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PATH COMPONENTS IN THE UNIFORM SPACES OF CONTINUOUS FUNCTIONS INTO A NORMED LINEAR SPACE

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Abstract. In this paper, we study the path components in relation to two uniform topologies on $C(X,Y)$, where $C(X,Y)$ denotes the space of all continuous functions from a Tychonoff space X to a normed linear space Y. As a consequence of this study, we show that these two uniform topologies on $C(X,Y)$ are not homeomorphic. In particular, we show that in one case $C(X,Y)$ is pathwise connected, and in the other, $C(X,Y)$ has uncountably many path components. But we show that these two uniform topologies on the space $C^*(X,Y)$, the set of all bounded members of $C(X,Y)$, coincide.

1. Introduction

For two topological spaces X and Y, let $C(X,Y)$ denote the set of continuous functions from X into Y. When $Y = \mathbb{R}$, the space of real numbers, we write $C(X)$ instead of $C(X,\mathbb{R})$. The set $C(X,Y)$ has a number of natural topologies, such as point-open, compact-open and uniform topologies (see, for example, [4] and [7]). The space $C(X,Y)$ with the point-open topology and compact-open topology is denoted by $C_p(X,Y)$ and $C_k(X,Y)$, respectively. In order to define a uniform topology, it is necessary for Y to have some uniform structure. When Y is a metric space with compatible bounded metric d, then $C(X,Y)$ with the uniform topology generated by d is denoted by $C_d(X,Y)$ and is a metric space. Also $C_d(X,Y)$ is a complete metric space if and only if Y is a complete metric space under d.

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In this paper, we study the properties of $C(X,Y)$ with two different uniform topologies, where X is a Tychonoff space and Y is a normed linear space. To define these two uniform topologies on $C(X,Y)$, we use two different compatible bounded metrics ρ and τ on Y. Metric ρ is the natural metric obtained from the norm on Y, but metric τ is a less natural bounded metric obtained by using the arctan function. To characterize the path components of the space $C_\rho(X,Y)$, we define a natural equivalence relation $\sim$ on $C_\rho(X,Y)$, and show that the path components of $C_\rho(X,Y)$ are precisely the equivalence classes of $\sim$; and that whenever X is not pseudocompact, there are uncountably many such equivalence classes. On the other hand, we show that the space $C_\tau(X,Y)$ is path connected, and that hence the uniform spaces $C_\rho(X,Y)$ and $C_\tau(X,Y)$ are not homeomorphic. This shows that the uniform topology $C_d(X,Y)$ depends on the choice of compatible bounded metric d used on Y. We end by showing that the uniform topologies on the subspace $C^*(X,Y)$ of bounded members of $C(X,Y)$ are the same with respect to the two metrics ρ and τ; that is, $C^*_\rho(X,Y)$ and $C^*_\tau(X,Y)$ are homeomorphic.

The definition of the uniform topology on $C(X,Y)$ is given in the second section for general metric space Y. Also in this section, we give the proof that the metric space $C_d(X,Y)$ is only separable for Y separable and X compact and metrizable, showing that the functions spaces in which we are working are in general non-separable metric spaces. In addition, in this section, the two metrics ρ and τ are defined for a normed linear space Y. Then in the third section, we study the path connectedness of spaces $C_\rho(X,Y)$ and $C_\tau(X,Y)$ and characterize their path components.

2. Preliminaries

Let X be a Tychonoff space, and let Y be a metric space with bounded metric d. Define metric d_∞ on C(X,Y) by

$$d_\infty(f,g) = \sup\{d(f(x),g(x)) : x \in X\}$$

for all $f,g \in C(X,Y)$. The space $C(X,Y)$ with the metric topology defined from d_∞ is denoted by $C_d(X,Y)$, and this topology on $C(X,Y)$ is called the uniform topology with respect to d. As we shall see, the topological properties of $C_d(X,Y)$ depend on the choice of compatible bounded metric d used on Y.

Since the space $C_d(X,Y)$ is a metric space, the conditions of its being second countable, being separable, being Lindelöf and having a countable chain condition are all equivalent. The following theorem characterizes these countability properties of $C_d(X,Y)$ in terms of properties of X and Y, and was partially proved in each of [2] and [3].

For a normed linear space Y, the notation $C^*(X,Y)$ refers to the subset of $C(X,Y)$ consisting of the members whose images are bounded sets in Y with respect to the norm.

Theorem 2.1. *If Y is a metric space with compatible bounded metric d and Y contains a nontrivial path, then $C_d(X,Y)$ is separable if and only if Y is separable and X is compact and metrizable. Furthermore, if Y is a normed linear space and the subspace $C_d^*(X,Y)$ is separable, then the entire space $C_d(X,Y)$ is separable.*

Proof. First let Y be separable and X be compact and metrizable. Then $C_d(X,Y)$ is equal to the space $C_k(X,Y)$ with compact-open topology (see [5]). Now suppose $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ are countable bases for X and Y respectively. Then consider the countable family of open sets $\{[\overline{U_n}, V_m] : n,m \in \mathbb{N}\}$ in $C_k(X,Y)$, where $[\overline{U_n}, V_m] = \{f \in C(X,Y) : f(\overline{U_n}) \subseteq V_m\}$. Then $\{[\overline{U_n}, V_m] : n,m \in \mathbb{N}\}$ is a subbase for $C_k(X,Y) = C_d(X,Y)$, and hence $C_d(X,Y)$ is separable.

Now let $C_d(X,Y)$ be separable and for each $y \in Y$, define a function $i_y : X \to Y$ such that $i_y(x) = y$ for all $x \in X$. Clearly, $i_y \in C(X,Y)$. By Theorem 2.1.1 of [4] the function $f : Y \to C_d(X,Y)$ defined by $f(y) = i_y$ is a closed embedding. So, Y is separable.

Now let βX denote the Čech-Stone compactification of X and let $e : X \to \beta X$ denote the embedding of X into βX such that $\overline{e(X)} = \beta X$. Then the function $e^* : C_d(\beta X, Y) \to C_d(X,Y)$ defined by $e^*(f) = f \circ e$ is an embedding by Theorem 2.2.9 in [4], consequently, $C_d(\beta X, Y) = C_k(\beta X, Y)$ is separable. Since Y contains the nontrivial path, there is an embedding $q : \mathbb{R} \to Y$. Then the function $q_* : C_k(\beta X) \to C_k(\beta X, Y)$ defined by $q_*(f) = q \circ f$ is an embedding (see Theorem 2.2.5 in [4]). Since $C_k(\beta X)$ is separable and metrizable, $C_k(\beta X)$ has a countable dense subset D.

Now the function $\psi : \beta X \to \mathbb{R}^D$ defined by $\psi(x)(f) = f(x)$ for all $x \in \beta X$ and $f \in D$ is a continuous injection. But βX is compact, so ψ must be a closed map, and thus an embedding. Therefore βX is metrizable but βX is metrizable only when X is already compact and metrizable. For the second part of the theorem we just need to observe that in the above proof each $i_y \in C_d^*(X,Y)$ and e^* maps $C_d(\beta X, Y)$ into $C_d^*(X,Y)$. □

Now we study the space $C_d(X,Y)$ when Y is a normed linear space with norm $||\cdot||$. For example, Y can be $\mathbb{R}^n$, $C_k([0,1])$ or $C_\infty^*(\mathbb{R})$. The norm on Y gives a compatible metric on Y defined by $d(x,y) = ||x-y||$ for all $x, y \in Y$. But this metric d, induced by the norm on Y is not a bounded metric, and d_∞ is not finite-valued. So d_∞ does not define a metric on $C(X,Y)$.

Now let us consider the subset $C^*(X,Y)$ of $C(X,Y)$ consisting of all bounded members of $C(X,Y)$. In this case, $C^*(X,Y)$ has a norm $||\cdot||_\infty$ defined by $||f||_\infty = \sup\{||f(x)|| : x \in X\}$. We denote the normed linear space $C^*(X,Y)$ with above norm by $C^*_\infty(X,Y)$. One can prove that whenever Y is a Banach space, the space $C^*_\infty(X,Y)$ is a Banach space.

But in this paper our focus is on two different uniform topologies on the full space $C(X,Y)$ that are generated by two different natural compatible bounded metrics on the normed linear space Y. The first of these two metrics is easy to define and is a commonly used metric.

Define the compatible bounded metric ρ on Y by $\rho(x,y) = \min\{1, ||x-y||\}$ for all $x, y \in Y$. Then $C_\rho(X,Y)$ is the space $C(X,Y)$ with uniform topology generated by ρ_∞. Let $C^*_\rho(X,Y)$ be $C^*(X,Y)$ with the subspace topology from $C_\rho(X,Y)$. Then $C^*_\rho(X,Y) = C^*_\infty(X,Y)$. So that $C^*_\infty(X,Y)$ is a subspace of $C_\rho(X,Y)$. In fact we show that it is both an open as well as closed subset of $C_\rho(X,Y)$.

To define another compatible bounded metric on Y, let us first consider continuous maps $T : [0, \frac{\pi}{2}) \to [1, \infty)$ and $A : [0, \infty) \to (0, 1]$ defined by

$$T(x) = \begin{cases} \frac{\tan x}{x} & 0 < x < \frac{\pi}{2} \\ 1 & x = 0 \end{cases},$$

and

$$A(x) = \begin{cases} \frac{\arctan x}{x} & 0 < x \\ 1 & x = 0 \end{cases}.$$

Define the map $\phi : Y \to B(\mathbf{0}, 1)$ such that $\phi(y) = \frac{2}{\pi} A(||y||)y$, where $B(\mathbf{0}, 1)$ is the unit open ball in Y centered at the zero element $\mathbf{0}$ of Y, that is, $B(\mathbf{0}, 1) = \{y \in Y : ||y|| < 1\}$.

Proposition 2.2. *The function $\phi : Y \to B(\mathbf{0}, 1)$ is a homeomorphism.*

Proof. The continuity of ϕ is immediate from the continuity of the function A, norm function and from the fact that composition of two continuous functions is again continuous. To show ϕ is a bijection it is enough to show that $\phi^{-1}(y) = \frac{\pi}{2} T(\frac{\pi}{2}||y||)y$ is the inverse of ϕ. Let $y \neq 0$, then $\phi^{-1}(y) = \tan(\frac{\pi}{2}||y||)\frac{y}{||y||}$. So

$$\begin{aligned} \phi(\phi^{-1}(y)) &= \frac{2}{\pi} \frac{\arctan ||\phi^{-1}(y)||}{||\phi^{-1}(y)||} \phi^{-1}(y) \\ &= \frac{2}{\pi} \frac{\arctan \tan(\frac{\pi}{2}||y||)}{\tan(\frac{\pi}{2}||y||)} \tan(\frac{\pi}{2}||y||) \frac{y}{||y||} \\ &= y \end{aligned}$$

Similarly $\phi^{-1}\phi(y) = y$ for all $y \in Y$. This shows that ϕ^{-1} is the inverse of ϕ. The continuity of ϕ^{-1} follows from the same argument as that of ϕ. $\square$

Define a bounded compatible metric τ on Y using homeomorphism ϕ as follows: $\tau(x,y) = ||\phi(x) - \phi(y)||$ for all $x, y \in Y$. Since ϕ is a homeomorphism, τ is compatible with the topology of Y. Then let $C_\tau(X,Y)$ be the space C(X,Y) with the uniform metric topology generated by τ_∞.

Let $\phi_* : C(X,Y) \to C(X, B(\mathbf{0},1))$ denote the induced map defined by $\phi_*(f) = \phi f$ for all $f \in C(X,Y)$. Since $C(X, B(\mathbf{0},1))$ is subset of the normed linear space $C^*_\infty(X,Y)$, we consider $C(X, B(\mathbf{0},1))$ with the subspace topology of $C^*_\infty(X,Y)$, and denote it by $C_\infty(X, B(\mathbf{0},1))$.

3. Path Connectedness of C(X,Y) with uniform topologies

Define an equivalence relation $\sim$ on $C_\rho(X,Y)$ by $f \sim g$ provided that $||f - g||_\infty = \sup\{||f(x) - g(x)|| : x \in X\} < \infty$. Let $E(g)$ denote the equivalence class of the above equivalence relation containing the element $g \in C(X,Y)$. Note that $E(f_0) = C^*_\infty(X,Y)$, where $f_0(x) = \mathbf{0}$ for all $x \in X$.

Proposition 3.1. *Any two distinct members of the family $\{E(g) : g \in C(X,Y)\}$ are homeomorphic as subspaces of $C_\rho(X,Y)$.*

Proof. We show that each equivalence class $E(f)$ is homeomorphic to $E(f_0) = C^*_\infty(X,Y)$. Define $\eta : E(f) \to C^*_\infty(X,Y)$, such that $\eta(h) = f - h$. Clearly $f - h \in C^*_\infty(X,Y)$. Note that inverse of η is defined by $\eta^{-1}(h) = f + h$ for any $h \in C^*_\infty(X,Y)$. To show η is continuous, let $B_\rho(f - h; \varepsilon)$ be a basic neighborhood of $\eta(h)$ in $C^*_\infty(X,Y)$. Consider the neighborhood $B_\rho(h;\varepsilon)$ of h, and let $g \in B_\rho(h;\varepsilon)$. Without loss of generality we can assume $0 < \varepsilon < 1$, so $\rho(g(x), h(x)) = ||g(x) - h(x)|| < \varepsilon$ for all $x \in X$. Thus $||g(x) - h(x)|| = ||(f - h)(x) - (f - g)(x)|| < \varepsilon$ for all $x \in X$. Consequently, $\rho((f - h)(x), (f - g)(x)) < \varepsilon$ for all $x \in X$. This shows that $\eta(B_\rho(h;\varepsilon)) \subseteq B_\rho(f - h;\varepsilon)$ and hence η is continuous. A similar argument shows that η^{-1} is continuous. Consequently, η is a homeomorphism. ☐

Proposition 3.2. *Each equivalence class is open as well as closed in $C_\rho(X,Y)$.*

Proof. Because of above proposition it is enough to show $C^*_\infty(X,Y)$ is both open and closed in $C_\rho(X,Y)$. Let $f \in C^*_\infty(X,Y)$ and $0 < \varepsilon < 1$. Consider basic neighborhood $B_\rho(f,\varepsilon)$ of f and let $g \in B_\rho(f,\varepsilon)$, which implies $\rho(f(x), g(x)) < \varepsilon$ for all x in X. So $\min\{1, ||f(x) - g(x)||\} < \varepsilon$ for all $x \in X$, and since $\varepsilon < 1$, we have $||f(x) - g(x)|| < \varepsilon$ for all $x \in X$. As $f \in C^*_\infty(X,Y)$, we have $||f(x)|| \leq M$ for all $x \in X$ and for some $M > 0$. So $||g(x)|| \leq ||f(x) - g(x)|| + ||f(x)|| \leq \varepsilon + M$ for all $x \in X$.

Hence $g \in C^*_\infty(X,Y)$. So $B_\rho(f,\varepsilon) \subseteq C^*_\infty(X,Y)$, and thus $C^*_\infty(X,Y)$ is open in $C_\rho(X,Y)$. Now to show $C^*_\infty(X,Y)$ is closed in $C_\rho(X,Y)$, let $f \in C_\rho(X,Y) \setminus C^*_\infty(X,Y)$. So there is no $M > 0$ such that $||f(x)|| \leq M$ for all $x \in X$. Let $0 < \varepsilon < 1$ and $g \in B_\rho(f,\varepsilon)$. Therefore $\rho(f(x), g(x)) < \varepsilon$ for all $x \in X$. This implies that $\min\{1, ||f(x) - g(x)||\} < \varepsilon$ for all $x \in X$; and since $\varepsilon < 1$, we have $||f(x) - g(x)|| < \varepsilon$ for all $x \in X$. If possible let $g \in C^*_\infty(X,Y)$, so $||g(x)|| \leq M$ for all $x \in X$ and for some $M > 0$. So $||f(x)|| \leq ||f(x) - g(x)|| + ||g(x)|| \leq \varepsilon + M$ for all $x \in X$. But then $f \in C^*_\infty(X,Y)$, which is a contradiction. Hence $B_\rho(f,\varepsilon) \subseteq C_\rho(X,Y) \setminus C^*_\infty(X,Y)$, and thus $C^*_\infty(X,Y)$ is closed in $C_\rho(X,Y)$. □

Corollary 3.3. *The space $C_\rho(X,Y)$ is a topological sum of distinct members of the family $\{E(h) : h \in C(X,Y)\}$.*

Proposition 3.4. *The path components (and components) of the space $C_\rho(X,Y)$ are precisely the distinct members of the family $\{E(g) : g \in C(X,Y)\}$.*

Proof. Because of Propositions 3.1 and 3.2 it is enough to show that $E(f_0)$ is path connected. Let $f_0 \neq f \in E(f_0)$. Define a function $p : [0,1] \to E(f_0)$ by $p(t) = tf$, where $p(t)(x) = tf(x)$. Clearly $p(0) = f_0$ and $p(1) = f$. To show p is a path from f_0 to f, consider the basic neighborhood $B(p(t),\varepsilon)$ of $p(t)$, where $0 < \varepsilon < 1$. Choose $\delta = \frac{\varepsilon}{2||f||_\infty}$. Then for any $t' \in (t-\delta, t+\delta) \cap [0,1]$, we have $||p(t)(x) - p(t')(x)|| = |t - t'|||f||_\infty < \varepsilon$, and hence p is continuous. Thus $E(f_0)$ is path connected. □

By Corollary 3.3 above we have the following result related to connectedness of $C_\rho(X,Y)$.

Theorem 3.5. *If X is not pseudocompact, then $C_\rho(X,Y)$ is equal to the topological sum of uncountably many subspaces, each homeomorphic to the non-separable space $C^*_\infty(X,Y)$.*

Proof. It is enough to prove that when X is not pseudocompact, then there are uncountably many distinct equivalence classes of the equivalence relation $\sim$. Since X is not pseudocompact there exists a function $f \notin C^*(X)$. Let $0 \neq y \in Y$, and define for each $r \in (1,\infty)$ a function $g_r : X \to Y$ as $g_r(x) = f^r(x)y$ where $f^r(x) = \big(f(x)\big)^r$. Then $E(g_r) \neq E(g_s)$ for $r \neq s$, and $r, s \in (1,\infty)$. This is because for each $M > 1$, there exists $x \in X$ such that $f(x) > M$; then $||g_r(x) - g_s(x)|| > M$ whenever $r > s$. So for $r, s \in (1,\infty)$, $E(g_r) \neq E(g_s)$, and thus there are uncountably many equivalence classes when X is not pseudocompact. By proposition 3.1 and Theorem 2.1, each such equivalence class is homeomorphic to non separable space $C^*_\infty(X,Y)$. □

Corollary 3.6. *If X is not pseudocompact then $C_\rho(X,Y)$ is not connected.*

Now we study the connectedness of the space $C(X,Y)$ with uniform topology generated by the bounded compatible metric τ. Since ϕ is a homeomorphism by Proposition 2.2, the next Proposition shows that ϕ_* is also a homeomorphism.

Proposition 3.7. *The function $\phi_* : C_\tau(X,Y) \to C_\infty(X, B(\mathbf{0},1))$ is a homeomorphism.*

Proof. Let $B(\phi_*(f),\varepsilon)$ be a neighborhood of $\phi_*(f)$ in $C_\infty(X,B(\mathbf{0},1))$, where $f \in C_\tau(X,Y)$. Consider a neighborhood $B_\tau(f,\delta)$ of f, where $0 < \delta < \varepsilon$. Let $g \in B_\tau(f,\delta)$, so that $\tau(f(x),g(x)) = ||\phi(f(x)) - \phi(g(x))|| < \delta < \varepsilon$ for all x belonging to X. So, $\phi_*(g) \in B(\phi_*(f),\varepsilon)$, and hence ϕ_* is continuous. Now the function $\phi_*^{-1} : C_\infty(X,B(\mathbf{0},1)) \to C_\tau(X,Y)$ defined by $\phi_*^{-1}(f) = \phi^{-1}f$ is the inverse of ϕ_*. Let $B_\tau(\phi_*^{-1}(f),\varepsilon)$ be any basic neighborhood of $\phi_*^{-1}(f)$ in $C_\tau(X,Y)$, and let $g \in B(f,\varepsilon)$. Then $\sup\{||f(x) - g(x)|| : x \in X\} = \sup\{||\phi\phi^{-1}(f(x)) - \phi\phi^{-1}(g(x))|| : x \in X\} < \varepsilon$ and consequently, $\tau(\phi_*^{-1}(f),\phi_*^{-1}(g)) < \varepsilon$; thus $\phi_*^{-1}(g) \in B_\tau(\phi_*^{-1}(f),\varepsilon)$. This proves that ϕ_*^{-1} is continuous, and thus ϕ_* is a homeomorphism. □

Theorem 3.8. *The uniform space $C_\tau(X,Y)$ is path connected.*

Proof. From the above proposition, $C_\tau(X,Y)$ and $C_\infty(X,B(\mathbf{0},1))$ are homeomorphic. So, we shall prove that $C_\infty(X,B(\mathbf{0},1))$ is path connected. Let $f \in C_\infty(X,B(\mathbf{0},1))$. Define a function $p : [0,1] \to C_\infty(X,B(\mathbf{0},1))$ such that $p(t) = tf$. We show that p is continuous and hence a path between f and constant function f_0. Thus $C_\infty(X,B(\mathbf{0},1))$ is path connected. To show p is continuous, consider a neighborhood $B(tf,\varepsilon)$ of $p(t) = tf$ in $C_\infty(X,B(\mathbf{0},1))$ where $\varepsilon > 0$. Now choose a $0 < \delta < \varepsilon$, and consider $t' \in [0,1]$ such that $|t-t'| < \delta$. Then $||tf - t'f||_\infty = \sup\{||tf(x) - t'f(x)|| : x \in X\} = |t-t'|\sup\{||f(x)|| : x \in X\} < \delta < \varepsilon$. So, whenever $t' \in [0,1]$ and $|t-t'| < \delta$, then $t'f \in B(tf,\varepsilon)$. Therefore p is continuous. □

From Corollary 3.6 and Theorem 3.8 we see that whenever X is not pseudocompact the topological spaces $C_\rho(X,Y)$ and $C_\tau(X,Y)$ are not homeomorphic. This shows how the topological properties of the uniform space $C_d(X,Y)$ depend upon the compatible bounded metric d on Y. We should remark here that whenever X is pseudocompact then any two uniformities on Y generates the same uniform topology on C(X,Y) (see, for example, [6]). But the interesting fact is that $C_\rho^*(X,Y)$ and $C_\tau^*(X,Y)$ are homeomorphic. In order to prove this result, we need the following proposition. In this proposition $B_\infty(f_0,1)$ denotes the open ball in $C_\infty^*(X,Y)$ of radius 1 centered at the constant $\mathbf{0}$ function f_0.

Proposition 3.9. *The ball $B_\infty(f_0,1)$ is equal to $\phi_*(C^*(X,Y))$.*

Proof. Let $f \in C^*(X,Y)$, which implies that $||f||_\infty = M < \infty$. Now if $||y|| < M < \infty$, then $\arctan(||y||) \leq S < \frac{\pi}{2}$, so that $||\phi(y)|| = \frac{2}{\pi}\frac{\arctan(||y||)}{||y||}||y|| < \frac{2}{\pi}S < 1$ if $||y|| \neq 0$ and $||\phi(y)|| = 0$ if y = 0. Since $||f(x)|| < M$ for all x , this implies $||\phi f||_\infty < 1$, and thus $\phi_*(f) \in B_\infty(f_0,1)$. Therefore $\phi_*(C^*(X,Y))$ is a subset of $B_\infty(f_0,1)$.

Now suppose $f \in B_\infty(f_0,1)$, and let $g = \phi^{-1}f \in C^*(X,Y)$. Then $\phi_*(g) = f$, and thus $\phi_*(C^*(X,Y)) = B_\infty(f_0,1)$ since $g \in C^*(X,Y)$. We have $||\phi^{-1}f(x)|| = ||\frac{\pi}{2}T(\frac{\pi}{2}||f(x)||f(x))|| = ||\tan(\frac{\pi}{2}||f(x)||)||$ if $f(x) \neq 0$ and $||\phi^{-1}f(x)|| = 0$ if $f(x) = 0$. Now since $||f(x)|| < S < 1$ for all $x \in X$, this implies that $||\phi^{-1}f(x)|| < M$ for all $x \in X$ and for some $M > 0$. Thus $g \in C^*(X,Y)$. □

Remark 3.10. From the above proposition, we get $B_\infty(f_0,1)$ is a subset of $C_\infty(X,B(\mathbf{0},1))$. In fact, $B_\infty(f_0,1)$ is a dense subset of $C_\infty(X,B(\mathbf{0},1))$.

The next theorem is a consequence of Propositions 3.7, 3.9 and 2.2.

Theorem 3.11. *The space $C^*_\tau(X,Y)$ is homeomorphic to the normed linear space $C^*_\rho(X,Y)$.*

Proof. Propositions 3.7 and 3.9 implies that $C^*_\tau(X,Y)$ is homeomorphic to $B_\infty(f_0,1)$. Then by replacing Y by $C^*_\infty(X,Y)$ in Proposition 2.2 we get $C^*_\infty(X,Y)$ is homeomorphic to $B_\infty(f_0,1)$. Therefore $C^*_\tau(X,Y)$ is homeomorphic to $C^*_\infty(X,Y)$. □

The difference between $C^*_\tau(X,Y)$ and $C^*_\rho(X,Y)$ is how they are embedded in $C_\tau(X,Y)$ and $C_\rho(X,Y)$. In particular, $C^*_\tau(X,Y)$ is a dense subspace of $C_\tau(X,Y)$, while $C^*_\rho(X,Y)$ is an open and closed subspace of $C_\rho(X,Y)$.

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