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ABSTRACT. Let X be a Tychonoff space and $C_p(X)$ and $C_b(X)$ denote the space $C(X)$ of all real-valued continuous functions on X provided with the pointwise convergence and the compact-bounded topology, respectively. Let us call an unbounded subspace Σ of $\mathbb{N}^{\mathbb{N}}$ σ-complete if each bounded sequence $\{\alpha_n\}_{n=1}^{\infty}$ in Σ verifies that $\sup_{n\in\mathbb{N}} \alpha_n \in \Sigma$, and let us call σ-complete every family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X such that Σ is σ-complete and $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$. We show that if there is a dense subspace of X covered by a σ-complete family consisting of bounded sets then $C_p(X)$ is angelic. This is used to prove our main theorem, which asserts that if X has a dense subspace Y covered by a σ-complete family consisting of Y-bounded sets, then $C_b(X)$ is angelic and every compact set in $C_b(X)$ is metrizable.

1. Preliminaries

Let us start by recalling that a subset A of a topological space X is called (functionally) *bounded* [1, Chapter 0] if $f(A)$ is a bounded set in $\mathbb{R}$ for every real-valued continuous function f on X. On the other hand, a subset B of a topological vector space E is (linearly) *bounded* [9, Chapter 3] if B is absorbed by every neighborhood of the origin. In what follows, unless otherwise stated, X will be a Hausdorff completely regular space and $C_p(X)$, $C_c(X)$ and $C_b(X)$ will denote the space $C(X)$ of all real-valued continuous functions defined on X equipped with the pointwise convergence topology, the compact-open and the compact-bounded topology, respectively. We denote by $L(X)$ the topological dual of $C_p(X)$ and by $L_p(X)$ the linear space $L(X)$ endowed with the weak* topology.

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A locally convex space E belongs to the *class* $\mathfrak{G}$ if its topological dual E' has a covering $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$ and for each $\alpha \in \mathbb{N}^\mathbb{N}$ every sequence in A_α is equicontinuous [7, Chapter 11]. A space X is called *web-compact* if there is a mapping T from a subspace Σ of $\mathbb{N}^\mathbb{N}$ into $\mathcal{P}(X)$ such that $\overline{\bigcup \{T(\alpha) : \alpha \in \Sigma\}} = X$ and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n=1}^\infty$ has a cluster point in X [7, Chapter 4]. It turns out that each *quasi-Suslin* space [7, Chapter 3] is web-compact, and every *Lindelöf* Σ*-space* [1, Chapter 0] is web-compact and Lindelöf. The space $\mathbb{R}^\mathbb{R}$ is web-compact but not Lindelöf. A topological space X is *angelic* [5] if relatively countably compact sets in X are relatively compact and for every relatively compact subset A of X each point of $\overline{A}$ is the limit of a sequence of A. In angelic spaces (relatively) compact sets, (relatively) countably compact sets and (relatively) sequentially compact sets are the same.

A uniform space $(X, \mathcal{N})$ is called *trans-separable* if for every vicinity N of $\mathcal{N}$ there is a countable subset Q of X such that $\bigcup_{x \in Q} U_N(x) = X$, where $U_N(x) = \{y \in X : (x, y) \in N\}$, see [7, Section 6.4]. Separable uniform spaces and Lindelöf uniform spaces are trans-separable but the converse statements are not true in general, although for uniform pseudometrizable spaces trans-separability is equivalent to separability. Equivalently, a (Hausdorff) uniform space $(X, \mathcal{N})$ is trans-separable if it is uniformly isomorphic to a subspace of a uniform product of separable (pseudo) metric spaces. Each locally convex space in its weak topology is trans-separable when equipped with the (unique) translation-invariant uniformity associated to that topology. The class of trans-separable uniform spaces is hereditary, productive and closed under uniform continuous images. Moreover, it is shown in [3] that every uniform web-compact space is trans-separable, but if $X := [0, \omega_1)$ (where ω_1 is the first ordinal of uncountable cardinality) and

$$N_\gamma := \{(\alpha, \beta) : \alpha = \beta \text{ or } (\alpha \geq \gamma \text{ and } \beta \geq \gamma)\} \subseteq X \times X$$

for $0 \leq \gamma < \omega_1$, then the family $\{N_\gamma : 0 \leq \gamma < \omega_1\}$ is a base of a uniformity $\mathcal{N}$ for X such that $(X, \mathcal{N})$ is a trans-separable uniform space but $(X, \tau_\mathcal{N})$, where $\tau_\mathcal{N}$ stands for the uniform topology of $\mathcal{N}$, is not a web-compact topological space. Regarding to the interest of this paper, let us recall that trans-separability is 'dually' related to metrizability of compact sets in the sense that trans-separability of the space $C_c(X)$ is equivalent to metrizability of all compact subsets of X [7, Lemma 6.5] (see also [8, Theorem 2] for a generalization of this result to spaces of vector-valued continuous functions).

The following theorem extends some earlier results concerning metrizability of precompact sets in locally convex spaces due to Cascales, Orihuela, Pfister and Valdivia (see [7, Chapter 10] for details).

Theorem 1.1 (Cascales-Orihuela)**.** [7, Theorem 11.1] *Every precompact set in a locally convex space in the class $\mathfrak{G}$ is metrizable.*

The best result on this subject is the next theorem, which characterize those locally convex spaces whose precompact are metrizable in terms of trans-separability.

Theorem 1.2 (Ferrando-Kąkol-López Pellicer)**.** [7, Theorem 6.4] *Precompact subsets of a locally convex space E are metrizable if and only if E' endowed with the topology of uniform convergence on the precompact sets of E is trans-separable.*

Theorem 1.1 easily follows from Theorem 1.2 if one uses the following facts: (i) according to [4, Theorem 5], if E is a locally convex space in the class $\mathfrak{G}$ and τ_P denotes the topology on E' of uniform convergence on the precompact sets of E, then (E', τ_P) is quasi-Suslin, (ii) every quasi-Suslin locally convex space is web-compact, and (iii) as observed above, every web-compact space is trans-separable.

In what follows, motivated by Orihuela's angelicity theorem (see Theorem 1.3 below), we introduce the notion of σ-complete family (*cf.* Definition 2.1) and show that if X contains a dense subspace covered by a σ-complete family consisting of bounded sets, then $C_p(X)$ is angelic (Theorem 2.6). Then we prove that if X has a dense subspace Y covered by a σ-complete family consisting of Y-bounded sets, every compact set in $C_b(X)$ is metrizable (Theorem 2.8). We shall later on use the two following results.

Theorem 1.3 (Orihuela)**.** [10], [7, Theorem 4.5] *If X is web-compact then $C_p(X)$ is angelic.*

Theorem 1.4 (Ferrando-Kąkol-López Pellicer)**.** [7, Theorem 6.3] *Compact subsets of a locally convex space E are metrizable if and only if E' endowed with the topology of uniform convergence on the compact sets of E is trans-separable.*

2. Compact sets in $C_b(X)$

If X is a Hausdorff completely regular space, let us denote by $\mathcal{D}$ the admissible uniform structure for X generated by the family of pseudometrics $\{d_f : f \in C(X)\}$ with $d_f(x, y) = |f(x) - f(y)|$ for $x, y \in X$. According to [6, 15.14 Corollary, (a)], the uniform space $(X, \mathcal{D})$ is complete if and only if X is realcompact. Alternatively, X is realcompact if and only if every net $\{x_d : d \in D\}$ in X converges if $\lim f(x_d)$ exists for each $f \in C(X)$ (see [2, Theorem 2]).

Observe that if $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of the metric space $\mathbb{N}^{\mathbb{N}}$ then $\{\alpha_n(i)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{N}$ for each $i \in \mathbb{N}$, so that $\gamma(i) = \sup_{n\in\mathbb{N}} \alpha_n(i)$ belongs to $\mathbb{N}$ and $\alpha_n \leq \gamma$ for all $i \in \mathbb{N}$. Consequently, if a set X contains a family of sets $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, for any bounded sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{N}^{\mathbb{N}}$ and any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \in A_{\alpha_n}$ for each $n \in \mathbb{N}$ it holds that $x_n \in A_\gamma$ for all $n \in \mathbb{N}$. This fact motivates the following definition.

Definition 2.1. An unbounded subspace Σ of $\mathbb{N}^{\mathbb{N}}$ is called σ-complete if for each bounded sequence $\{\alpha_n\}_{n=1}^{\infty}$ in Σ it holds that $\sup_{n\in\mathbb{N}} \alpha_n \in \Sigma$. If Σ is a σ-complete subspace of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{A} = \{A_\alpha : \alpha \in \Sigma\}$ a family of subsets of a set X such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, we say that $\mathcal{A}$ is a σ-complete family of sets.

Example 2.2. *Some σ-complete subspaces of* $\mathbb{N}^{\mathbb{N}}$. Note that every σ-complete subspace Σ of $\mathbb{N}^{\mathbb{N}}$ is a directed subset of the ordered set $\left(\mathbb{N}^{\mathbb{N}}, \leq\right)$. The whole space $\mathbb{N}^{\mathbb{N}}$ is clearly σ-complete. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and every $n > 1$ the set $\Sigma(\alpha, n) = \{\gamma \in \mathbb{N}^{\mathbb{N}} : \gamma(i) \leq \alpha(i), \, i \geq n\}$ is also a σ-complete subspace of $\mathbb{N}^{\mathbb{N}}$. Setting $\gamma_n = (n, n+1, n+2, \ldots)$ then $\{\gamma_n : n \in \mathbb{N}\}$ is a countable σ-complete subspace of $\mathbb{N}^{\mathbb{N}}$.

Example 2.3. *Examples of σ-complete family of sets.* If Σ is a σ-complete subspace of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{B} = \{B_\alpha : \alpha \in \Sigma\}$ is an arbitrary family of subsets of a set X, setting

$$A_\alpha := \bigcup \{B_\beta : \beta \in \Sigma, \, \beta \leq \alpha\}$$

the family $\mathcal{A} = \{A_\alpha : \alpha \in \Sigma\}$ is clearly σ-complete and $B_\alpha \subseteq A_\alpha$ for every $\alpha \in \Sigma$.

Lemma 2.4. *If X contains a σ-complete family consisting of bounded sets whose union is dense in X, then the Hewitt realcompactification υX of X is web-compact.*

Proof. Let $\{A_\alpha : \alpha \in \Sigma\}$ be a σ-complete family of X-bounded sets in X covering a dense subspace Y of X. Let us start with some useful remarks. If $\delta : X \to L_p(X)$ stands for the canonical homomorphism $\delta(x) = \delta_x$ that embeds X into a (closed) subspace $\delta(X)$ of $L_p(X)$, the uniformity $\mathcal{D}$ for X is (uniformly) isomorphic to the relative uniformity on $\delta(X)$ induced by the uniform structure for $L(X)$ associated to its weak* topology, i.e. the (unique) translation invariant uniformity for $L(X)$ defined by the locally convex structure of $L_p(X)$. So, if necessary, we can identify X with $\delta(X)$ and $\mathcal{D}$ with the uniformity for $\delta(X)$ induced by the weak* topology of

$L(X)$. On the other hand, note that clearly A is a (functionally) bounded subset of X if and only if $\delta(A)$ is a (linearly) bounded set in $L_p(X)$. Hence, given that the weak* topology of $L(X)$ is a weak locally convex topology and every weakly bounded set in a locally convex space is precompact [9, 20.9 (3)], it follows that A is bounded if and only if $\delta(A)$ is a precompact subset of the locally convex space $L_p(X)$. In other words, a subset A of the topological space X is X-bounded if and only if A is a $\mathcal{D}$-precompact set of the uniform space $(X, \mathcal{D})$.

Let $(\widetilde{X}, \widetilde{\mathcal{D}})$ denote the completion of $(X, \mathcal{D})$ and $\tau_{\widetilde{\mathcal{D}}}$ be the uniform topology on $\widetilde{X}$ induced by the uniformity $\widetilde{\mathcal{D}}$ for $\widetilde{X}$. According to [6, 15.13 Theorem, (a)], the topological space $(\widetilde{X}, \tau_{\widetilde{\mathcal{D}}})$ coincides with υX. We claim that υX is a web-compact space.

If we define the map $T : \Sigma \to \mathcal{P}(\widetilde{X})$ by $T(\alpha) = A_\alpha$, on the one hand we have that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = Y$. Moreover, since Y is dense in X and X is dense in $(\widetilde{X}, \tau_{\widetilde{\mathcal{D}}})$, it follows that that $\overline{\bigcup\{T(\alpha) : \alpha \in \Sigma\}}^{\tau_{\widetilde{\mathcal{D}}}} = \widetilde{X}$. On the other hand, if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, since Σ is a σ-complete subspace of $\mathbb{N}^{\mathbb{N}}$ one has that $\sup \alpha_n \in \Sigma$. Hence, setting $\gamma(i) := \sup\{\alpha_n(i) : n \in \mathbb{N}\}$, then $\gamma \in \Sigma$ and $\alpha_n \leq \gamma$ for every $n \in \mathbb{N}$. Consequently, $A_n \subseteq A_\gamma$ and $x_n \in A_\gamma$ for every $n \in \mathbb{N}$. So, the fact that A_γ is $\mathcal{D}$-precompact, ensures that the sequence $\{x_n\}_{n=1}^{\infty}$ has a $\mathcal{D}$-Cauchy subnet in X which converges in $\widetilde{X}$ under the uniform topology $\tau_{\widetilde{\mathcal{D}}}$. Hence $(\widetilde{X}, \tau_{\widetilde{\mathcal{D}}})$ is web-compact. □

Remark 2.5. *It follows from the previous lemma that every realcompact space X that contains a σ-complete family consisting of bounded sets covering X is web-compact. Every non-separable Banach space E with closed unit ball B contains a σ-complete family of sets, namely $\{\alpha(1) B : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ (consisting of linearly bounded but not functionally bounded sets) which covers E. But E is not web-compact since it is not trans-separable.*

Theorem 2.6. *If X contains a σ-complete family consisting of bounded sets whose union is dense in X, then $C_p(X)$ is angelic.*

Proof. Lemma 2.4 and Theorem 1.3 combine to get that the space $C_p(\upsilon X)$ is angelic, which as is well known forces the space $C_p(X)$ to be angelic as well. □

The following lemma extends the beautiful main theorem of [11].

Lemma 2.7. *Let $(X, \mathcal{N})$ be a uniform space. If the space X has a σ-complete covering consisting of precompact sets, then $(X, \mathcal{N})$ is trans-separable.*

Proof. Let $\{A_\alpha : \alpha \in \Sigma\}$ be a σ-complete covering of X consisting of precompact sets. If $(\widetilde{X}, \widetilde{\mathcal{N}})$ stands for the completion of $(X, \mathcal{N})$ and $T : \Sigma \to \mathcal{P}(\widetilde{X})$ is defined in the usual way by $T(\alpha) = A_\alpha$, then $\overline{\bigcup\{T(\alpha) : \alpha \in \Sigma\}}^{\tau_{\widetilde{\mathcal{N}}}} = \widetilde{X}$. On the other hand, if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, the σ-completeness of Σ yields that $\sup \alpha_n \in \Sigma$. Setting $\gamma(i) := \sup\{\alpha_n(i) : n \in \mathbb{N}\}$ as in the proof of Lemma 2.4, then $\gamma \in \Sigma$ and $\alpha_n \leq \gamma$ for all $n \in \mathbb{N}$. Thus $x_n \in A_\gamma$ for all $n \in \mathbb{N}$ which, by virtue of the precompactness of A_γ, implies that $\{x_n\}_{n=1}^{\infty}$ has a convergent subnet in $(\widetilde{X}, \tau_{\widetilde{\mathcal{N}}})$. This shows that $(\widetilde{X}, \tau_{\widetilde{\mathcal{N}}})$ is web-compact, so that $(\widetilde{X}, \widetilde{\mathcal{N}})$ is trans-separable. Thus $(X, \mathcal{N})$ is trans-separable too. □

Theorem 2.8. *If there is a dense subspace Y of X covered by a σ-complete family consisting of Y-bounded sets, then $C_b(X)$ is angelic and every compact set in $C_b(X)$ is metrizable.*

Proof. First note that every Y-bounded subset of Y is X-bounded, so Theorem 2.6 ensures that $C_p(X)$ is angelic. Since the compact-bounded topology is stronger than the pointwise convergence topology, the angelic lemma [5] guarantees that the space $C_b(X)$ is angelic.

For the second statement let $\{A_\alpha : \alpha \in \Sigma\}$ be a σ-complete family of Y-closed and Y-bounded sets in X covering Y. Let τ_p and τ_b denote the pointwise convergence and the compact-bounded topology on $C(Y)$, respectively. Since Σ is unbounded in $\mathbb{N}^{\mathbb{N}}$ there is $k \in \mathbb{N}$ such that $\sup\{\alpha(k) : \alpha \in \mathbb{N}\} = +\infty$. Define

$$U_\alpha = \{f \in C(Y) : \sup_{y \in A_\alpha} |f(y)| \leq \alpha(k)^{-1}\}$$

for $\alpha \in \Sigma$ and set $\mathcal{U} := \{U_\alpha : \alpha \in \Sigma\}$. Since $(\Sigma, \leq)$ is a directed set and $\bigcup\{A_\alpha : \alpha \in \Sigma\} = Y$, we can see that $\mathcal{U}$ is a family of absolutely convex and absorbing sets in $C(Y)$ that compose a filter base. It can be easily seen that $\bigcap\{U_\alpha : \alpha \in \Sigma\} = \{0\}$ and that for each $\alpha \in \Sigma$ and $\epsilon > 0$ there is $\gamma \in \Sigma$ with $U_\gamma \subseteq \epsilon \cdot U_\alpha$. For instance, for the latter statement choose $\beta \in \Sigma$ such that $\beta(k) \geq \alpha(k) \cdot \epsilon^{-1}$ and then select $\gamma \in \Sigma$ with $\gamma(i) = \max\{\alpha(i), \beta(i)\}$ for every $i \in \mathbb{N}$. If $f \in U_\gamma$, since $A_\alpha \subseteq A_\gamma$ we have

$$\sup_{y \in A_\alpha} |\epsilon^{-1} f(y)| \leq \epsilon^{-1} \cdot \sup_{y \in A_\gamma} |f(y)| \leq \epsilon^{-1} \cdot \gamma(k)^{-1} \leq \epsilon^{-1} \cdot \beta(k)^{-1} \leq \alpha(k)^{-1}$$

which means that $f \in \epsilon \cdot U_\alpha$. Hence τ is a Hausdorff locally convex topology τ on $C(Y)$ for which $\mathcal{U}$ is a base of neighborhoods of the origin which satisfies that $\tau_p \leq \tau \leq \tau_b$.

Setting $F := (C(Y), \tau)'$, if $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$ it turns out that $U_\beta \subseteq U_\alpha$, which implies that $U_\alpha^0 \subseteq U_\beta^0$, the polars being taken in F. Thus $\{U_\alpha^0 : \alpha \in \Sigma\}$ is a σ-complete covering of F formed by equicontinuous sets. Since the topology ζ on F of uniform convergence on the compact sets of $(C(Y), \tau)$ agrees with the weak* topology $\sigma(F, C(Y))$ of F on every equicontinuous set, it follows that each U_α^0 is a ζ-compact set. This implies that the uniform space $(F, \mathcal{N})$, where $\mathcal{N}$ is the uniform structure for F associated to the locally convex topology ζ, has a σ-complete covering consisting of precompact (in fact, compact) sets, which according to Lemma 2.7 means that $(F, \mathcal{N})$ is trans-separable. Hence (F, ζ) is a trans-separable locally convex space and Theorem 1.4 guarantees that every compact set in $(C(Y), \tau)$ is metrizable.

Now observe that the restriction map $S : C_b(X) \to (C(Y), \tau)$ defined $S(f) = f|_Y$ is a continuous (linear) injection from $C_b(X)$ into $(C(Y), \tau)$. So if K is a compact set in $C_b(X)$ its image $S(K)$ is a compact set in $(C(Y), \tau)$, hence metrizable. Given that S restricts itself to an homeomorphism on K, it follows that K is metrizable in $C_b(X)$ as required. $\square$

Example 2.9. *For $C_p(X)$ the previous theorem does not hold.* If X is a non-metrizable Talagrand compact space then $C_p(X)$ is K-analytic. Hence, according to a classic result of [12], there is a family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact sets covering $C_p(X)$ such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$. Therefore, although $C_p(X)$ has a σ-complete covering consisting of compact sets (hence bounded sets), the space $C_p(C_p(X))$ contains a non-metrizable Fréchet-Urysohn compact set.

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