 covering map $P_h : {}^{Y}\!/_{\Delta^h} \to {}^{Y}\!/_{A_p}$, since A_p is acting freely on X (and hence on $A_p Y \subset X$).

If the point $x \in \Delta^h y$, then the trivial, degenerate arc would suffice, so without loss of generality assume that $x \notin \Delta^h y$. Denote by $x_1 := \pi_h(x)$ and $x_2 := \pi_h(y)$. Since $x \notin \Delta^h y$, immediately $x_1 \neq x_2$. Since P_h is a covering map and ${}^{Y}\!/_{\Delta^h}$ has property S, there is a $\delta > 0$ such that, for every $w \in {}^{Y}\!/_{\Delta^h}$, the distance $d(w, \tau^m(w)) \geq \delta$ for all $1 \leq m < p^h$.

Since ${}^{Y}\!/_{A_p}$ has property S, there is a brick partition, ν of ${}^{Y}\!/_{A_p}$ such that, for each $N \in \nu$, the pullback $P_h^{-1}(N)$ has exactly p^h components each of which having diameter less than δ in ${}^{Y}\!/_{\Delta^h}$. Moreover the collection of these components forms the induced brick partition μ of ${}^{Y}\!/_{\Delta^h}$ where each partition member $M \in \mu$ has exactly order p^h under the free $\mathbb{Z}_{p^h}$ group action induced by A_p. This can be done so that $x_1 \in Int(M)$ for some unique $M \in \mu$.

Since the space ${}^{Y}\!/_{\Delta^h}$ is connected and has property S, the space is arcwise connected. Let $\{M_i\}_{i=1}^n \subset \mu$ be a minimal (with respect to n) chain, in order, connecting the points x_1 to x_2. Thus $x_1 \in M_1$, $x_1 \notin M_i$ for $i > 1$, $x_2 \in M_n$, $x_2 \notin M_j$ for $j < n$, and if $\partial M_k \cap \partial M_j \neq \emptyset$, then $|k - j| \leq 1$.

Since A_p acts on μ as a free $\mathbb{Z}_{p^h}$ action, for any $0 < k < p^h$ and any $1 \leq i, j \leq n$, if $\tau^k(M_j) \cap M_i$ contains a non-empty open set, then $\tau^k(M_j) = M_i$.

Let $B_0 := \{x_1\}$. For $1 \leq i \leq n$, let $B_i \subset \partial M_i \cap \partial M_{i+1}$ be the largest set such that if $b \in B_i$ and $b \in \partial M$, then either $M = M_i$ or $M = M_{i+1}$. Since μ is a brick partition, each set B_i is non-empty and a local separator.

Let $B'_i = \bigcup_{j=1}^{i} B_j$.

Let $K_0 := \{x_2\}$ and $L_0 := \{\tau^l(K_0)\}_{l=1}^{p^h-1}$. Since L_0 is a finite collection of points, it is not a local separator thus $B_1 \setminus L_0$ is non-empty. Pick K_1 to be an arc in M_1 from the point $k_0 := x_1$ to some other point $k_1 \in B_1 \setminus L_0$ such that $K_1 \cap B'_1 = \{k_1\}$. Define $L_1 := L_0 \cup \bigcup_{l=1}^{p^h-1} \tau^l(K_1)$. Since $L_1 \setminus L_0 \subset {}^{Y}\!/_{\Delta^h} \setminus M_1$, $K_1 \cap L_0 = \emptyset$, and $K_1 \cap \partial M_1 = \{k_1\}$, the intersection $L_1 \cap K_1 = \emptyset$.

Suppose, for some k with $1 \leq k < n$, there are, for all $1 \leq i \leq k$, arcs $K_i \subset M_i$ going from $k_{i-1} \in B_{i-1}$ to $k_i \in B_i$ such that the set of obstacles $L_k := \bigcup_{j=0}^{k} \bigcup_{l=1}^{p^h-1} \tau^l(K_j)$ consists of a finite number of disjoint arcs and isolated points, the union $K'_k := \bigcup_{j=1}^{k} K_j$ is an arc from x_0 to k_k, the intersection of the two $K'_k \cap L_k = \emptyset$, and $K'_k \cap B'_k = \{k_j\}_{j=1}^{k}$.

Since

$$\big(L_k \setminus L_0\big) \cap B'_{n-1} \subseteq \bigcup_{l=1}^{p^h-1} \tau^l(K'_k \cap B'_{n-1}) = \bigcup_{l=1}^{p^h-1} \tau^l(K'_k \cap B'_k) = \bigcup_{l=1}^{p^h-1} \bigcup_{j=1}^{k} \tau^l(k_j)$$

is a finite set of points, it is not a local separator. The remainder of the set $B_{k+1} \setminus L_k$ is non-empty.

Pick $k_{k+1} \in B_{k+1} \setminus L_k$.

Since L_k is a finite number of arcs and isolated points, it does not locally separate M_{k+1}. Define K_{k+1} to be an arc from $k_k \in B_k$ to $k_{k+1} \in B_{k+1}$ such that $K_{k+1} \cap L_k = \emptyset$ and $K_{k+1} \cap B'_{k+1} = \{k_k, k_{k+1}\}$. Denote by $L_{k+1} := L_k \cup \bigcup_{l=1}^{p^h-1} \tau^l(K_{k+1})$. Observe that $L_{k+1} \cap K'_{k+1} = \emptyset$ follows from the union $L_k \cup K'_k$ being the full A_p-orbit of K'_k.

By finite induction, define $K' := K'_n$, where the endpoint k_n is taken to be x_2.

Let J be the arc (obtained from Corollary 4.2) that is the lift of K' starting at the point $x \in Y$ and the lemma is done. Moreover, given some finite set of obstacle arcs L, simply realize that the set $Y \setminus L \cup \{\pi_0(x), \pi_0(y)\}$ has property S and has the same local separator conditions so $Y \setminus L \cup \{\pi_0(x), \pi_0(y)\}$ could have been used in the proof instead of Y. $\square$

While the prior lemma could only draw an arc within a subgroup of the desired target, the following one sequentially uses the first to obtain an arc going to the exact desired endpoint.

Lemma 5.2. *If a p-adic group, A_p acts freely on a space X, then for any $x, y \in X$ such that there is a connected property S sub-space $Y \subset X$ with $x, y \in Y$ whose quotient ${}^{Y}\!/_{A_p}$ cannot be locally separated by a finite number of arcs, there is an arc $J \hookrightarrow X$ from x to y with $J \cap (A_p x \cup A_p y) = \{x, y\}$ and $g(J) \cap J \subset \{x, y\}$ for every $g \in A_p$. Moreover, if $L \subset {}^{X}\!/_{A_p}$ is a finite union of arcs, then the arc J can be chosen such that $\pi_0(J) \cap L \subset \{\pi_0(x), \pi_0(y)\}$.*

Proof. Let $x, y \in Y \subset X$ as above. The statement is trivial if $x = y$, so without loss of generality assume that $x \neq y$ and let $d_0 := d(x, y)$. Since Y has property S, Y is partitionable. Let ν_0 be a ${}^{d_0}\!/_{2}$-partition of Y such that $y \in \mathring{Y}_1$ for some $Y_1 \in \nu_0$.

Pick a subgroup $\Delta^{h_0} \leq A_p$ such that $\Delta^{h_0} y \subset \mathring{Y}_1$. By Lemma 5.1, there is an arc J_0 from x to some point $y_0 \in \Delta^{h_0} y \subset \mathring{Y}_1$.

Define J_0' to be a sub-arc of J_0 from x to ∂Y_1 such that $J_0' \cap \partial Y_1 =: \{x_1\}$ is a singleton.

Let $d_1 := \min\{\frac{d_0}{2}, d(x_1, y)\}$. Since Y_1 has property S, there is a $\frac{d_1}{2}$-partition, ν_1 of Y_1, such that $y \in \mathring{Y}_2$ for some $Y_2 \in \nu_1$. Pick a subgroup $\Delta^{h_1} \leq \Delta^{h_0}$ such that $\Delta^{h_1} y \subset \mathring{Y}_2$ and again apply Lemma 5.1 to obtain an arc J_1 from x_1 to some point $y_1 \in \Delta^{h_1} y \subset \mathring{Y}_2$ such that $J_0' \cap A_p J_1 = \{x_1\}$.

Now define J_1' to be the arc J_0' together with the sub-arc of J_1 from x_1 to ∂Y_2 such that $J_1' \cap \partial Y_2$ is a singleton, $\{x_2\}$ and repeat this process ad infinitum to obtain a sequence of arcs $(J_k')_{k=0}^{\infty}$, diameters $(d_k)_{k=0}^{\infty}$, and endpoints $(y_k)_{k=0}^{\infty}$ thereof.

Since the sequence (d_k) is summable, the limit $J := \lim J_n'$ is an arc. Since $y_k \to y$, the arc J runs from x to y. By construction, the arc J has the desired properties. $\square$

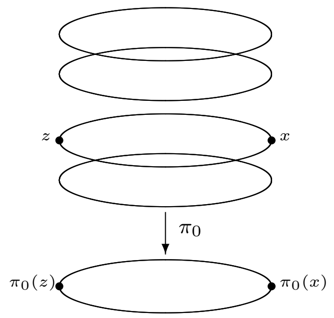
Applying this lemma when the endpoint y is in the orbit of x and then seeing the full orbit of the resulting arc, some interesting sub-continua can be formed upon which A_p, perforce, acts freely.

Theorem 5.3. *If a p-adic group, A_p acts freely on a space X with the property that for every pair of points $x, y \in X$ there exists a Peano continuum $Y \subset X$ containing both points and the quotient ${}^{Y}\!/_{A_p}$ cannot be locally separated by a finite number of arcs, then for every $x \in X$, there is a space $Z_x \subset A_p Y \subset X$ with $x \in Z_x$, the restriction $(A_p)_{|Z_x}$ is a free action, and can be constructed so that Z_x and ${}^{Z_x}\!/_{A_p}$ are homeomorphic to:*

(1) *the product $A_p \times S^1$ and the circle S^1 respectively, where the group A_p simply acts on the first factor,*
(2) *a solenoid and a circle respectively, or*
(3) *p^k disjoint solenoids and a circle respectively.*

Proof. Let $\tau \in A_p$ be a generator for the group.

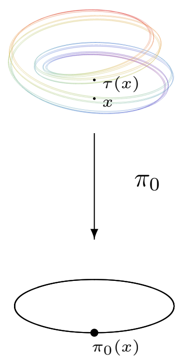
(1) Pick $z \notin A_p(x)$. Applying Lemma 5.2, there is an arc J_1 from x to z such that $\pi_0(J_1)$ is an arc. Applying the lemma again, there is an arc J_2 from z to x such that $\pi_0(J_2)$ is an arc, the intersection $J_1 \cap J_2 = \{x, z\}$, and $\pi_0(J_1 \cup J_2)$ is a simple closed curve. Take $Z_x := A_p(J_1 \cup J_2)$. The space $Z_x \cong A_p \times S^1$ and the orbit space ${}^{Z_x}/_{A_p} \cong S^1$.



$S^1 \times A_p \cong Z_x \subset X$

$S^1 \cong \pi_0(Z_x) \subset {}^X/_{A_p}$

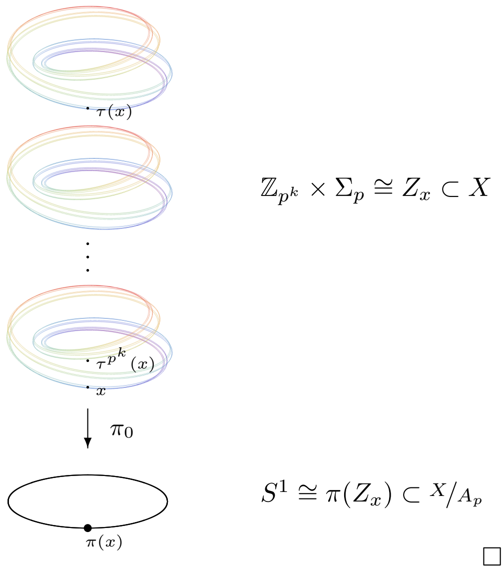
(2) Using Lemma 5.2, there is an arc J from x to $\tau(x)$ where $Z_x := A_p(J) \cong \Sigma_p$ is a p-adic solenoid. Moreover, the orbit space ${}^J/_{A_p} = {}^{Z_x}/_{A_p}$ is a simple closed curve.



$\Sigma_p \cong Z_x \subset X$

$S^1 \cong \pi_0(Z_x) \subset {}^X/_{A_p}$

(3) Using Lemma 5.2, there is an arc J from x to $\tau^{p^k}(x)$. The full image $Z_x := A_p(J) \cong \mathbb{Z}_{p^k} \times \Sigma_p$ forms p^k distinct p-adic solenoids. Again the orbit space ${}^J/_{A_p} \cong S^1$ is a simple closed curve.



□

A more complicated example is constructed below using, as building blocks, the invariant spaces formed in parts 2 and 3 of Theorem 5.3 (namely p-adic solenoids). Given a free p-adic action on a Peano continuum, X, where X cannot be locally separated by a one dimensional subset, we show that at each and every point, x of the continua, there is a group invariant Menger curve μ_x. Lessening the condition that the action be free to a merely effective action on X but one such that the space X cannot be locally separated by any one dimensional subset together with the set of periodic points would only reduce the result to invariant Menger curves existing at every point $x \in X$ where the action was free. It is interesting that the quotient ${}^{\mu_x}\!/_{A_p}$ is one dimensional even in the case should the map $\pi_0 : X \to {}^{X}\!/_{A_p}$ raise dimension.

Theorem 5.4. *If a p-adic group, A_p, acts freely on a Peano continuum X such that ${}^{X}\!/_{A_p}$ cannot be locally separated by a finite number of arcs, then, for each point $x \in X$, there is a Menger curve μ containing x, where the restriction $(A_p)_{|\mu}$ is a free p-adic action.*

Proof. We label some types of continua: a subspace, V, of ${}^{X}\!/_{A_p}$ will be of type I, if it is homeomorphic to an arc with two distinct endpoints in the ambient space, of type Q if it is homeomorphic to a space of type I together with a circle that is tangent at a point in the interior of that arc, and of type X if it is homeomorphic to the union of two spaces of type I that have a single point of intersection and that single point occurs in the interior of both arcs. The naming scheme for the types should be indicative of the appearance of the respective V.

Let $x \in X$. From Theorem 5.3, there is a simple closed curve, $W_0 \subset {}^{X}/_{A_p}$ such that $\Sigma_x := \pi_0^{-1}(W_0)$ is a solenoid upon which A_p acts freely. For sake of a later induction, let $\Omega'_{-1} := \{{}^{X}/_{A_p}\}$ be the trivial brick partition of ${}^{X}/_{A_p}$, let $B_{-1} = C_0 = \emptyset$, and let $L_0 = 0$.

Since W_0 is a simple closed curve in a Peano continuum ${}^{X}/_{A_p}$, choose Ω'_0 to be a brick 1-partition of ${}^{X}/_{A_p}$ such that there is a sub-collection Ω_0 with the following properties:

(1) $W_0 \cap \bar{\omega} = \emptyset$ for any $\omega \in (\Omega'_0 \setminus \Omega_0)$.

(2) For each $\omega \in \Omega_0$, the intersection $W_0 \cap \omega$ is of type I, having boundary points α_ω and β_ω.

Let $B_0 := \{\alpha_\omega | \omega \in \Omega_0\} \cup \{\beta_\omega | \omega \in \Omega_0\}$.

For each $\omega \in \Omega_0$, let $K_\omega \subset \omega$ be the core element in a core refinement of ω constructed such that $K_\omega \cap W_0 \neq \emptyset$. Pick $c_\omega \in K_\omega \cap W_0$. There is an $l_\omega \geq 0$ such that each component of the pre-image $\pi_0^{-1}(K_\omega)$ is invariant under Δ^{l_ω} . Pick $L_1 \geq 1$ such that $L_1 \geq \max\{l_\omega | \omega \in \Omega_0\}$. From Theorem 5.3, there is a simple closed curve $J_\omega \subset \bar{K}_\omega \subset \omega \subset {}^{X}/_{A_p}$ with base point c_ω such that $J_\omega \cap W_0 = \{c_\omega\}$ and $\pi_0^{-1}(J_\omega)$ is exactly p^{L_1} solenoids that permute under A_p.

Let $C_1 := \{c_\omega | \omega \in \Omega_0\}$, and let $W_1 := W_0 \cup \bigcup_{\omega \in \Omega_0} J_\omega$.

For some $k \geq 1$, assume that for all $1 \leq n \leq k$:

(1) There is a set $C_n \subset {}^{X}/_{A_p}$ that is finite, with $C_{n-1} \subset C_n$ and $C_{n-1} \neq C_n$.

(2) There is a natural number $L_n > L_{n-1}$.

(3) The space $W_n \neq W_{n-1}$, with $W_{n-1} \subset W_n \subset {}^{X}/_{A_p}$, is a continuum decomposed into a finite number of simple closed curves J_i, such that if $J_i \subset W_{n-1}$ and $J_i \neq W_{n-1}$, then $J_i \cap W_{n-1} \subset C_n \setminus C_{n-1}$ is a singleton and $\pi_0^{-1}(J_i) \subset X$ is p^{L_n} solenoids.

(4) The collection Ω'_{n-1} is a brick 2^{1-n}-partition of ${}^{X}/_{A_p}$ that refines Ω'_{n-2} such that the sub-collection $\Omega_{n-1} := \{\omega \in \Omega'_{n-1} | \omega \cap W_{n-1} \neq \emptyset\}$ satisfies $W_{n-1} \cap \bar{\omega} = \emptyset$ for any $\omega \in (\Omega'_{n-1} \setminus \Omega_{n-1})$

(5) Moreover, the partition Ω'_{n-1} is constructed such that, for any $\omega \in \Omega_{n-1}$, the sub-space $W_n \cap \omega$ is a space that is either of type Q, or of type X having boundary $W_n \cap \partial\omega$ being respectively either two or four isolated points.

(6) The set of all these boundary points B_{n-1}, where $B_{n-2} \subset B_{n-1} := \cup_{\omega \in \Omega_{n-1}} \{x \in W_n | x \in \partial\omega\}$, locally separate W_n.

Let Ω'_k be a brick 2^{-k}-partition refining Ω'_0 such that the sub-collection $\Omega_k := \{\omega \in \Omega'_k |\ \omega \cap W_k \neq \emptyset\}$ satisfies:

(1) $W_k \cap \bar{\omega} = \emptyset$, for any $\omega \in (\Omega'_k \setminus \Omega_k)$.

(2) If $\omega \in \Omega_k$, then $\omega \cap W_k$ is either of type I or of type X.

(3) If $\omega_1, \omega_2 \in \Omega_k$ such that $\bar{\omega}_1 \cap \bar{\omega}_2 \cap W_k \neq \emptyset$, then the intersection is a singleton and at most one of these two sets is of type X.

Let $B_k := B_{k-1} \cup \bigcup_{\omega \in \Omega_k} W_k \cap \partial\omega$ be a finite number of points that obviously locally separates W_k.

Let $O_k := \{\omega \in \Omega_k | \omega \cap W_k \text{ is of type I}\}$. For each $\omega \in O_k$, let $K_\omega \subset \omega$ be the core element in a core refinement of ω such that $K_\omega \cap W_k \neq \emptyset$. Pick $c_\omega \in K_\omega \cap W_k$. Choose a number $L_{k+1} > L_k$ such that, for every K_ω, each component of the pre-image $\pi_0^{-1}(K_\omega)$ is invariant under $\Delta^{L_{k+1}}$. Using Theorem 5.3, there is a simple closed curve, $J_\omega \subset \bar{K}_\omega \subset \omega \subset X/A_p$ with base point c_ω such that $J_\omega \cap W_k = \{c_\omega\}$ and $\pi_0^{-1}(J_\omega)$ is exactly $p^{L_{k+1}}$ solenoids that permute under A_p.

Let $W_{k+1} := W_k \cup \bigcup_{\omega \in \Omega_k} J_\omega$, and let $C_{k+1} := C_k \cup \{c_\omega | \omega \in O_k\}$. This completes the induction, and so there is a strictly increasing sequence of natural numbers $\{L_n\}$ as well as sets $W := \bigcup_{k=0}^{\infty} W_k$, $C := \bigcup_{k=0}^{\infty} C_k$, and $B := \bigcup_{k=0}^{\infty} B_k$.

Let $\mu := \pi_0^{-1}(W)$. Bestvina's characterization of a Menger curve is that it is the unique connected and locally connected compactum of dimension 1 having the disjoint arc property[3].

Claim 1. The space μ has covering dimension $\dim \mu = 1$.

Define $\mathcal{B} := \{\{\omega \in \Omega_i\} \cup \bigcup_{b \in B_i} St(b, \Omega_i) | i \in \mathbb{N}\}$. Since $\mathrm{mesh}(\Omega_i) \to 0$ as $i \to \infty$, the collection $\mathcal{B}$ forms a property S basis for W. For any $B \in \mathcal{B}$, the boundary ∂B is a finite number of points, thus W has covering dimension $\dim W = 1$.

For each $B \in \mathcal{B}$, the pre-image $\pi_0^{-1}(B)$ consists of a finite number of open sets and the collection $\mathcal{B}' := \{B' \subset X | B' \text{ is a component of } \pi_0^{-1}(B)$ for some $B \in \mathcal{B}\}$ forms a basis for μ. Since, for each $B \in \mathcal{B}$, the boundary ∂B is a finite set, for any $B' \in \mathcal{B}'$ the boundary $\partial B'$ is a finite number of Cantor sets and is thus 0-dimensional. Therefore, we have $\dim \mu = 1$.

Claim 2. The space μ is connected.

Pick points $y_1, y_2 \in \mu$. There is an $n \in \mathbb{N}$ such that $\pi_0(y_i) \in W_n$ for both $i = 1, 2$. There is an arc joining $\pi_0(y_i)$ to W_0 which lifts to arcs in μ starting at y_i for both $i = 1, 2$, and since the pre-image $\pi_0^{-1}(W_0) = \Sigma_x$ is the solenoid which is connected, the points y_1 and y_2 lie in the same component of μ. Since y_1, y_2 were an arbitrary pair of points in μ, the space μ is connected.

Claim 3. The space μ is locally connected.

Using the bases $\mathcal{B}$ and $\mathcal{B}'$ defined in claim 1, it suffices to show that each $B' \in \mathcal{B}'$ has only finitely many components. This further reduces to showing that each $B \in \mathcal{B}$ has only finitely many components, which in turn reduces to showing that for every $i \in \mathbb{N}$, the intersection $\omega \cap W$ has finitely many components for every $\omega \in \Omega_i$.

For an arbitrary $\omega \in \Omega_i$, either $\omega \cap W_i$ is of type I (and thus is an arc) or of type X (and thus four arcs meeting in a common point). In either case, $\omega \cap W_i$ is connected. Let $w \in \omega \cap W$, then $w \in W_n$ for some $n \geq i$. From the construction of W_n, finitely many arcs in ω join w to W_i. Since w was chosen arbitrarily, $\omega \cap W$ is connected. Hence every $B \in \mathcal{B}$ is connected, and thus every $B' \in \mathcal{B}$ has finitely many components and μ is locally connected.

Claim 4. The space μ has the Disjoint Arc Property (DD^1P).

Let $f_1, f_2 : [0,1] \to \mu$ be arcs in μ. Given $\epsilon > 0$, there is an $n \in \mathbb{N}$ that is so large that the following hold true:

(1) The projection of the starting point $\pi_0 f_i(0) \in W_n$ for $i = 1, 2$.
(2) The loop size (i.e. $\mathrm{mesh}(\Omega'_{n-1})$) is at most $2^{-n} < \frac{\epsilon}{2}$.
(3) The subgroup $\Delta^{L_n} \leq A_p$ is such that $d(x, gx) < \frac{\epsilon}{2}$ for every $x \in X$ and $g \in \Delta^{L_n}$.

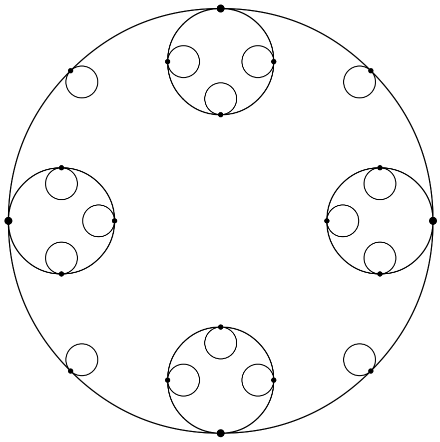
Consider the arc projections $\pi_0 f_i : [0,1] \to W$ for $i = 1, 2$, then approximate each by W_n obtaining arcs $f'_i : [0,1] \to W_n$ for $i = 1, 2$. For any $g \in \Delta^{L_n}$ and $i = 1, 2$, any lift $\hat{f}_i : [0,1] \to \mu$ of f'_i taking $f'_i(0)$ to $g f_i(0)$ is such that $d(f_1, \hat{f}_i) < \epsilon$.

Since the image of each f'_i can be decomposed into finitely many simply connected pieces in W_n which lift uniquely up to choice of base-point and there are uncountably many choices for lifts by choosing a base-point by $g \in \Delta^{L_n}$, it follows that there are lifts $\hat{f}_1, \hat{f}_2$ such that $d(f_i, \hat{f}_i) < \epsilon$ and $\hat{f}_1 \cap \hat{f}_2 = \emptyset$. Thus μ has the Disjoint Arc Property (DD^1P).

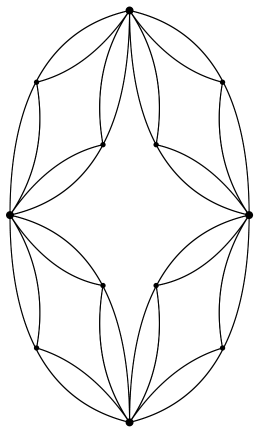
Since μ is a compact, connected, locally connected 1-dimensional metric space with the disjoint arc property, using Bestvina's characterization of the Menger curve [3], the space μ is a Menger curve. $\square$

Thus Theorem 5.4 produces, for each $x \in X$, a A_p-invariant Menger curve. If we restrict the free A_p-action to one of these Menger curves, we obtain a free A_p action on μ^1 with a one-dimensional orbit space. This is similar, but not identical, to the free A_p action on μ^1 obtained by A. N. Dranishnikov [4], and later described by Zhiqing Yang [11]. To readily see that the two actions are different, simply compare the orbit spaces of the respective actions. Below is a representation of each orbit space up to the third stage of each of their constructions. The orbit space ${}^{\mu^1}/_{A_p}$ from Theorem 5.4 on the left has cut points, while the orbit space from Dranishnikov's action on the right only has local cut points.

${}^{\mu^1}\!/_{A_p}$ from Theorem 5.4



Dranishnikov's ${}^{\mu^1}\!/_{A_p}$ [4, 11]



With slight modifications to the construction in Theorem 5.4, the following can be constructed without difficulty:

Corollary 5.5. *The Menger curve μ can be constructed so that the inherited free A_p action on it is precisely the one described by A.N. Dranishnikov* [4, 11].

Corollary 5.6. *The Menger curve, μ, can be constructed in such a fashion that there is a point $x \in \mu$ such that $A_p x$ locally separates μ but no other orbit does so.*

Corollary 5.7. *The Menger curve, μ, can be constructed in such a fashion that the orbit space of this sub-action, ${}^{\mu}\!/_{A_p} \cong \mu^1$ is a Menger curve as well.*

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