C AND C^* AMONG INTERMEDIATE RINGS

by

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ABSTRACT. Given a completely regular Hausdorff space X, an intermediate ring $A(X)$ is a ring of real valued continuous functions between $C^*(X)$ and $C(X)$. We discuss two correspondences between ideals in $A(X)$ and z-filters on X, both reviewing old results and introducing new results. One correspondence, $\mathcal{Z}_A$, extends the well-known correspondence between ideals in $C^*(X)$ and z-filters on X. The other, $\mathfrak{Z}_A$, extends the natural correspondence between ideals in $C(X)$ and z-filters on X. This paper highlights how these correspondences help clarify what properties of $C^*(X)$ and $C(X)$ are shared by all intermediate rings and what properties of $C^*(X)$ and $C(X)$ characterize those rings among intermediate rings. Using these correspondences, we introduce new classes of ideals and filters for each intermediate ring that extend the notion of z-ideals and z-filters for $C(X)$, and with $\mathcal{Z}_A$, a new class of filters for each intermediate ring $A(X)$ that extends the notion of e-filter for $C^*(X)$.

1. INTRODUCTION

Let X be a completely regular topological space, $C(X)$ the ring of all continuous real-valued functions on X, and $C^*(X)$ the ring of bounded continuous real-valued functions on X. A ring $A(X)$ of continuous functions on X is called an *intermediate ring* if

$$C^*(X) \subseteq A(X) \subseteq C(X) .$$

The rings $C^*(X)$ and $C(X)$ have been studied extensively, and the theory is beautifully presented in the book *Rings of Continuous Functions* by Leonard Gillman and Meyer Jerison [8]. Although the rings $C(X)$ and $C^*(X)$ are very different rings, they do share some crucial properties. The properties they do share are proved in different ways in each case. Much research on these rings can be described in terms of explaining the similarities and difference between these rings. In this article we organize some of the results of this research, as well as new results, around the following two questions.

- Which properties of $C^*(X)$ and $C(X)$ are shared by all intermediate rings?
- Which properties of $C^*(X)$ and $C(X)$ characterize these rings among intermediate rings?

With these questions in mind we give new proofs of some results and extend others. A class of ideals for all intermediate rings, that extends the class of e-ideals in C^*, was defined in [2, p. 49]. We introduce a complementary class of filters for each intermediate ring $A(X)$, that coincides with the class of e-filters when $A(X) = C^*(X)$. We furthermore introduce new classes of ideals and filters for each intermediate ring $A(X)$ that coincide with the classes of z-ideals and z-filters when $A(X) = C(X)$. In general, the results we present are those that are in the same spirit as the Gillman and Jersion book—that is, those results that relate the algebraic structure of the ring $A(X)$ to the topology of X. We also give examples and counterexamples to help motivate the various definitions on intermediate rings $A(X)$. Wherever appropriate, we state open problems whose solution would help complete the answers to the two questions stated above.

One of the main differences between $C^*(X)$ and $C(X)$ is the way in which the ideals of each ring correspond to z-filters on X. These correspondences are used in proving many of the properties of $C^*(X)$ and $C(X)$. The key to extending these correspondences to intermediate rings is a type of local invertibility of functions called E-regularity. In Section 2 we present the definition of E-regularity and state some of its main properties. In Section 3 we review how the correspondence for $C^*(X)$ can be extended to intermediate rings $A(X)$ by using E-regularity. We then introduce for each intermediate ring $A(X)$, a new class of filters, called $\mathcal{Z}_A$-filters, and we establish relationships between maximal ideals in $A(X)$ and $\mathcal{Z}_A$-ultrafilters on X (Theorem 3.9). We also provide a bijective correspondence between z-ultrafilters on X and $\mathcal{Z}_A$-ultrafilters on X (Proposition 3.11). In Section 4 we review how the correspondence for $C(X)$ can be extended to intermediate rings $A(X)$ by using E-regularity.

We then introduce for each intermediate ring $A(X)$, a new class of ideals and filters, called $\mathfrak{Z}_A$-ideals and $\mathfrak{Z}_A$-filters, and we establish relationships between maximal ideals in $A(X)$ and $\mathfrak{Z}_A$-ultrafilters on X (Theorem 4.9). We also show that every maximal ideal in $A(X)$ is a $\mathfrak{Z}_A$-ideal (Corollary 4.8), and we provide a bijective correspondence between z-ultrafilters on X and $\mathfrak{Z}_A$-ultrafilters on X (Proposition 4.10). In Section 5 we compare the correspondences that extend those of C^* and C, and present an explicit formula that relates them. In Section 6, we discuss properties of C^* and C that characterize them among intermediate rings, and we state several open problems relating to this.

Intermediate rings of continuous functions have been studied by several authors. D. Plank [10] gives a description of their maximal ideals. In [11], [12], and [2] the correspondence between ideals and z-filters for $C^*(X)$ is generalized to intermediate rings. In [9], the correspondence between ideals in $C(X)$ and z-filters on X is generalized to all intermediate rings. In [5] it is shown that intermediate rings can be realized as certain rings of fractions of $C^*(X)$. In [1] a description is given for the intersection of the free maximal ideals in such rings. Some examples and methods of constructing intermediate rings of continuous functions can be found in [6, 7].

2. Local invertibility in $A(X)$ on subsets of X

The sets $C(X)$ and $C^*(X)$ form commutative rings under pointwise addition and pointwise multiplication [8]. Any intermediate ring $A(X)$ is a lattice with respect to the operations of pointwise minimum and maximum:

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\} \ .$$

This result is from [2, Theorem 1.1] and the remark that immediately follows.

One of the main tools we use for rings of continuous functions is the analysis of those subsets of X on which f is "locally invertible" with respect to $A(X)$. Specifically, we have the following definition.

Definition 2.1 (*E*-regularity)**.** If $f \in A(X)$ and E is a set in X, we say that f is *E-regular in* $A(X)$ if there exists $g \in A(X)$, such that $fg(x) = 1$ for all $x \in E$.

The following is adapted from [11].

Lemma 2.2. *Let* $f, g \in A(X)$ *and let* $E, F \subseteq X$.

   (a) *If* f *is* E*-regular and* F*-regular, then* f *is* $E \cup F$*-regular.*

(b) *If f is bounded away from 0 on E, then f is E regular.*

(c) *If $0 < f(x) \leq g(x)$ for all $x \in E$, and if f is E-regular, then g is E-regular.*

(d) *If f is E-regular and g is F-regular, then fg is $E \cap F$-regular and $f^2 + g^2$ is $E \cup F$-regular.*

Whenever we apply E-regularity, E will be either a zero set or a co-zero set. A *zero set* is a set of the form $\boldsymbol{Z}(f) = \{x \mid f(x) = 0\}$ for some $f \in C(X)$. A *co-zero set* is the complement of some zero set in X. The set of *zero sets* of X is $\boldsymbol{Z}[X] = \{\boldsymbol{Z}(f) \mid f \in C(X)\}$. A *z-filter* on X is the intersection of a filter on X with the set $\boldsymbol{Z}[X]$ of zero-sets on X. Recall that a filter is a non-empty set of subsets of X closed under the formation of supersets and finite intersections and that does not include the empty set. A *z-ultrafilter* is a z-filter $\mathcal{F}$ such that any zero set $A \notin \mathcal{F}$ is disjoint from some zero set in $\mathcal{F}$. Given a set $E \subseteq X$, let $\langle E \rangle$ be the collection of z-supersets of E, and given a collection $\mathcal{E} \subseteq \boldsymbol{Z}[X]$, let $\langle \mathcal{E} \rangle$ be the smallest set containing $\mathcal{E}$ closed under z-supersets and finite intersections. Note that $\langle \mathcal{E} \rangle$ is either a z-filter or is $\boldsymbol{Z}[X]$.

3. From $C^*(X)$ to $A(X)$

The well-known correspondence between ideals in $C^*(X)$ and z-filters on X is described as follows. Let $\boldsymbol{E}_\epsilon(f) = \{x : |f(x)| \leq \epsilon\}$, and let $\boldsymbol{E}(f) = \{\boldsymbol{E}_\epsilon(f) : \epsilon > 0\}$; then the correspondence is given by

$$I \to \boldsymbol{E}[I] = \bigcup_{f \in I} \boldsymbol{E}(f).$$

We have the following important property of $\boldsymbol{E}$ from [8].

Proposition 3.1. *$f \in C^*(X)$ is noninvertible in $C^*(X)$ if and only if $\langle \boldsymbol{E}(f) \rangle$ is a z-filter on X.*

Although $\boldsymbol{E}$ can be defined on any intermediate ring $A(X)$, we show that the property in Proposition 3.1 does not hold whenever $A(X)$ properly contains $C^*(X)$.

Proposition 3.2. *If $A(X)$ properly contains $C^*(X)$, then there is a function $g \in A(X)$ that is invertible in $A(X)$, but where $\langle \boldsymbol{E}(g) \rangle$ is a z-filter.*

Proof. Let $f \in A(X) \setminus C^*(X)$. Then let $h = |f| \vee 1$. Since $A(X)$ is a lattice, $|f| = f \vee -f \in A(X)$ and hence $h \in A(X)$. Let $g = h^{-1}$. Note that g is bounded, and hence $g \in C^*(X)$. Now since h is unbounded, g is not invertible in $C^*(X)$, and hence by Proposition 3.1, $\langle \boldsymbol{E}(g) \rangle$ is a z-filter. But $g \in A(X)$ has an inverse in $A(X)$. $\qquad\square$

Toward extending $\boldsymbol{E}$, we point out that proposition 3.1 is connected to the fact that a function f is invertible in $C^*(X)$ if and only if f is bounded away from zero. Now, each $\boldsymbol{E}_\epsilon(f)$ is a set whose complement bounds $|f|$ away from ϵ, and hence is such that f is "locally invertible". More formally, f is $(\boldsymbol{E}_\epsilon(f))^c$-regular in $C^*(X)$ (Definition 2.1). This leads to the following reformulation of $\boldsymbol{E}$, which follows from [9, Theorem 2.1]:

(3.1) $\qquad \langle \boldsymbol{E}(f) \rangle = \{E \in \boldsymbol{Z}[X] : f \text{ is } E^c\text{-regular in } C^*(X)\}.$

We now generalize this with the following definition.

Definition 3.3. Let $A(X)$ be an intermediate ring. For $f \in A(X)$, we define

$$\mathcal{Z}_A(f) = \{E \in \boldsymbol{Z}[X] : f \text{ is } E^c\text{-regular in } A(X)\}.$$

For an ideal $I \subset A(X)$, we define $\mathcal{Z}_A[I] = \bigcup_{f \in I} \mathcal{Z}_A(f)$, and for a z-filter $\mathcal{F}$ on X, we define $\mathcal{Z}_A^\leftarrow(\mathcal{F}) = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$.

The map $\mathcal{Z}_A$ extends Proposition 3.1 from $C^*(X)$ to any intermediate ring $A(X)$, as the following theorem from [11] shows:

Theorem 3.4. $f \in A(X)$ is noninvertible in $A(X)$ if and only if $\mathcal{Z}_A(f)$ is a z-filter on X.

The following lemma, which follows from [11, Lemmas 2 and 3], gives a basic tool in proving many of the properties of the z-filter $\mathcal{Z}_A(f)$ in Lemma 3.6.

Lemma 3.5. Let $f \in A(X)$, and let $\mathcal{F}$ be a z-filter on X. Then $\mathcal{Z}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_\mathcal{F} fh = 0$ for all $h \in A(X)$.

The next lemma shows to what extent $\mathcal{Z}_A$ maps products and sums of functions to respectively meets and joins on the lattice of z-filters augmented with the set of all zero sets.

Lemma 3.6. Let $f, g \in A(X)$.
  (a) If $0 \le f \le g$, then $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(g)$.
  (b) $\bigcap \mathcal{Z}_A(f) = \boldsymbol{Z}(f)$.
  (c) $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$.
  (d) $\mathcal{Z}_A(f + g) \subseteq \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$.
  (e) If $f, g \ge 0$, then $\mathcal{Z}_A(f + g) = \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$.

Item (a) is proved in [11, Theorem 1 and Lemma 1(d)]. Item (b) is shown in [12, Proposition 2.2]. Items (c), (d), and (e) are proved in [9, Lemma 1.5]. Furthermore, observe that parts (c) and (e) show that the collection of elements of the form $\mathcal{Z}_A(f)$ for $f \ge 0$ forms a lattice.

Notice that Lemma 3.6(c) implies that for any $f \in A(X)$ we have

$$\mathcal{Z}_A(f) = \mathcal{Z}_A(f^2).$$

So when dealing with the z-filter $\mathcal{Z}_A(f)$ we often can assume, without loss of generality, that $f \geq 0$.

Parts (a) and (b) of the next two theorem are proved in [11] and [2] respectively; we include shorter proofs here for completeness.

Theorem 3.7. *Let $A(X)$ be an intermediate ring of continuous functions.*

(a) *If I is an ideal in $A(X)$ then $\mathcal{Z}_A[I]$ is a z-filter on X.*

(b) *If $\mathcal{F}$ is a z-filter on X then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is an ideal in $A(X)$.*

Proof. (a) If $E, F \in \mathcal{Z}_A[I]$, there exist $f, g \in I$ such $E \in \mathcal{Z}_A(f)$ and $F \in \mathcal{Z}_A(g)$. We may assume $f, g \geq 0$ by the above remark. Now by Lemma 3.6(e), $\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g) = \mathcal{Z}_A(f + g) \subseteq \mathcal{Z}_A[I]$. Since $E \cap F \in \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$, it follows that $E \cap F \in \mathcal{Z}_A[I]$.

(b) If $f, g \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$, then by Lemma 3.6(d) $\mathcal{Z}_A(f + g) \subseteq \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g) \subseteq \mathcal{F}$, so $f + g \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$. Now if $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ and $g \in A(X)$, then by Lemma 3.6(c) $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f)$, so $fg \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$. $\square$

Part (a) of this theorem shows that $\mathcal{Z}_A$ maps ideals in $A(X)$ to z-filters on X. Furthermore, $\mathcal{Z}_A$ does indeed extend $\boldsymbol{E}$; according to [9, Corollary 1.3], for any ideal $I \in C^*(X)$,

$$\mathcal{Z}_{C^*}[I] = \boldsymbol{E}[I].$$

The behavior of $\mathcal{Z}_A$ on maximal ideals and $\mathcal{Z}_A^{\leftarrow}$ on z-ultrafilters is addressed in Theorem 3.10 in the next section.

3.1. $\mathcal{Z}_A$-ideals and $\mathcal{Z}_A$-filters. In general, if I is an ideal in $A(X)$ and $\mathcal{F}$ is a z-filter on X, it is easy to see that

$$(3.2) \qquad \mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[I]] \supseteq I \quad \text{and} \quad \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]] \subseteq \mathcal{F}.$$

Equality does not hold in general; when it does we have the following definition.

Definition 3.8. An ideal I is called a $\mathcal{Z}_A$-*ideal* if $\mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[I]] = I$ and a z-filter $\mathcal{F}$ is called a $\mathcal{Z}_A$-*filter* if $\mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$. *A maximal $\mathcal{Z}_A$-filter (i.e., not properly contained in any other $\mathcal{Z}_A$-filter) is called a $\mathcal{Z}_A$-ultrafilter.*

Clearly, every maximal ideal is a $\mathcal{Z}_A$-ideal. However, not every ideal is a $\mathcal{Z}_A$-ideal, and not every z-filter is a $\mathcal{Z}_A$-filter. For example, in $C(\mathbb{R})$ let M_0 be the ideal consisting of all functions vanishing at 0 and O_0 be the ideal consisting of all functions vanishing on a neighborhood of 0. It is easy to see that $\mathcal{Z}_C[M_0] = \mathcal{Z}_C[O_0]$ so that $\mathcal{Z}_C^{\leftarrow}[\mathcal{Z}_C[O_0]] = M_0$, and clearly M_0 strictly contains O_0. So O_0 is not a $\mathcal{Z}_C$-ideal. For $A(X) = C^*(\mathbb{R})$, the z-ultrafilter $\mathcal{U}_0$ consisting of all zero sets in $\mathbb{R}$ containing 0 is not a $\mathcal{Z}_A$-filter. In fact, $\mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_0]]$ is the z-filter consisting of all zero-set neighborhoods of 0.

It is easy to see that the following identities always hold

$$(3.3) \qquad \mathscr{Z}_A[\mathscr{Z}_A^{\leftarrow}[\mathscr{Z}_A[I]]] = \mathscr{Z}_A[I] \quad \text{and} \quad \mathscr{Z}_A^{\leftarrow}[\mathscr{Z}_A[\mathscr{Z}_A^{\leftarrow}[\mathcal{F}]]] = \mathscr{Z}_A^{\leftarrow}[\mathcal{F}].$$

It follows from these that if I is an ideal in $A(X)$ then $\mathscr{Z}_A[I]$ is a $\mathscr{Z}_A$-filter, and if $\mathcal{F}$ is a z-filter on X then $\mathscr{Z}_A^{\leftarrow}[\mathcal{F}]$ is a $\mathscr{Z}_A$-ideal. It is shown in [6, Theorem 3.13] that $\mathscr{Z}_A$-ideals are the intersections of maximal ideals. If $A(X) = C^*(X)$ then $\mathscr{Z}_A$-ideals and $\mathscr{Z}_A$-filters are called e-ideals and e-filters, respectively, as given in [8, p. 33].

Theorem 3.9. *Let $A(X)$ be an intermediate ring of continuous functions.*

(a) *If M is a maximal ideal then $\mathscr{Z}_A[M]$ is a $\mathscr{Z}_A$-ultrafilter.*
(b) *If $\mathcal{V}$ is a $\mathscr{Z}_A$-ultrafilter then $\mathscr{Z}_A^{\leftarrow}[\mathcal{V}]$ is a maximal ideal.*

Proof. (a) Clearly $\mathscr{Z}_A[M]$ is a $\mathscr{Z}_A$-filter. If $\mathcal{F}$ is a $\mathscr{Z}_A$-filter and $\mathscr{Z}_A[M] \subseteq \mathcal{F}$, then by (3.2)

$$M \subseteq \mathscr{Z}_A^{\leftarrow}[\mathscr{Z}_A[M]] \subseteq \mathscr{Z}_A^{\leftarrow}[\mathcal{F}].$$

Since M is maximal, $M = \mathscr{Z}_A^{\leftarrow}[\mathcal{F}]$. It follows that $\mathscr{Z}_A[M] = \mathscr{Z}_A[\mathscr{Z}_A^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$. So $\mathscr{Z}_A[M]$ is a $\mathscr{Z}_A$-ultrafilter.

(b) Clearly $\mathscr{Z}_A^{\leftarrow}[\mathcal{V}]$ is a $\mathscr{Z}_A$-ideal. Suppose N is an ideal and $\mathscr{Z}_A^{\leftarrow}[\mathcal{V}] \subseteq N$. Then, because $\mathcal{V}$ is a $\mathscr{Z}_A$-ultrafilter, we have

$$\mathcal{V} = \mathscr{Z}_A[\mathscr{Z}_A^{\leftarrow}[\mathcal{V}]] \subseteq \mathscr{Z}_A[N].$$

Thus $\mathscr{Z}_A[N]$ is a $\mathscr{Z}_A$-filter containing the $\mathscr{Z}_A$-ultrafilter $\mathcal{V}$, and so $\mathcal{V} = \mathscr{Z}_A[N]$. It follows that $\mathscr{Z}_A^{\leftarrow}[\mathcal{V}] = \mathscr{Z}_A^{\leftarrow}[\mathscr{Z}_A[N]] \supseteq N$. Thus $\mathscr{Z}_A^{\leftarrow}[\mathcal{V}] = N$ and so $\mathscr{Z}_A^{\leftarrow}[\mathcal{V}]$ is maximal. $\qquad\square$

It is known [8] that there is a one-to-one correspondence between the z-ultrafilters on X and the maximal ideals of $C^*(X)$. It is also known [8] that there is a one-to-one correspondence between the z-ultrafilters on X and the maximal ideals of $C(X)$. It was noted in [8, p. 82] that "It is a remarkable fact that these two problems not only have solutions, but a common one." The next theorem shows that the single map $\mathscr{Z}_A$ provides an injective correspondence between maximal ideals in $A(X)$ and z-filters on X, including the cases where $A(X)$ is $C(X)$ or $C^*(X)$. It is stated in [11] and proved in [2], though its proof uses special properties of $C(X)$. Here we provide a self contained proof, based only on the properties of $\mathscr{Z}_A$ and the Stone-Čech compactification of X. The Stone-Čech compactification βX of X is an extension of X with the Stone topology; we identify every point $p \in \beta X$ with a unique z-ultrafilter $\mathcal{U}_p$ on X.

Theorem 3.10. *Let $A(X)$ be an intermediate ring of continuous functions.*

(a) If $\mathcal{U}$ is a z-ultrafilter on X, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $A(X)$. Furthermore, the map $\mathcal{Z}_A^{\leftarrow}$ is one-one from the z-ultrafilters on X onto the maximal ideals of $A(X)$.

(b) If M is a maximal ideal then $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter.

Proof. (a) Given a z-ultrafilter $\mathcal{U}_p$, note from (3.2), (3.3), and the definitions that $\mathcal{V} = \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]]$ is the largest subset of $\mathcal{U}_p$ that is a $\mathcal{Z}_A$-filter. Furthermore, suppose S is a zero set that is not in $\mathcal{U}_p$. We show that S is disjoint from some set in $\mathcal{V}$, and hence (by contrapositive) any z-filter containing $\mathcal{V}$ is contained in $\mathcal{U}_p$. As $S \notin \mathcal{U}_p$, then $p \notin cl_{\beta X} S$. By normality of βX and Urysoln's Lemma, there exists a function $g' \in C(\beta X)$ that is 0 on a neighborhood of p and 1 on $cl_{\beta X}(S)$. If g is the restriction of g' to X, then by the density of X in βX, $p \in cl_{\beta X} \mathbf{Z}(g)$, and hence $\mathbf{Z}(g) \in \mathcal{U}_p$. It follows that $\lim_{\mathcal{U}_p} gh = 0$ for all $h \in A(X)$. By Lemma 3.5, $g \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]$, and so $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]] = \mathcal{V}$. Finally, if $R = \{x \in X : |g(x)| \leq 1/2\}$, then $R \in \mathcal{Z}_A(g) \subseteq \mathcal{V}$ and $R \cap S = \emptyset$. Since $\mathcal{V}$ is the largest $\mathcal{Z}_A$-filter contained in $\mathcal{U}$, and every z-filter that contains $\mathcal{V}$ is contained in $\mathcal{U}$, we have that $\mathcal{V}$ is a $\mathcal{Z}_A$-ultrafilter. By Theorem 3.9(b), $\mathcal{Z}_A^{\leftarrow}[\mathcal{V}]$ is a maximal ideal. As $\mathcal{U} \supseteq \mathcal{V}$, we have that $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is an ideal containing $\mathcal{Z}_A^{\leftarrow}[\mathcal{V}]$, and hence is maximal.

Suppose $\mathcal{U}_p \neq \mathcal{U}_q$, so p and q are distinct points of βX. By the normality of βX and Urysohn's Lemma, there exists a continuous function g' on βX such that g' is zero on a neighborhood of p and 1 on a neighborhood of q. If g is the restriction of g' to X, then $\lim_{\mathcal{U}_p} gh = 0$ for all $h \in A(X)$ but $\lim_{\mathcal{U}_q} g = 1$. So by Lemma 3.5, $g \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]$ but $g \notin \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_q]$. This establishes the injectivity of $\mathcal{Z}_A^{\leftarrow}$. Surjectivity follows from a routine exercise involving (3.2).

(b) Suppose $\mathcal{Z}_A[M]$ is contained in the distinct z-ultrafilters $\mathcal{U}_p$ and $\mathcal{U}_q$. Then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p] \supseteq \mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[M]] \supseteq M$, so $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p] = M$. Similarly, $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_q] = M$. So $\mathcal{U}_p = \mathcal{U}_q$ by part (a). $\qquad\square$

Proposition 3.11. *The correspondence $\mathcal{U} \longrightarrow \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]]$ is one-one from the set of z-ultrafilters on X onto the set of $\mathcal{Z}_A$-ultrafilters.*

Proof. If $\mathcal{U}_1 \neq \mathcal{U}_2$ then by Theorem 3.10(a) $M_1 = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_1]$ and $M_2 = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_2]$ are distinct maximal ideals. By (3.3), M_1 and M_2 are $\mathcal{Z}_A$-ideals. Thus $\mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[M_1]] = M_1 \neq M_2 = \mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[M_2]]$. This shows that the map is one-one.

Now if $\mathcal{V}$ is a $\mathcal{Z}_A$-ultrafilter then $\mathcal{V} = \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{V}]]$, so $\mathcal{V}$ is contained in a unique z-ultrafilter $\mathcal{U}$ by 3.10(b). Since $\mathcal{Z}_A^{\leftarrow}[\mathcal{V}]$ is a maximal ideal by Theorem 3.9(b) and $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] \supseteq \mathcal{Z}_A^{\leftarrow}[\mathcal{V}]$ it follows that $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] = \mathcal{Z}_A^{\leftarrow}[\mathcal{V}]$. Thus $\mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]] = \mathcal{V}$ and this shows that the correspondence is onto. $\quad\square$

4. FROM $C(X)$ TO $A(X)$

The natural correspondence between ideals I in $C(X)$ and z-filters on X is given by

$$I \to \mathbf{Z}[I] = \{\mathbf{Z}(f) : f \in I\}.$$

The following important property of $\mathbf{Z}$ follows from [8].

Proposition 4.1. $f \in C(X)$ is noninvertible in $C(X)$ if and only if $\langle \mathbf{Z}(f) \rangle$ is a z-filter on X.

Although $\mathbf{Z}$ can be defined on any intermediate ring $A(X)$, we show that the property in Proposition 4.1 does not hold whenever $A(X)$ is properly contained in $C(X)$.

Proposition 4.2. If $A(X)$ is properly contained in $C(X)$, then there is a function $g \in A(X)$ that is not invertible in $A(X)$, but where $\langle \mathbf{Z}(g) \rangle$ is not a z-filter.

Proof. Let $f \in C(X) \setminus A(X)$. Note that $f = (f \wedge 1)(f \vee 1)$. Hence either $(f \wedge 1) \notin A(X)$ or $(f \vee 1) \notin A(X)$. First assume that $(f \wedge 1) \notin A(X)$. Then $h = (f \wedge 1) - 2 \notin A(X)$. Let $g = h^{-1}$. Then g is bounded, and is hence in $A(X)$. But as $h \notin A(X)$, g is not invertible in $A(X)$. But $\mathbf{Z}(g) = \emptyset$, and hence $\langle \mathbf{Z}(g) \rangle = \mathbf{Z}[X]$, and is not a z-filter. If $(f \wedge 1) \in A(X)$, then $(f \vee 1) \notin A(X)$. We then set $h = (f \vee 1)$ and follow the same argument. $\square$

We would like to extend $\mathbf{Z}$ to all intermediate rings in such a way that ensures that the property in Proposition 4.1 is preserved. To this end, let us compare how $\mathbf{Z}$ and $\mathcal{Z}_C$ act on, for example, the function $f(x) = x$. Note that

$$\mathcal{Z}_C(f) = \{E \in \mathbf{Z}[X] : f \text{ is } E^c\text{-regular in } C(X)\}$$
$$= \{E \in \mathbf{Z}[X] : \text{ there is an open set } U, \text{ with } 0 \in U \subseteq E\}.$$

To better compare $\mathbf{Z}$ to the definition of $\mathcal{Z}_A$, we want to describe $\langle \mathbf{Z}(f) \rangle$ in terms of E-regularity. We then note that

$$\langle \mathbf{Z}(f) \rangle = \{E \in \mathbf{Z}[X] : 0 \in E\}$$
$$= \{E \in \mathbf{Z}[X] : \text{for all } \textit{zero sets } H \subseteq E^c, f \text{ is } H\text{-regular in } C(X)\}.$$

We now generalize this with the following definition.

Definition 4.3. Let $A(X)$ be an intermediate ring. For $f \in A(X)$, we define

$$\mathfrak{Z}_A(f) = \{E \in \mathbf{Z}[X] : f \text{ is } H\text{-regular in } A(X) \text{ for all zero sets } H \subseteq E^c\}.$$

For an ideal $I \subset A(X)$, we define $\mathfrak{Z}_A[I] = \bigcup_{f \in I} \mathfrak{Z}_A(f)$, and for a z-filter $\mathcal{Z}$ on X, we define $\mathfrak{Z}_A^{\leftarrow}(\mathcal{F}) = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathcal{F}\}$.

It is easy to see that for $f \in A(X)$

$$(4.1) \qquad \qquad \mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f).$$

To compare the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ on ideals, it is clear that $\mathcal{Z}_A[I] \subseteq \mathfrak{Z}_A[I]$.

The map $\mathfrak{Z}_A$ extends Proposition 4.1 from $C(X)$ to any intermediate ring $A(X)$, as the following theorem from [9, Proposition 2.2] shows:

Theorem 4.4. $f \in A(X)$ *is noninvertible in $A(X)$ if and only if $\mathfrak{Z}_A(f)$ is a z-filter on X.*

Question 1. Let $f \in A(X)$, and let $\mathcal{F}$ be a z-filter on X. Is it the case that $\mathfrak{Z}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} fg = 0$ for every $g \in A(X)$?

Observe that the left-to-right direction of Question 1 immediately follows from (4.1) together with Lemma 3.5. The right-to-left direction remains open.

Question 2. Let $f, g \in A(X)$. Which properties analogous to those of Lemma 3.6 hold with $\mathfrak{Z}_A$ in place of $\mathcal{Z}_A$?

We will see in Section 5 how the analog of Lemma 3.6(a) does hold. The following is from [9, Theorem 4.3].

Theorem 4.5. *If I is an ideal in $A(X)$, then $\mathfrak{Z}_A[I]$ is a z-filter on X.*

Theorem 4.5 shows that $\mathfrak{Z}_A$ maps ideals in $A(X)$ to z-filters in X. Furthermore, $\mathfrak{Z}_A$ does extend $\mathbf{Z}$; according to [9, Corollary 2.4], for any ideal $I \in C(X)$,

$$\mathfrak{Z}_C[I] = \mathbf{Z}[I].$$

Question 3. Is it the case that if $\mathcal{F}$ is a z-filter on X, then $\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$ is an ideal in $A(X)$?

4.1. $\mathfrak{Z}_A$-ideals and $\mathfrak{Z}_A$-filters. Note that $\mathfrak{Z}_A$ and $\mathfrak{Z}_A^{\leftarrow}$ are well-defined on all subsets of $A(X)$ and subsets of $\mathcal{P}(X)$ respectively. In general, if I is a subset of $A(X)$ and $\mathcal{F}$ is a collection of subsets of X, it is easy to see that

$$(4.2) \qquad \mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[I]] \supseteq I \quad \text{and} \quad \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] \subseteq \mathcal{F}.$$

Definition 4.6. An ideal I is called a $\mathfrak{Z}_A$-*ideal* if $\mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[I]] = I$ and a z-filter $\mathcal{F}$ is called a $\mathfrak{Z}_A$-*filter* if $\mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$. A maximal $\mathfrak{Z}_A$-*filter* is called a $\mathfrak{Z}_A$-*ultrafilter.*

Not every ideal is a $\mathfrak{Z}_A$-ideal and not every z-filter is a $\mathfrak{Z}_A$-filter. For example, if $A(X) = C(\mathbb{R})$ then the principal ideal generated by the function $f(x) = x$ is not a z-ideal and hence not a $\mathfrak{Z}_A$-ideal ([8, p. 26]).

From [9, Example 4.5 and Theorem 4.7], any free z-ultrafilter $\mathcal{U}$ on $[0, \infty)$ that contains the natural numbers is not a $\mathfrak{Z}_{C^*}$-filter.

It is easy to see that the following identities always hold

(4.3) $\qquad \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[I]]] = \mathfrak{Z}_A[I] \quad$ and $\quad \mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]]] = \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$.

It follows that if I is an ideal in $A(X)$ then $\mathfrak{Z}_A[I]$ is $\mathfrak{Z}_A$-filter. If $A(X) = C(X)$ then $\mathfrak{Z}_A$-*ideals* and $\mathfrak{Z}_A$-*filters* are simply z-ideals and z-filters, respectively, as given in [8, p. 26–27].

The next theorem shows that the single map $\mathfrak{Z}_A$ provides another (see Theorem 3.10) injective correspondence between maximal ideals in $A(X)$ and z-filters on X, including the case where $A(X)$ is $C(X)$ or $C^*(X)$.

Theorem 4.7. *Let $A(X)$ be an intermediate ring.*

(a) *If $\mathcal{U}$ is a z-ultrafilter on X, then $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $A(X)$. Furthermore, there is a bijective correspondence between z-ultrafilters on X and maximal ideals in $A(X)$, given by*

$$\mathcal{U} \mapsto \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}].$$

(b) *If M is a maximal ideal then $\mathfrak{Z}_A[M]$ is contained in a unique z-ultrafilter.*

Proof. (a) The proof is in [9, Theorems 4.7 and 4.8].

(b) This follows from Theorem 3.10(b) and the fact that $\mathcal{Z}_A[M] \subseteq \mathfrak{Z}_A[M]$. $\qquad \square$

Corollary 4.8. *Every maximal ideal is a $\mathfrak{Z}_A$-ideal.*

Proof. Let M be a maximal ideal in $A(X)$. By Theorem 4.7(b), $\mathfrak{Z}_A[M]$ is contained in a unique z-ultrafilter $\mathcal{U}$. By (4.2) and the monotocity of $\mathfrak{Z}_A^{\leftarrow}$

$$M \subseteq \mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[M]] \subseteq \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}].$$

By Theorem 4.7(a), $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal, and hence must be equal to M. Thus $M = \mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[M]]$. $\qquad \square$

It was shown in Theorem 3.9 that $\mathcal{Z}_A^{\leftarrow}$ provides a correspondence between $\mathfrak{Z}_A$-ultrafilters on X and maximal ideals in $A(X)$, for all intermediate rings, including $C(X)$ and $C^*(X)$. Here we show the the same is true for $\mathfrak{Z}_A^{\leftarrow}$.

Theorem 4.9. *Let $A(X)$ be an intermediate ring.*

(a) *If M is a maximal ideal then $\mathfrak{Z}_A[M]$ is a $\mathfrak{Z}_A$-ultrafilter.*
(b) *If $\mathcal{V}$ is a $\mathfrak{Z}_A$-ultrafilter then $\mathfrak{Z}_A^{\leftarrow}[\mathcal{V}]$ is a maximal ideal.*

Proof. The proof is similar to that of Theorem 3.9, though we present a detailed version here, that makes it clear that we do not depend on knowing that $\mathfrak{Z}_A^{\leftarrow}$ takes z-filters to ideals (see Question 3).

(a) Clearly by (4.3), $\mathfrak{Z}_A[M]$ is a $\mathfrak{Z}_A$-filter. If $\mathcal{U}$ is a $\mathfrak{Z}_A$-ultrafilter and $\mathfrak{Z}_A[M] \subseteq \mathcal{U}$, then by Corollary 4.8

$$M = \mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[M]] \subseteq \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}].$$

By Theorem 4.7(a), $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal, and hence must be equal to M. It follows that $\mathfrak{Z}_A[M] = \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]] = \mathcal{U}$. So $\mathfrak{Z}_A[M]$ is a $\mathfrak{Z}_A$-ultrafilter.

(b) Suppose N is an ideal and $\mathfrak{Z}_A^{\leftarrow}[\mathcal{V}] \subseteq N$. Then, because $\mathcal{V}$ is a $\mathfrak{Z}_A$-ultrafilter, we have

$$\mathcal{V} = \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{V}]] \subseteq \mathfrak{Z}_A[N].$$

Thus $\mathfrak{Z}_A[N]$ is a $\mathfrak{Z}_A$-filter containing the $\mathfrak{Z}_A$-ultrafilter $\mathcal{V}$, and so $\mathcal{V} = \mathfrak{Z}_A[N]$. Then by (4.2), $\mathfrak{Z}_A^{\leftarrow}[\mathcal{V}] = \mathfrak{Z}_A^{\leftarrow}[\mathfrak{Z}_A[N]] \supseteq N$. Thus $\mathfrak{Z}_A^{\leftarrow}[\mathcal{V}] = N$ and so $\mathfrak{Z}_A^{\leftarrow}[\mathcal{V}]$ is a maximal ideal.   □

Proposition 4.10. *The correspondence* $\mathcal{U} \longrightarrow \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]]$ *is one-one from the set of z-ultrafilters on X onto the set of $\mathfrak{Z}_A$-ultrafilters.*

Proof. The proof is similar to the proof of Proposition 3.11.   □

5. Comparing the correspondences

In this section we compare the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$. These give us a way of comparing $\boldsymbol{E}$ with $\boldsymbol{Z}$, which cannot be compared directly, since they are defined on different rings. But the question of comparison is not new, as it is stated in [8, p. 30] that the correspondence between ideals and z-filters in $C(X)$ occurs in a "rudimentary form" in $C^*(X)$.

Given a z-filter $\mathcal{F}$, we write $h\mathcal{F}$ for the hull of $\mathcal{F}$:

$$h\mathcal{F} = \{\mathcal{U} : \mathcal{U} \text{ is a } z\text{-ultrafilter, and } \mathcal{F} \subseteq \mathcal{U}\}.$$

Given a collection $\mathfrak{U}$ of of z-ultrafilters, we write $k\mathfrak{U}$ to denote the kernel of $\mathfrak{U}$:

$$k\mathfrak{U} = \bigcap_{\mathcal{U} \in \mathfrak{U}} \mathcal{U}.$$

The following theorem from [9] provides a comparison between these correspondences.

Theorem 5.1. *For any intermediate ring $A(X)$ and non-invertible function $f \in A(X)$,*

$$\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f).$$

Observe that kh is monotone, whence a $\mathfrak{Z}_A$ analog to Lemma 3.6(a) immediately follows from Lemma 3.6(a) itself.

For the inverse maps, we have the following theorem from [9, Theorem 4.7], which shows that $\mathcal{Z}_A^{\leftarrow}$ and $\mathfrak{Z}_A^{\leftarrow}$ agree on z-ultrafilters on X.

Theorem 5.2. *If $A(X)$ is an intermediate ring and $\mathcal{U}$ is a z-ultrafilter on X, then $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}] = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$.*

We know from Theorem 3.10(b) that if M is a maximal ideal in $A(X)$, then $\mathcal{Z}_A(M)$ is contained in a unique z-ultrafilter on X. Let us denote that z-ultrafilter by $\overline{\mathcal{Z}}_A(M)$. Similarly, we know from Theorem 4.7(b) that $\mathfrak{Z}_A(M)$ is contained in a unique z-ultrafilter on X, and we denote this by $\overline{\mathfrak{Z}}_A(M)$. From [2, Theorem 3.4], we have that given any two intermediate rings $A(X)$ and $B(X)$, the map $M \mapsto \mathcal{Z}_B^{\leftarrow}\overline{\mathcal{Z}}_A[M]$ is a bijective map between maximal ideals in $A(X)$ and maximal ideals in $B(X)$. Note that because $\mathcal{Z}_A[M] \subseteq \mathfrak{Z}_A[M]$ is contained in a unique z-ultrafilter and hence $\overline{\mathcal{Z}}_A[M] = \overline{\mathfrak{Z}}_A[M]$ for every maximal ideal $M \in A(X)$. From this together with Theorem 5.2, we have that $\mathfrak{Z}_B^{\leftarrow}\overline{\mathfrak{Z}}_A$ is the exact same map as $\mathcal{Z}_B^{\leftarrow}\overline{\mathcal{Z}}_A$ on maximal ideals, and hence is also such a bijection.

6. Characterizing C and C^* among intermediate rings

In this section, we investigate properties of C and C^* that characterize them among intermediate rings. It is shown in [9, Theorem 1.2] that the correspondence $\boldsymbol{E}$ characterizes $C^*(X)$ among its intermediate rings in the following sense.

Theorem 6.1. *Let $A(X)$ be an intermediate ring. Then $A(X) = C^*(X)$ if and only if $\mathcal{Z}_A(f) = \langle \boldsymbol{E}(f) \rangle$ for all $f \in A(X)$.*

The proof in [9] of this theorem makes an unstated assumption that $f \geq 0$. It is worth noting that this is a valid assumption, since for any function $f \in A(X)$, it holds that $|f| \in A(X)$ (from [2, Theorem 1.1]), that $\mathcal{Z}_A(f) = \mathcal{Z}_A(|f|)$ (from Lemma 3.6(c)), and that $\boldsymbol{E}(f) = \boldsymbol{E}(|f|)$ (directly from the definition of $\boldsymbol{E}$).

Similar to Theorem 6.1, the correspondence $\boldsymbol{Z}$ characterizes $C(X)$ among its intermediate rings in the following sense, as was shown in [9, Theorem 2.3].

Theorem 6.2. *Let $A(X)$ be an intermediate ring. Then $A(X) = C(X)$ if and only if $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$ for all $f \in A(X)$.*

We know from [9, Proposition 4.5] that $\mathfrak{Z}_A(M)$ need not be a z-ultrafilter for every maximal ideal M. We also know from [8] that $\mathfrak{Z}_C(M)$ is a z-ultrafilter for every maximal ideal M. This leads to the following question.

Question 4. Is it the case that $A(X) = C(X)$ if and only if $\mathfrak{Z}_A(M)$ is a z-ultrafilter for every maximal ideal M?

If $A(X) = C(X)$ then every z-filter is a $\mathfrak{Z}_A$-filter because in this case $\mathfrak{Z}_A = \mathbf{Z}$ and it is known that $\mathbf{Z}[\mathbf{Z}^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$ for every z-filter $\mathcal{F}$ ([8], p. 26). We ask if this property characterizes $C(X)$.

Question 5. Is it the case that $A(X) = C(X)$ if and only if every z-filter on X is a $\mathfrak{Z}_A$-filter on X?

References

[1] S. K. Acharyya, K. C. Chattopadhyay, and D. P. Ghosh, *On a class of subalgebras of $C(X)$ and the intersection of their free maximal ideals.*, Proc. Amer. Math. Soc. **125** (1997), 611–615.

[2] H. L. Byun and S. Watson, *Prime and maximal ideals in subrings of $C(X)$*, Topology Appl. **40** (1991), 45–62.

[3] H. L. Byun, L. Redlin, and S. Watson, *A-compactifications and rings of continuous functions between C^* and C*, Topological Vector Spaces, Algebras, and Related Areas, Pitman Research Notes in Mathematics Series **316** (1995), 130–139.

[4] ———, *Local bounded inversion in rings of continuous functions*, Comment. Math. **37** (1997), 55–63.

[5] J. M. Dominguez, J. Gomez, and M. A. Mulero, *Intermediate algebras between $C^*(X)$ and $C(X)$ as rings of fractions of $C^*(X)$*, Topology Appl. **77** (1997), 115–130.

[6] J. M. Dominguez and J. Gomez-Perez, *Intersections of maximal ideals in algebras between $C^*(X)$ and $C(X)$*, Topology Appl. **98** (1999), 149–165.

[7] ———, *There do not exist minimal algebras between $C^*(X)$ and $C(X)$ with prescribed real maximal ideal spaces*, Acta Math. Hungar. **94** (2002), 351–355.

[8] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, 1978.

[9] P. Panman, J. Sack, S. Watson, *Correspondences between ideals and z-filters for rings of continuous functions between C^* and C*, Comment. Math. **52** (2012), 11–20.

[10] D. Plank, *On a class of subalgebras of $C(X)$ with applications to $\beta X - X$*, Fund. Math. **64** (1969), 41–54.

[11] L. Redlin and S. Watson, *Maximal ideals in subalgebras of $C(X)$*, Proc. Amer. Math. Soc. **100** (1987), 763–766.

[12] ———, *Structure spaces for rings of continuous functions with applications to realcompactifications*, Fund. Math. **152** (1997), 151–163.

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