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PROGRAMMING SEMANTICS TO TOPOLOGICAL SYSTEMS TO LATTICE-VALUED TOPOLOGY

JEFFREY T. DENNISTON, AUSTIN MELTON, AND STEPHEN E. RODABAUGH

Abstract. This paper examines the synergism emerging from three historically distinctive traditions: theory of locales; programming semantics and topological systems; and point-set lattice-theoretic (poslat) topology, both fixed-basis and variable-basis. Many gaps are discovered and filled with new results; and open questions are posed.

1. Introduction and plan of paper

This paper extends the presentation [67] made by the third author and traces the emerging synergism of these three historically distinct developments:

(1) the study of locales, motivated in part by the Stone representation theorems and the subsequent and underlying sobriety-spatiality representation theorem based upon D. Papert and S. Papert [46] and J. R. Isbell [28];
(2) the study of programming as a discipline begun in 1976 by E. W. Dijkstra [14] and further developed in a topology related direction by M. Smyth [70], culminating in the topological systems of S. J. Vickers [75]; and

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(3) the study of fuzzy sets as introduced by L. A. Zadeh [77], motivated in part by control theory in engineering, followed by lattice-valued topology in the sense of C. L. Chang, J. A. Goguen, and R. Lowen [6, 18, 41], the openness predicate motivated lattice-valued fuzzy topology in the sense of U. Höhle, T. Kubiak, and A. P. Šostak [21, 34, 74], and culminating in the first variable-basis categories for both lattice-valued topology and lattice-valued fuzzy topology in S. E. Rodabaugh [62].

What is striking is that these three developments did not know of each other initially. The programming development became aware of the localic tradition fairly early in its history, at least by the 1980's as judged by [70, 75], while the "fuzzy" development took relatively longer in its history to become aware of locales (and similar structures such as MV-algebras, quantales, residuated lattices), doing so in the 1980's [22, 53] and culminating in [25, 32, 33, 48, 49, 50, 62] in the 1990's and beyond. What is most striking is that (2) and (3) above were apparently unaware of each other until J. T. Denniston and Rodabaugh [7] in 2009 showed how both fixed-basis and variable-basis topology are intimately connected to topological systems, which seemed to spark developments in several directions. These include: attachment relations in C. Guido [19, 20]; algebraic varieties and powerset theories in S. A. Solovjov [71, 72, 73]; and nondeterminism, formal concept analysis, and enriched categories in Denniston, A. Melton, and Rodabaugh [9, 10, 11, 12]. Not only are (2) and (3) invigorating each other, but this linkage is reshaping the relationship between (1) and (3) as well.

It is the purpose of this paper to extend the presentation of [67], update and broaden the linkage of (1), (2), (3) above given in [9], and generally examine the growing synergism of (1), (2), (3). While many of the results of this paper are known, our examination uncovers and fills many gaps with new results. Known constructions and results are cited without proof, while new constructions and results are given proofs. Several open questions are posed.

Unless stated otherwise, categorical notions are from [1]. Also, a frame L is *consistent* if it has at least two elements, in which case $\perp \neq \top$; otherwise, L is *inconsistent*. We find the following notation, commonly attributed to P. Halmos, to be frequently convenient: if $f : X \to Y$ is a function and P is a possible predicate of members of Y, then

$$[f \text{ has } P] := \{x \in X : f(x) \text{ has } P\}.$$

An outline of the rest of this paper is as follows:

2. From programming to topological systems and TopSys

2.1. **Dijkstra's programming principles and adjointness of programs.** In 1976 [14], E. W. Dijkstra laid down principles the goal of which was to improve programming methodology. Two key ideas are the following:

Out. Focus more on outputs than inputs.
Pred. Focus on predicates or properties of outputs (*postcondition predicates*) and predicates or properties of inputs (*precondition predicates*).

Letting X and Y be the sets, respectively, of inputs and outputs, the above principles can be illustrated in the special case in which precondition predicates comprise a family $\mathcal{P}$ of subsets of X and postcondition predicates a family $\mathcal{Q}$ of subsets of Y. In this case, we say that input x *satisfies* $P \in \mathcal{P}$, output y *satisfies* $Q \in \mathcal{Q}$ if $x \in P$, $y \in Q$; so that inputs and outputs are related to, and satisfy, predicates via the membership relation.

These notions can be packaged as a "system", which, in the case of the input side, comprises an ordered triple $(X, \mathcal{P}, \in)$, where X is a set (perhaps of bitstrings), $\mathcal{P} \subset \wp(X)$ is a family of predicates, and $\in \subset X \times \mathcal{P}$ acts as a "satisfaction relation" indicating when a given string satisfies a given predicate. Later we shall consider generalizations of the notions of predicates and satisfaction, but for now we continue to work within the special case of predicates as subsets and satisfaction as membership.

In addition to the above notions, one can distinguish *deterministic* from *nondeterminstic* relationships between inputs of X and outputs of Y:

- In the deterministic case, the input-to-output correspondence is a well-defined (partial) function $f : X \to Y$.
- In the nondeterministic case, the input-to-output correspondence is a relation $R \subset X \times Y$.

The nondeterministic case is carefully studied in [11]. Focusing now on the deterministic case with a total function f and continuing to use the special case of predicates and satisfaction as above, a *deterministic program* $(f, \varphi) : (X, \mathcal{P}, \in) \to (Y, \mathcal{Q}, \in)$ comprises:

- a "forward" map $f : X \to Y$ converting inputs to outputs, and
- a "backwards" map $\varphi^{op} : \mathcal{Q} \to \mathcal{P}$ converting postcondition predicates to precondition predicates.

Additionally, Dijkstra goes on to describe an "optimal" deterministic program (f, φ) by imposing the following axiom which some call *adjointness*:

$$\forall Q \in \mathcal{Q}, \forall x \in X,\ x \in \varphi^{op}(Q) \Leftrightarrow f(x) \in Q.$$

Each deterministic program in the sequel is assumed to have adjointness and may be said to be "adjoint". It is instructive to motivate each direction of the biconditional predicate of the adjointness axiom in our special case, a biconditionality with far-reaching categorical consequences (cf. Proposition 3.1.1 below):

(1) A desirable postcondition predicate Q for outputs is chosen. Next, the map $\varphi^{op} : \mathcal{Q} \to \mathcal{P}$ is applied to pull Q back to a precondition predicate $\varphi^{op}(Q)$ for inputs. Then for each input x satisfying this precondition predicate, it is mandated that the corresponding output $f(x)$ satisfies the originally chosen postpredicate Q. This approach to improved program quality implements Dijkstra's Out and Pred conditions above and motivates the "only if" direction of adjointness.

(2) To have the most applicable program possible, Dijkstra also wants each $\varphi^{op}(Q)$ to be the optimal pullback of Q, namely that it should be the weakest or largest possible pullback to a precondition predicate. This means that if an input x does not satisfy the pullback $\varphi^{op}(Q)$, then the program output $f(x)$ does not satisfy Q. This motivates (the contrapositive of) the "if" direction of adjointness and further implements Dijkstra's philosophy.

It is worthwhile to motivate the term "adjointness" describing Dijkstra's philosophy for high quality, optimal programs. It should be recalled [29] that if $f : L \to M$, $g : L \leftarrow M$ are isotone maps of preordered sets, then $f \dashv g$ whenever

$$\forall b \in M, \forall a \in L,\ a \leq g(b) \Leftrightarrow f(a) \leq b.$$

The relationship $f \dashv g$ is termed *adjunction* and the displayed condition is dubbed *adjointness*, and its similarity to the adjointness condition for deterministic programs should be apparent—see the first display above and Definition 2.3.2(2) below.

It is also worthwhile to note that the adjointness condition for Dijkstra's programs is the same condition required for Chu transforms as morphisms between Chu spaces (or Chu systems) introduced by M. Barr [2] in 1979.

Finally, we note that the special case we have been tracing above determines uniquely for each input-output function f a compatible φ^{op} predicate map so that the program (f, φ) satisfies adjointness, as indicated in the following proposition.

Proposition 2.1.1. *Assume predicates are subsets and that inputs/outputs relate to predicates by membership. Then (f, φ) satisfies adjointness if and only if*

$$\varphi^{op} = (f^{\leftarrow})_{|\,\mathcal{Q}},$$

where $f^{\leftarrow} : \wp(Y) \to \wp(X)$ is the usual preimage operator for the mapping $f : X \to Y$.

Throughout this paper we adopt—or modify as appropriate—T. S. Blyth's arrow notation [3] for the image and preimage operators of a function.

2.2. **Program semantics with open predicates.** M. Smyth [70] in 1983 advocated viewing predicates as open sets. Continuing with predicates as subsets and satisfaction of predicates as membership, as in the previous subsection, the application of *finite observational logic*, described by S. J. Vickers [75] and J. T. Denniston, A. Melton, and S. E. Rodabaugh [9], to this special case forces the precondition predicates $\mathcal{P}$ and the postcondition predicates $\mathcal{Q}$ to respectively form topologies on the input set X and the output set Y. Thus, finite observational logic gives us what we heuristically call "topological" systems $(X, \mathcal{P}, \in), (Y, \mathcal{Q}, \in)$, where $(X, \mathcal{P}), (Y, \mathcal{Q})$ are topological spaces. We point out that "topological system" will be formally defined later. The relationship of programs to continuous maps is given by the following proposition:

Proposition 2.2.1. $(f, \varphi) : (X, \mathcal{P}, \in) \to (Y, \mathcal{Q}, \in)$ *is a deterministic program if and only if $f : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ is continuous.*

Up to this point, "topological" systems are simply a repackaging of topological spaces—literally a rewriting of spaces with the membership relation; and in that setting, Proposition 2.2.1 points out that deterministic programs correspondingly become repackaged continuous maps. We now construct some example classes which motivate consideration of "topological" systems which cannot be repackaged topological spaces. Such example classes justify the formal and general definition of topological systems and the category **TopSys** given in [75].

Example 2.2.2 (restriction examples). Let $(Z, \mathfrak{T}, \in)$ be a "topological" system as considered above, but with $\mathfrak{T}$ an infinite topology on Z—many such systems exist. Now let $X \subset Z$ be any finite subset of Z, and let "satisfaction" relation $\vDash \ \subset X \times \mathfrak{T}$ be the restriction $\in_{| X \times \mathfrak{T}}$ of the membership relation $\in \ \subset Z \times \mathfrak{T}$. From the standpoint of programming, the system $(X, \mathfrak{T}, \vDash)$ makes good sense as either an input or output system: this is true, in part, because there will in fact be a finite set of inputs or outputs with a potentially unlimited family of predicates; this is consistent with the "finite-unlimited" paradox existing in computer science. Since the predicates of $(X, \mathfrak{T}, \vDash)$ are open sets (from the space $(Z, \mathfrak{T})$), it is reasonable to speak of $(X, \mathfrak{T}, \vDash)$ as a "topological" system in the sense of [70]; however, $(X, \mathfrak{T}, \vDash)$ cannot be a repackaged topological space—a finite set cannot have an infinite topology. Further, it is important to note for such systems that the following "interchange" laws hold:

$$x \vDash \bigcup_{\gamma \in \Gamma} U_\gamma \ \Leftrightarrow \ \exists \beta \in \Gamma,\, x \vDash U_\beta; \ x \vDash \bigcap_{\gamma \in \Gamma} U_\gamma \ \Leftrightarrow \ \forall \gamma \in \Gamma,\, x \vDash U_\gamma \ \ (\Gamma \text{ finite})$$

Borrowing from the formalism to come later in this paper, $(X, \mathfrak{T}, \vDash)$ is a "non-spatial" topological system, which is rather remarkable since its family $\mathfrak{T}$ of predicates is a spatial locale. Finally, a deterministic program (f, φ) between such restricted "topological" systems cannot be the repackaging of a continuous map between topological spaces; and these programs are more general than those covered by Proposition 2.1.1—the backwards map φ^{op} cannot be a restriction to the codomain topology of the preimage operator $f^{\leftarrow}$ of the forward map f of the deterministic program between restricted systems. Thus the simple notion of restricted systems generates a huge example class of systems we want to regard as "topological"systems, but each of which is not the rewriting of a topological space as a system.

A second, huge class of systems which should be regarded as "topological" systems, but each of which is not the rewriting of a topological space as a system, can be constructed, analogously to the above class, by beginning with a "topological" system $(Z, \mathfrak{T}, \in)$ as above, but with $\mathfrak{T}$ having cardinality greater than the continuum, and then letting X be any countable subset of Z. Both of these example classes are included by assuming

$$|\wp(X)| < |\mathfrak{T}| .$$

Example 2.2.3 (prefix ordering examples). We give a subclass of the example class of Example 2.2.2 above which is directly related to programming. Let $\mathbf{2}^*$ be set of all finite (and empty) strings with values from $\{0, 1\}$.

(1) Put $s \sqsubseteq t$ in $\mathbf{2}^*$ if $\exists r \in \mathbf{2}^*$, $s :: r = t$, where $s :: r$ is the concatenation s followed by r. Then $(\mathbf{2}^*, \sqsubseteq)$ is a poset.
(2) For $s \in \mathbf{2}^*$ and $U \subset \mathbf{2}^*$, put

$$\mathbf{starts}\,(s) = \uparrow\!(s) = \{t \in \mathbf{2}^* : s \sqsubseteq t\}\,,$$

$$\mathbf{starts}\,(U) = \bigcup \{\mathbf{starts}\,(s) : s \in U\}\,.$$

Then

$$\mathcal{A}\,(\mathbf{2}^*) := \{U \subset \mathbf{2}^* : U = \mathbf{starts}\,(U)\}$$

is an Alexandrov topology on $\mathbf{2}^*$ with basis $\{\mathbf{starts}\,(s) : s \in U\}$.
(3) We now have a "topological" system $(\mathbf{2}^*, \mathcal{A}\,(\mathbf{2}^*)\,, \in)$ in the sense used just above 2.2.1.
(4) For $s \in \mathbf{2}^*$, $\neg\, s$ is defined to be the bitstring in $\mathbf{2}^*$ formed by interchanging all 0's and 1's; and for $U \subset \mathbf{2}^*$,

$$\neg\, U := \{\neg\, s \in \mathbf{2}^* : s \in U\}\,.$$

Now consider

$$(f, \varphi) : (\mathbf{2}^*, \mathcal{A}\,(\mathbf{2}^*)\,, \in) \to (\mathbf{2}^*, \mathcal{A}\,(\mathbf{2}^*)\,, \in)$$

given by

$$f\,(s) = \neg\, s, \quad \varphi^{op}\,(U) = \neg\, U.$$

Then $(f, \varphi) : (\mathbf{2}^*, \mathcal{A}\,(\mathbf{2}^*)\,, \in) \to (\mathbf{2}^*, \mathcal{A}\,(\mathbf{2}^*)\,, \in)$ is a deterministic program as defined in Subsection 2.1 above and can also be called a *complementation* program or morphism.
(5) Continuing with the complementation morphism (f, φ) from (4) above, let X be a finite subset of $\mathbf{2}^*$, let Y be the corresponding finite subset $f^{\to}\,(X)$ of $\mathbf{2}^*$, and put

$$\vDash_1 \;= \in_{|\, X \times \mathcal{A}(\mathbf{2}^*)}, \quad \vDash_2 \;= \in_{|\, Y \times \mathcal{A}(\mathbf{2}^*)}\,.$$

Then each of $(X, \mathcal{A}\,(\mathbf{2}^*)\,, \vDash_1)$ and $(Y, \mathcal{A}\,(\mathbf{2}^*)\,, \vDash_2)$ are (restricted) "topological" systems as in 2.2.2 above which cannot be repackaged topological spaces, and

$$\left(f_{|\,X}\,, \varphi\right) : (X, \mathcal{A}\,(\mathbf{2}^*)\,, \vDash_1) \to (Y, \mathcal{A}\,(\mathbf{2}^*)\,, \vDash_2)$$

is a deterministic program which is not the repackaging of a continuous map between topological spaces. This "morphism" $\left(f_{|\,X}\,, \varphi\right)$ can also be called a *complementation* program or morphism.

2.3. **Topological systems and TopSys.** A few preliminary notions are needed before topological systems are formally defined.

We first recall that a *complete lattice* is a poset closed under arbitrary $\bigvee$ and $\bigwedge$, including those of the empty set, which means that each complete lattice has a universal lower bound $\bot$ and a universal upper bound $\top$. A *frame* or *locale* A is a complete lattice satisfying the *first infinite distributive law*: $\forall a \in A$, $\forall \{b_\gamma\}_{\gamma \in \Gamma} \subset A$,

$$a \wedge \left(\bigvee_{\gamma \in \Gamma} b_\gamma \right) = \bigvee_{\gamma \in \Gamma} (a \wedge b_\gamma).$$

Frame morphisms are mappings between frames which preserve arbitrary $\bigvee$ and finite $\wedge$; and *localic morphisms* are morphisms between locales which are in a bijection with, and in the opposite direction of, corresponding frame morphisms between the same locales. This information about frames and locales and their associated morphisms are respectively packaged as the categories **Frm** and **Loc** $\equiv$ **Frm**^{op}. We point out that frames and locales are appropriate for computer science because of the role of finite observational logic [75, 9] referred to in Subsection 2.2.

Definition 2.3.1 (ground category **Set** $\times$ **Loc**)**.** The category **Set** $\times$ **Loc** is a product category and comprises the following data:

(1) *Objects:* (X, A), where X is a set, A is a locale.
(2) *Morphisms:* $(f, \varphi) : (X, A) \to (Y, B)$, where $f : X \to Y$ is in **Set** and $\varphi : A \to B$ is in **Loc**, i.e., $\varphi^{op} : B \to A$ is in **Frm**.
(3) *Composition, identities:* component-wise from **Set** and **Loc**.

It can be shown that **Set** $\times$ **Loc** is both complete and cocomplete.

Definition 2.3.2 (category of topological systems)**.** The category **TopSys** of *topological systems* and *continuous mappings* has ground category **Set** $\times$ **Loc** and comprises data subject to axioms as follows:

(1) *Objects:* $(X, A, \vDash)$, where $(X, A) \in |\mathbf{Set} \times \mathbf{Loc}|$ and $\vDash \;\subset X \times A$ is a *satisfaction relation* possessing the arbitrary $\bigvee$ and finite $\wedge$ *interchange laws*:

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \; x \vDash \bigvee_{\gamma \in \Gamma} a_\gamma \;\Leftrightarrow\; \exists \beta \in \Gamma, x \vDash a_\beta;$$

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \; x \vDash \bigwedge_{\gamma \in \Gamma} a_\gamma \;\Leftrightarrow\; \forall \gamma \in \Gamma, x \vDash a_\gamma \;\; (\Gamma \text{ finite})$$

The set X in some examples could be interpreted as bitstrings; A may be interpreted as a locale of (open) predicates; and if $x \vDash a$, then it may be said that (bitstring) x *satisfies* (predicate) a.

(2) *Morphisms:* $(f, \varphi) : (X, A, \vDash_1) \to (Y, B, \vDash_2)$, where $(f, \varphi) : (X, A) \to (Y, B)$ in **Set** $\times$ **Loc** and (f, φ) satisfies *adjointness:*

$$\forall b \in B, \forall x \in X, \ x \vDash_1 \varphi^{op}(b) \Leftrightarrow f(x) \vDash_2 b.$$

(3) *Composition, identities:* from **Set** $\times$ **Loc**.

The reader can check that **TopSys** is indeed a category. The categorical isomorphisms of **TopSys** are called *homeomorphisms*. A ground morphism (f, φ) is a homeomorphism if and only if f and φ^{op} are bijections, (f, φ) is continuous, and $\left(f^{-1}, \left((\varphi^{op})^{-1}\right)^{op}\right)$ is continuous.

As discussed in the previous subsection, the objects of **TopSys** are more general than topological spaces rewritten as "topological" systems and include all the examples in Example 2.2.2 and Example 2.2.3; and topological systems as defined in Definition 2.3.2 include much more than all the systems considered in Subsection 2.2, as will be seen in the next section. Similar comments may be made for morphisms: those defined in 2.3.2 include all those considered in Subsection 2.2 and much more besides.

Given topological systems $(X, A, \vDash_1)$, $(Y, B, \vDash_2)$ respectively interpreted as input and output systems, a **TopSys** morphism $(f, \varphi) : (X, A, \vDash_1) \to (Y, B, \vDash_2)$ is then a deterministic program as discussed in Subsection 2.1. Thus, **TopSys** may be viewed as the category of all systems having open predicates in a generalized way (from locales and not just topologies) and satisfaction relations generalizing (restricted) membership relations, together with all deterministic programs between them.

This completes our trajectory from Dijkstra's programming principles [14] through Smyth's topological point of view [70] to topological systems in the sense of [75], aided by the example classes of [75, 9] and Examples 2.2.2–2.2.3 above.

3. Categorical behavior of TopSys

3.1. Basic categorical properties of TopSys.

It is helpful to specify the *forgetful functor* T : **TopSys** $\to$ **Set** $\times$ **Loc** given by

$$T(X, A, \vDash) = (X, A), \quad T(f, \varphi) = (f, \varphi).$$

The categorical behavior of **TopSys** is tantamount in many cases to the behavior of T.

Proposition 3.1.1 [7, 71]. **TopSys** *is quasi-algebraic over* **Set** $\times$ **Loc** *w.r.t.* T*, i.e.,* T *reflects isomorphisms: thus, if* (f, φ) *is a* **TopSys** *morphism, then* (f, φ) *is a* **Set** $\times$ **Loc** *isomorphism if and only if it is a* **TopSys** *homeomorphism.*

The term "quasi-algebraic" stems from [12]. The proof of 3.1.1 [7, 71] hinges around the fact that, given ground isomorphism (f, φ) with (f, φ) continuous, the biconditionality defining adjointness implies that $\left(f^{-1}, \left((\varphi^{op})^{-1}\right)^{op}\right)$ is also continuous, thus simplifying the definition of homeomorphism given in Subsection 2.3 above. And there is more along this line.

Lemma 3.1.2 [71]**.** *T is transportable and (generating, mono-source)-factorizable in the sense of* [1]*.*

Theorem 3.1.3 [71]**. TopSys** *is essentially algebraic over* **Set** $\times$ **Loc** *w.r.t. T.*

Corollary 3.1.4. TopSys *is complete and cocomplete.*

When we investigate the topological behavior of **TopSys**, a very different story emerges.

Lemma 3.1.5 [7]**.** *T-structured sources [sinks] need not have unique initial [final] lifts; not even singleton T-structured sources need have lifts.*

This lemma means that topological systems lack the initial and final structures typical of classical topological spaces and catalogued in [5] and [30]. In fact, we have the following theorem:

Theorem 3.1.6 [9]**. TopSys** *is not topological over* **Set** $\times$ **Loc** *w.r.t. T. Further,* **TopSys** *is neither mono-, nor epi-, nor (small) existentially, nor (small) essentially topological over* **Set** $\times$ **Loc** *w.r.t. T.*

The proof of Theorem 3.1.6 relies on Lemma 3.1.5; but we note that the first statement of Theorem 3.1.6 follows independently from Example 23.6(4) of [1]: if **TopSys** were topological over **Set** $\times$ **Loc** w.r.t. T, then T would be an isomorphism, which is manifestly not the case.

The rather surprising bottom line is that topological systems are algebraic and not topological. On the other hand, topological systems are intimately related to topology in various ways which will be seen below.

3.2. **TopSys as supercategory of Top and Loc.** The categorical relationships presented below detail how topological systems are related to topological spaces and locales, often in ways which illumine the insights of Dijkstra and Smyth discussed in Section 1 above. We recall that **Top** is the category of topological spaces and continuous maps. Results stated without proof are a blend of [75, 7, 9, 72]; but new results are given proofs.

Theorem 3.2.1. **TopSys** *is a supercategory up to isomorphism of* **Top** *via the full functorial embedding* $E_V : \mathbf{Top} \rightarrowtail \mathbf{TopSys}$ *given by*

$$E_V\left(X, \mathfrak{T}\right) = \left(X, \mathfrak{T}, \in\right),$$

$$E_V\left(f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})\right) = \left(f, \left(\left(f^{\leftarrow}\right)_{|\mathfrak{S}}\right)^{op}\right) : (X, \mathfrak{T}, \in) \to (Y, \mathfrak{S}, \in).$$

This embedding is essentially Smyth's insight in a systems setting. Of course, as documented by the Examples 2.2.2, 2.2.3, there is much more in **TopSys** than the image $E_V^{\rightarrow}(\mathbf{Top})$ as a subcategory of **TopSys**; indeed, these examples are not even homeomorphic in **TopSys** to systems in $E_V^{\rightarrow}(\mathbf{Top})$. Thus the subcategory $E_V^{\rightarrow}(\mathbf{Top})$ of **TopSys** is distinctive and, as it turns out, in a manner parallel to the distinctiveness of the subcategory **SpatLoc** of **Loc**; and this leads to the next discussion and definition used throughout the sequel.

We recall the first Stone comparison map associated with spectra of locales—see [29] and below. Recording that

$$\mathbf{2} = \{\bot, \top\} = \{\mathbf{false}, \mathbf{true}\}$$

is a frame, and given locale A, put

$$pt\left(A\right) = \mathbf{Frm}\left(A, \mathbf{2}\right) = \left\{p : A \to \mathbf{2} \mid p \text{ preserves arbitrary } \bigvee, \text{ finite } \wedge\right\},$$

$$\Phi : A \to \wp\left(pt\left(A\right)\right) \quad \text{by} \quad \Phi\left(a\right) = \left\{p \in pt\left(A\right) : p\left(a\right) = \top\right\}.$$

Then Φ is a frame map, $\Phi^{\rightarrow}(A)$ is a topology on $pt(A)$, and $Pt(A) := (pt(A), \Phi^{\rightarrow}(A))$ is a topological space which is the *spectrum* of A. A locale A is *spatial* if Φ is injective, in which case A is order isomorphic to the topology $\Phi^{\rightarrow}(A)$. It can be shown from the (co)universality of Φ that A is spatial if and only if it is order-isomorphic to some topology. This suggests the following definition for topological systems.

Definition 3.2.1.1 (spatial topological systems)**.** A topological system is *spatial* if it is homeomorphic (in **TopSys**) to some system in $E_V^{\rightarrow}(\mathbf{Top})$. Equivalently, a topological system $(X, A, \models)$ is spatial if and only if there exists a topological space $(Y, \mathfrak{T})$ such that $(X, A, \models)$ is homeomorphic (in **TopSys**) to $(Y, \mathfrak{T}, \in)$.

Proposition 3.2.1.2. *Let* $(X, A, \models)$ *be a topological system.*

(1) *If* $(X, A, \models)$ *is spatial, then the locale* A *is spatial.*
(2) *The converse to (1) need not hold.*

Proof. If $(f, \varphi) : (X, A, \models) \to (Y, \mathfrak{T}, \in)$ is a **TopSys** homeomorphism for some topological space $(Y, \mathfrak{T})$, then $\varphi^{op} : \mathfrak{T} \to A$ is a bijective frame map, and hence a frame isomorphism. Hence A is a spatial locale, and (1) is confirmed. Examples 2.2.2 and 2.2.3 verify (2). □

There is more to the relationship between **Top** and **TopSys** than the embedding E_V, and, indeed, E_V is part of a well-behaved, two way relationship between topological spaces and topological systems using our next functor Ext. The functor Ext is built using a variation of the (first) Stone comparison map Φ recalled above.

Theorem 3.2.2. **TopSys** *functorially maps to* **Top** *via* $Ext : \mathbf{TopSys} \to \mathbf{Top}$ *constructed as follows:*

(1) *Given* $(X, A, \vDash) \in |\mathbf{TopSys}|$, *put* $ext : A \to \wp(X)$ *by*

$$ext(a) = \{x \in X : x \vDash a\}.$$

Then ext is a frame map and $(X, ext^{\to}(A)) \in |\mathbf{Top}|$.

(2) $Ext : \mathbf{TopSys} \to \mathbf{Top}$, *defined by*

$$Ext(X, A, \vDash) = (X, ext^{\to}(A)), \quad Ext(f, \varphi) = f,$$

is a functor.

Theorem 3.2.3. *Ext is both the right adjoint and left inverse of* E_V; *i.e.,*

$$E_V \dashv Ext, \quad Ext \circ E_V = Id_{\mathbf{Top}}.$$

The sense in which $E_V \circ Ext$ is essentially the identity (i.e., up to homeomorphism) characterizes spatiality of systems, as seen in the next proposition.

Theorem 3.2.3.1. *A topological system* $(X, A, \vDash)$ *is spatial if and only if it is homeomorphic (in* **TopSys***) to* $E_V(Ext((X, A, \vDash)))$ *via a homeomorphism of the form* (id_X, φ).

Proof. Sufficiency follows by the definition of spatial systems. For necessity, suppose $(g, \psi) : (X, A, \vDash) \to (Y, \mathfrak{T}, \in)$ is a **TopSys** homeomorphism for some topological space $(Y, \mathfrak{T})$. We are to find $\varphi^{op} : ext^{\to}(A) \to A$ so that $(id_X, \varphi) : (X, A, \vDash) \to (X, ext^{\to}(A), \in)$ is a homeomorphism. We already have that $ext^{|ext^{\to}(A)} : A \to ext^{\to}(A)$ is surjective, so we now show that $ext^{|ext^{\to}(A)}$ is injective. Let $a \neq b$ in A. Then by the bijectivity of ψ^{op}, it follows that $U \neq V$ in $\mathfrak{T}$, where

$$U := (\psi^{op})^{-1}(a), \quad V := (\psi^{op})^{-1}(b).$$

Now let $x \in X$. Then the adjointness of (g, ψ) implies

$$x \vDash a \Leftrightarrow x \vDash \psi^{op}(U) \Leftrightarrow g(x) \in U,$$

$$x \vDash b \Leftrightarrow x \vDash \psi^{op}(V) \Leftrightarrow g(x) \in V.$$

It follows from the bijectivity of g that

$$\begin{aligned} ext\,(a) = ext\,(b) &\Leftrightarrow [\forall x \in X,\ x \vDash a \Leftrightarrow x \vDash b] \\ &\Leftrightarrow [\forall x \in X,\ g\,(x) \in U \Leftrightarrow g\,(x) \in V] \\ &\Leftrightarrow U = V, \end{aligned}$$

a contradiction. Hence $ext\,(a) \neq ext\,(b)$. Hence $ext^{\,|\, ext^{\rightarrow}(A)}$ is injective and hence an order-isomorphism, whose inverse we now denote as $\varphi^{op} : ext^{\rightarrow}(A) \rightarrow A$. The proof is finished by checking the adjointness of $(id_X, \varphi) : (X, A, \vDash) \rightarrow (X, ext^{\rightarrow}(A), \in)$; and this is trivial since it amounts to saying that for $x \in X$ and $ext\,(a) \in ext^{\rightarrow}(A)$, $x \vDash a$ iff $x \in ext\,(a)$. □

The difference $\mathbf{TopSys} - E_V^{\rightarrow}(\mathbf{Top})$ is much larger than documented in Subsection 2.2, and this claim rests in part on the two-way relationship now outlined between locales and topological systems. We begin by seeing how **TopSys** is a supercategory up to isomorphism of **Loc**.

For a locale A, recall the carrier set $pt\,(A)$ of its spectrum $Pt\,(A)$ discussed above, and put

$$\vDash_A \ \subset pt\,(A) \times A \quad \text{by} \quad p \vDash_A a \Leftrightarrow p\,(a) = \mathbf{true}.$$

It can be shown that $\vDash_A$ satisfies the arbitrary join and finite meet interchange laws, yielding a topological system $(pt\,(A), A, \vDash_A)$ similar to the spectrum $Pt\,(A)$ of A, a similarity which we later examine in more detail.

Theorem 3.2.4. TopSys *is a supercategory up to isomorphism of* **Loc** *via the full functorial embedding* $E_{\mathbf{Loc}} : \mathbf{Loc} \rightarrowtail \mathbf{TopSys}$ *given by*

$$E_{\mathbf{Loc}}(A) = (pt\,(A), A, \vDash_A),$$
$$E_{\mathbf{Loc}}(\varphi : A \rightarrow B) = ((\quad) \circ \varphi^{op}, \varphi) : (pt\,(A), A, \vDash_A) \rightarrow (pt\,(B), B, \vDash_B).$$

We now construct counterparts to Examples 2.2.2, 2.2.3 above, namely an example class of topological systems which are not in $E_{\mathbf{Loc}}^{\rightarrow}(\mathbf{Loc})$.

Example 3.2.4.1 (restriction examples)**.** Let A be a locale and choose $X \subset pt\,(A)$ such that $|X| < |pt\,(A)|$. Take the restriction $\vDash_X$ of the satisfaction relation $\vDash_A$ given by

$$\vDash_X \ = (\vDash_A)_{\,|\, X \times A}.$$

Then $(X, A, \vDash_X)$ is a topological system which is not homeomorphic to any system in $E_{\mathbf{Loc}}^{\rightarrow}(\mathbf{Loc})$. To see the latter part of this claim, suppose there is a locale B such that $(X, A, \vDash_X)$ is homeomorphic to $(pt\,(B), B, \vDash_B)$. Then X is bijective with $pt\,(B)$ and A is order-isomorphic to B; and this order-isomorphism implies that $pt\,(A)$ is bijective with $pt\,(B)$, making X bijective with $pt\,(A)$, a contradiction to $|X| < |pt\,(A)|$. It remains to see that the assumption $X \subset pt\,(A)$ such that $|X| < |pt\,(A)|$ is frequently satisfied. We indicate two examples, which the reader can easily expand:

(1) Let A be the four-element Boolean algebra $\{\bot, a, b, \top\}$ with $\bot$ the universal lower bound, $\top$ the universal upper bound, and a, b unrelated. Then it can be shown that $pt(A)$ has precisely two members, call them p, q, in which case, setting $X = \{p\}$ or $X = \{q\}$, $(X, A, \vDash_X)$ is a topological system as claimed above.

(2) Let $A = [0, 1]$ with the usual ordering. Then it can be shown that $pt(A)$ is bijective with $[0, 1)$ and so is uncountable. In this case, X can be chosen as any finite or countable subset of $pt(A)$, and then $(X, A, \vDash_X)$ is a topological system as claimed above.

Example 3.2.4.1 documents that there is much more in **TopSys** than the image $E^{\rightarrow}_{\mathbf{Loc}}(\mathbf{Loc})$ as a subcategory of **TopSys**; indeed, these examples are not homeomorphic in **TopSys** to systems in $E^{\rightarrow}_{\mathbf{Loc}}(\mathbf{Loc})$. Thus the subcategory $E^{\rightarrow}_{\mathbf{Loc}}(\mathbf{Loc})$ of **TopSys** is distinctive and, as it turns out, in a manner parallel to the distinctiveness of the subcategory **SobTop** of **Top**; and this leads to the next discussion and definition.

For a topological space $(X, \mathfrak{T})$, we consider the second Stone comparison map $\Psi : X \to pt(\mathfrak{T})$ constructed as follows—see [29]:

$$\begin{array}{rll} x & \mapsto & \text{irreducible closed } \overline{\{x\}} \\ & \mapsto & \text{prime open } X - \overline{\{x\}} \\ & \mapsto & \text{prime principal ideal } \downarrow\left(X - \overline{\{x\}}\right) \\ & \mapsto & \text{frame map } \chi_{\mathfrak{T} - \downarrow\left(X - \overline{\{x\}}\right)} : \mathfrak{T} \to \mathbf{2}. \end{array}$$

Then $(X, \mathfrak{T})$ is *sober* if $\Psi : X \to pt(\mathfrak{T})$ is a bijection—injectivity is equivalent to $(X, \mathfrak{T})$ being T_0, and sobriety is unrelated to T_1 and implied by Hausdorff separation. Considered as a map between spaces, $\Psi : (X, \mathfrak{T}) \to Pt(\mathfrak{T})$—the latter space being the spectrum of the topology of the first space—is continuous and relatively open; hence $(X, \mathfrak{T})$ is T_0 if and only if Ψ is a homeomorphic embedding, and sober if and only if Ψ is a homeomorphism. Finally, it can be shown, using the universality of Ψ, that a space $(X, \mathfrak{T})$ is sober if and only if it is homeomorphic to the spectrum of some locale. This suggests the following definition [75] for topological systems.

Definition 3.2.4.2 (sober topological systems)**.** A topological system is *localic* or *sober* if it is homeomorphic (in **TopSys**) to some system in $E^{\rightarrow}_{\mathbf{Loc}}(\mathbf{Loc})$. Equivalently, a topological system $(X, A, \vDash)$ is sober if and only if there exists a locale B such that $(X, A, \vDash)$ is homeomorphic (in **TopSys**) to $(pt(B), B, \vDash_B)$.

Proposition 3.2.4.3. *A topological system* $(X, A, \vDash)$ *is sober if and only if* $(X, A, \vDash)$ *is homeomorphic (in* **TopSys***) to* $(pt(A), A, \vDash_A)$ *via some* (f, φ) *with* $\varphi^{op} = id_A$.

Proof. Sufficiency is immediate, and necessity is implicit in the first part of Example 3.2.4.1. To make necessity explicit, suppose there is a locale B such that $(X, A, \models_X)$ is homeomorphic to $(pt(B), B, \models_B)$ via (g, ψ). Then $g : X \to pt(B)$ is a bijection and $\psi^{op} : B \to A$ is an order isomorphism. Put $f : X \to pt(A)$ by

$$f(x) = g(x) \circ (\psi^{op})^{-1} : A \to \mathbf{2}.$$

The claim is that $(f, id_A^{op}) : (X, A, \models) \to (pt(A), A, \models_A)$ is the needed **TopSys** homeomorphism. Now the bijectivity of f follows from that of g and ψ^{op}. To check the adjointness of (f, id_A^{op}), let $x \in X$, $a \in A$. Then $\exists!\, b \in B$, $\psi^{op}(b) = a$. It follows that

$$\begin{aligned} x \models a &\Leftrightarrow x \models \psi^{op}(b) \\ &\Leftrightarrow g(x) \models_B b \\ &\Leftrightarrow g(x)(b) = \top \\ &\Leftrightarrow g(x)\left((\psi^{op})^{-1}(a)\right) = \top \\ &\Leftrightarrow f(x)(a) = \top \\ &\Leftrightarrow f(x) \models_A a. \end{aligned}$$

□

We are now in a position to give further results and characterizations of spatial and sober systems.

Lemma 3.2.4.4. *Let $(X, A, \models)$ be a topological system.*

(1) *If $(X, A, \models)$ is sober, then the space $Ext(X, A, \models)$ is sober.*
(2) *The converse to (1) need not hold.*

Proof. Ad(1). Applying Proposition 3.2.4.3, it may be assumed that $(f, id_A^{op}) : (X, A, \models) \to (pt(A), A, \models_A)$ is a **TopSys** homeomorphism; and, in particular, we have f is a bijection. We are to show $(X, ext^{\to}(A))$ is sober, i.e., that $\Psi : X \to pt(ext^{\to}(A))$ is bijective. It can be shown that the action of $\Psi : X \to pt(ext^{\to}(A))$ may be summarized thusly:

$$\Psi(x)(ext(a)) = \chi_{ext(a)}(x) = \top \Leftrightarrow x \models a.$$

Now applying the adjointness of (f, id_A^{op}), we have, for $x \in X$, $a \in A$, that

$$x \models a \Leftrightarrow f(x) \models_A a \Leftrightarrow f(x)(a) = \top.$$

Hence, $\forall x \in X$, $a \in A$,

$$\Psi(x)(ext(a)) = f(x)(a).$$

Now if $x \neq y$, then the injectivity of f implies that $\exists\, a \in A$, $f(x)(a) \neq f(y)(a)$, so that $\Psi(x)(ext(a)) \neq \Psi(y)(ext(a))$, so Ψ is injective. To show Ψ is onto, let $p \in pt(ext^{\to}(A))$. Then $p \circ ext \in pt(A)$.

Invoking the surjectivity of f, there is $x \in X$, $f(x) = p \circ ext$. Now let $ext(a) \in ext^{\rightarrow}(A)$. Then

$$p(ext(a)) = \top \Leftrightarrow f(x)(a) = \top \Leftrightarrow \Psi(x)(ext(a)) = \top,$$

and so $\Psi(x) = p$.

Ad(2). The example of Example 3.2.4.1(1) suffices to confirm (2). In that example, $pt(A) = \{p, q\}$, and, setting $X = \{p\}$, we have $(X, A, \models_X)$ is a topological system using the restricted satisfaction $\models_X$. But, as shown in 3.2.4.1, this system is not sober. Now the action of p must be either

$$p(a) = p(\top) = \top, \quad p(b) = p(\bot) = \bot$$

or

$$p(b) = p(\top) = \top, \quad p(a) = p(\bot) = \bot.$$

W.L.O.G. assume the former. Then the restricted satisfaction $\models_X$ is as follows: p satisfies precisely a and $\top$. This implies that

$$ext(a) = ext(\top) = \{p\}, \quad ext(b) = ext(\bot) = \varnothing.$$

Hence

$$ext^{\rightarrow}(A) = \{\varnothing, \{p\}\},$$

so that

$$Ext(X, A, \models_X) = (X, \{\varnothing, \{p\}\}),$$

which is a sober topological space. □

Theorem 3.2.4.5. *The following hold:*

(1) *A locale A is spatial if and only if the system $E_{\mathbf{Loc}}(A)$ is spatial.*
(2) *A space $(X, \mathfrak{T})$ is sober if and only if the system $E_V(X, \mathfrak{T})$ is sober.*

Proof. Ad(1). Sufficiency follows from Proposition 3.2.1.2 above. As for necessity, assume that A is spatial, which means that $\Phi^{|\Phi^{\rightarrow}(A)} : A \rightarrow \Phi^{\rightarrow}(A)$ is an order-isomorphism. To finish the proof that

$$\left(id_{pt(A)}, \left(\left(\Phi^{|\Phi^{\rightarrow}(A)}\right)^{-1}\right)^{op}\right) : (pt(A), A, \models_A) \rightarrow (pt(A), \Phi^{\rightarrow}(A), \in)$$

is a **TopSys** morphism, we need only check adjointness. Given $p \in pt(A)$ and $\Phi(a) \in \Phi^{\rightarrow}(A)$, it is trivially the case that

$$p \models_A a \Leftrightarrow p(a) = \top \Leftrightarrow p \in \Phi(a).$$

Ad(2). Assume a space $(X, \mathfrak{T})$ is sober, i.e., $\Psi : X \rightarrow pt(\mathfrak{T})$ is a bijection. For $(\Psi, id_{\mathfrak{T}}^{op}) : (X, \mathfrak{T}, \in) \rightarrow (pt(\mathfrak{T}), \mathfrak{T}, \models_{\mathfrak{T}})$ to be a **TopSys** homeomorphism, adjointness should be checked: given $x \in X$, $U \in \mathfrak{T}$,

$$x \in U \Leftrightarrow \Psi(x)(U) = \top \Leftrightarrow \Psi(x) \models_{\mathfrak{T}} U,$$

finishing necessity. Sufficiency follows from Lemma 3.2.4.4(1) by applying the identity $Ext \circ E_V = Id_{\mathbf{Top}}$ from Theorem 3.2.3. □

$E_{\mathbf{Loc}}$ is part of a well-behaved, two way relationship between topological spaces and topological systems, as seen in Theorem 3.2.6, using the functor Ω_V introduced in Theorem 3.2.5.

Theorem 3.2.5. **TopSys** *functorially maps to* **Loc** *via* $\Omega_V : \mathbf{TopSys} \to \mathbf{Loc}$ *by*

$$\Omega_V\left(X, A, \models\right) = A, \quad \Omega_V\left(f, \varphi\right) = \varphi.$$

Theorem 3.2.6. Ω_V *is both a left adjoint and left inverse of* $E_{\mathbf{Loc}}$; *i.e.,*

$$\Omega_V \dashv E_{\mathbf{Loc}}, \quad \Omega_V \circ E_{\mathbf{Loc}} = Id_{\mathbf{Loc}}.$$

We have presented **TopSys** as a supercategory (up to categorical isomorphism) of both **Top** and **Loc**. But **TopSys** is much more than a supercategory of these categories. In the following discussion, we give a rather complete description of the internalization within **TopSys** of fundamental representation theorems.

Discussion 3.2.7 (internalization of sobriety-spatiality representation)**.** To internalize representation *within* **TopSys**, we need two functors from Stone representation theory:

$$\begin{aligned}
\Omega : \mathbf{Top} \to \mathbf{Loc} \quad &\text{by} \quad \Omega\left(X, \mathfrak{T}\right) = \mathfrak{T}, \\
&\Omega\left[f : \left(X, \mathfrak{T}\right) \to \left(Y, \mathfrak{S}\right)\right] = \left[\left(\left(f^{\leftarrow}\right)_{|\mathfrak{S}}\right)^{op} : \mathfrak{T} \to \mathfrak{S}\right], \\
Pt : \mathbf{Loc} \to \mathbf{Top} \quad &\text{by} \quad Pt\left(A\right) = \left(pt\left(A\right), \Phi^{\rightarrow}\left(A\right)\right), \\
&Pt\left[\varphi : A \to B\right] = \left[\left(\ \right) \circ \varphi^{op} : Pt\left(A\right) \to Pt\left(B\right)\right].
\end{aligned}$$

The most important functorial relationship in representations à la Stone is the Dowker-Papert-Isbell adjunction $\Omega \dashv Pt$ [46, 15, 28, 29], which when restricted on both sides to the category **SobTop** of sober spaces and continuous maps and the category **SpatLoc** of spatial locales and localic morphisms, respectively, yields the foundational representation theorem

$$\mathbf{SobTop} \sim \mathbf{SpatLoc},$$

namely, that **SobTop** and **SpatLoc** are categorically equivalent and hence behave similarly in categorical terms (but they are not isomorphic). When this categorical equivalence is suitably restricted, it eventually yields the Stone representation theorems for distributive lattices and Boolean algebras [29]. The adjunction $\Omega \dashv Pt$ is also important because it converts a choice-free and point-free Čech-Stone compactification into the point-set version (using AC) [29], as well as a choice-free and point-free Hahn-Banach Theorem into its point-set analogue (again using AC) [45]. Now $\Omega \dashv Pt$ and **SobTop** $\sim$ **SpatLoc** are relationships *between* categories, indeed between categories and their subcategories which can be embedded into **TopSys**. What is nice about **TopSys** is that these relationships are internalized within **TopSys** as well. In particular, using the functors of this subsection (cf. [75]), it can be shown that

$$[\Omega \dashv Pt] \;=\; [(\Omega_V \circ E_V) \;\dashv\; (Ext \circ E_{\mathbf{Loc}})]\,. \qquad \square$$

4. Topological systems and fixed-basis lattice-valued topology

Having motivated and positioned topological systems with respect to programming semantics, traditional topological spaces, and locales, we shift our attention to many-valued topologies and show that topological systems are intimately connected to this part of topology as well. This section looks at the relationship between topological systems and "fixed-basis" lattice-valued topology, while the next section examines the relationship between topological systems and "variable-basis" lattice-valued topology. The terms "basis" and "base" in these contexts refer to a lattice of truth or membership values which occurs in the base of an expression of the form L^X. And in this and the next sections, lattices of truth values are assumed to be frames; generalizations are briefly considered in the last section.

4.1. **Fixed-basis lattice-valued powerset monad.** This subsection comes from [77, 18, 60, 61, 65]. Fix a set X and a frame L. Then the *L-powerset* of X, comprising all its *L-subsets*, is

$$L^X = \{a \mid a : X \to L\}\,,$$

equipped with the order lifted pointwise from that of L, and hence equipped also with all least upper bounds and greater lower bounds lifted pointwise from L; and so L^X is a frame. For $a \in L^X$ and $x \in X$, $a\,(x)$ is interpreted as the *degree of membership of x in the L-subset a*.

Let $f : X \to Y$ be a function. Then the (L-)*image* and (L-)*preimage* and (L-)*lower image* operators of f are as follows:

$$f_L^{\to} : L^X \to L^Y \quad \text{by} \quad f_L^{\to}(a)(y) = \bigvee_{f(x)=y} a(x) = \bigvee_{x \in f^{\leftarrow}\{y\}} a(x)\,,$$

$$f_L^{\leftarrow} : L^X \leftarrow L^Y \quad \text{by} \quad f_L^{\leftarrow}(b) = b \circ f\,,$$

$$f_{L\to} : L^X \to L^Y \quad \text{by} \quad f_{L\to}(a) = \bigvee_{f_L^{\leftarrow}(b) \le a} b.$$

Theorem 4.1.1. $\left(L^{(\;)}, f\,, f_L^{\to} \dashv f_L^{\leftarrow} \dashv f_{L\to}\right)$ *is the L-valued powerset monad associated with $f : X \to Y$. More precisely, the following hold:*

(1) *The adjunctions $f_L^{\to} \dashv f_L^{\leftarrow} \dashv f_{L\to}$ hold.*

(2) $f_L^{\rightarrow}$ *preserves order and arbitrary* $\bigvee$ *and universal lower bounds*; $f_L^{\leftarrow}$ *preserves order and arbitrary* $\bigvee$ *and arbitrary* $\bigwedge$ *and universal upper and lower bounds*; *and* $f_{L\rightarrow}$ *preserves order and arbitrary* $\bigwedge$ *and universal upper bounds.*

(3) $\left(L^{(\ )}, f, f_L^{\rightarrow} \dashv f_L^{\leftarrow} \dashv f_{L\rightarrow}\right)$ *lifts the traditional powerset monad* $(\wp(\ ), f, f^{\rightarrow} \dashv f^{\leftarrow} \dashv f_{\rightarrow})$. *In particular:*

(a) $\forall A \in \wp(X)$,

$$f_L^{\rightarrow}(\chi_A) = \chi_{f^{\rightarrow}(A)}; \quad f_L^{\rightarrow} \circ \rightarrowtail = \rightarrowtail \circ f^{\rightarrow}.$$

(b) $\forall B \in \wp(Y)$,

$$f_L^{\leftarrow}(\chi_B) = \chi_{f^{\leftarrow}(B)}; \quad f_L^{\leftarrow} \circ \rightarrowtail = \rightarrowtail \circ f^{\leftarrow}.$$

4.2. **L-Topological spaces and categories L-Top's and their behavior.** The main definition below is essentially from [6, 18], while the notation and results are from [25, 62].

Definition 4.2.1. The category L-**Top** of (L-)*topological spaces* and (L-)*continuous mappings* has ground category **Set** and comprises data subject to axioms as follows:

(1) *Objects:* (X, τ), where X is a set and τ is a subframe of L^X, i.e., $\tau \subset L^X$ is closed under arbitrary $\bigvee$ and finite $\wedge$.

(2) *Morphisms:* $f : (X, \tau) \rightarrow (Y, \sigma)$, where $f : X \rightarrow Y$ be a function and $(f_L^{\leftarrow})^{\rightarrow}(\sigma) \subset \tau$, i.e.,

$$\forall v \in \sigma, \ f_L^{\leftarrow}(v) \in \tau.$$

(3) *Composition, Identities:* from **Set**.

Let forgetful functor $V_L : L$-**Top** $\rightarrow$ **Set** be given by

$$V_L(X, \tau) = X, \quad V_L(f) = f.$$

Theorem 4.2.2. *Each* V_L*-structured source [sink] in* **Set** *has a unique initial [final] lift to* L-**Top**. *So* L-**Top** *is topological over* **Set** *w.r.t.* V_L.

Corollary 4.2.3. *For each* $L \in |\mathbf{Frm}|$, L-**Top** *is complete and cocomplete.*

4.3. **Example classes of L-topological spaces.**

Example 4.3.1 (first class of examples—characteristic functors)**.** These are functor-generated examples. For L consistent, G_χ : **Top** $\rightarrowtail$ L-**Top**, defined by

$$G_\chi(X, \mathfrak{T}) = (X, \{\chi_U : U \in \mathfrak{T}\}), \ \ G_\chi(f) = f,$$

is a concrete embedding; and it is an isomorphism when $L = \mathbf{2}$, i.e., **Top** $\approx$ **2-Top**. Hence for each consistent L, G_χ generated examples include all traditional topological spaces and continuous mappings as a subcategory.

This is part of a two-way relationship, the other direction being M_χ : **Top** $\leftarrow$ L-**Top** [43], given by

$$M_\chi(X,\tau) = (X, \{U \subset X : \chi_U \in \tau\}), \quad M_\chi(f) = f$$

which is a concrete functor. Further,

$$M_\chi \dashv G_\chi,$$

and this adjunction is a monocoreflection, but not an equivalence. G_χ will be referenced throughout the sequel.

Example 4.3.2 (second class of examples—lower semi-continuity)**.** These are functor-generated examples [41, 55, 36]. The correspondence ω_L : **Top** $\to$ L-**Top**, defined by

$$\omega_L(X,\mathfrak{T}) := (X, \omega_L(\mathfrak{T})) := (X, \langle\langle \{u : X \to L \mid \forall \alpha \in L, [u \not\leq \alpha] \in \mathfrak{T}\}\rangle\rangle),$$
$$\omega_L(f) = f,$$

is a concrete functor for which the L-topology of the image is generated by the (L-valued) subbase written within the double brackets $\langle\langle\ \ \rangle\rangle$. It should be noted that this subbasis comprises all continuous maps with respect to the given topology $\mathfrak{T}$ on X and the upper topology on L. Now ω_L is part of a two-way relationship, the other direction being ι_L : L-**Top** $\to$ **Top**, given by

$$\iota_L(X,\tau) = (X, \langle\langle\{[u \not\leq \alpha] : u \in \tau,\ \alpha \in L\}\rangle\rangle), \quad \iota_L(f) = f,$$

which is also a concrete functor. It is the case that

$$\omega_L \dashv \iota_L,$$

both functors reflect lifted morphisms [9], and ω_L preserves products (unusual for a left-adjoint). Further, if L is completely distributive, then the following improvements result [36]:

(1) The L-subbases of the ω_L image objects given above are L-topologies.
(2) ω_L : **Top** $\rightarrowtail$ L-**Top** is a categorical embedding.
(3) $\omega_L \dashv \iota_L$ is also an epicoreflection with ι_L as left-inverse of ω_L.
(4) The action of ω_L is "stratification" in the following sense: for each topological space $(X,\mathfrak{T})$, the L-topology $\omega_L(\mathfrak{T})$ of $\omega_L(X,\mathfrak{T})$ is given by

$$\omega_L(\mathfrak{T}) = G_\chi(\mathfrak{T}) \vee \{\underline{\alpha} : X \to L \mid \alpha \in L, (\forall x \in X, \underline{\alpha}(x) = \alpha)\},$$

where "$\vee$" indicates the smallest L-topology containing both families of mappings. Restated,

$$\omega_L(\mathfrak{T}) = \langle\langle\{\chi_U : U \in \mathfrak{T}\} \cup \{\underline{\alpha} : X \to L \mid \alpha \in L\}\rangle\rangle.$$

Any L-topology τ on X which contains $\{\underline{\alpha} : X \to L \mid \alpha \in L\}$ is said to be *stratified.*

It should be noted that the right hand side of the preceding display defines an L-topology for any given L, in which case it is denoted $G_\omega(\mathfrak{T})$, defining yet another concrete functor $G_\omega : \mathbf{Top} \to L\text{-}\mathbf{Top}$ which is a categorical embedding and coincides with ω_L when L is completely distributive [36]. It remains an open question, after more than 25 years [55], whether in general $\omega_L = G_\omega$. The functor G_ω plays its own critical role in Subsection 5.4 below.

Example 4.3.3 (third class of examples—the L-spectrum)**.** These examples include all L-spectra of locales and even complete lattices—a generalization of traditional spectra—and these are generated by the L-spectrum functor [53, 22, 56, 32, 57, 58, 59, 63, 48, 33, 50]. Put

$$L\Omega : L\text{-}\mathbf{Top} \to \mathbf{Loc} \quad \text{by} \quad L\Omega(X,\tau) = \tau,$$
$$L\Omega[f : (X,\tau) \to (Y,\sigma)] = \left[\left(\left(f_L^{\leftarrow}\right)_{|\sigma}\right)^{op} : \tau \to \sigma\right];$$

$$Lpt(A) = \mathbf{Frm}(A,L) = \left\{p : A \to L \mid p \text{ preserves arbitrary } \bigvee, \text{ finite } \wedge\right\};$$

$$\Phi_L : A \to L^{Lpt(A)} \quad by \quad \Phi_L(a)(p) = p(a);$$
$$LPt : L\text{-}\mathbf{Top} \leftarrow \mathbf{Loc} \quad \text{by} \quad LPt(A) := (Lpt(A), (\Phi_L)^{\rightarrow}(A)),$$
$$LPt[\varphi : A \to B] = [(\ ) \circ \varphi^{op} : LPt(A) \to LPt(B)].$$

Then the following hold:

(1) $L\Omega$ and LPt are functors.

(2) $L\Omega \dashv LPt$, with counits $\left((\Phi_L)^{|(\Phi_L)^{\rightarrow}(A)}\right)^{op} : (\Phi_L)^{\rightarrow}(A) \to A$ in **Loc**, and units $\Psi_L : (X,\tau) \to LPt(\tau)$ defined by
$$\Psi_L : X \to Lpt(\tau) \quad \text{by} \quad \Psi_L(x) : \tau \to L \quad \text{by} \quad \Psi_L(x)(u) = u(x).$$

(3) $\Phi_L : A \to L^{Lpt(A)}$ is a frame map—so that $(Lpt(A), (\Phi_L)^{\rightarrow}(A))$ is an L-topological space, and $(\Phi_L)^{|(\Phi_L)^{\rightarrow}(A)} : A \to (\Phi_L)^{\rightarrow}(A)$ is injective if and only if it is an order-isomorphism—in which case A is L-*spatial*.

(4) $\Psi_L : (X,\tau) \to LPt(\tau)$ is L-continuous and relatively L-open, an L-homeomorphic embedding if and only if Ψ_L is injective—in which case (X,τ) is L-T_0, and an L-homeomorphism if and only if Ψ_L is bijective—in which case (X,τ) is L-*sober*.

(5) The restriction of $L\Omega \dashv LPt$, respectively, to L-sober topological spaces and L-spatial locales yields a categorical equivalence between L-**SobTop** and L-**SpatLoc**, i.e.,
$$L\text{-}\mathbf{SobTop} \sim L\text{-}\mathbf{SpatLoc}.$$

(6) There are schemata of "Stone" representation theorems and compactifications based upon (5) indexed by categories for L.

(7) The above functors give more relationships between **Top** and L-**Top**: $LPt \circ \Omega : \mathbf{Top} \to L\text{-}\mathbf{Top}$; $Pt \circ L\Omega : L\text{-}\mathbf{Top} \to \mathbf{Top}$; $LPt \circ \Omega \dashv Pt \circ L\Omega$; and when restricted respectively to L-**SobTop** and **SobTop**, $LPt \circ \Omega \dashv Pt \circ L\Omega$ restricts to a categorical equivalence, in which case the reverse adjunction $Pt \circ L\Omega \dashv LPt \circ \Omega$ also holds.

An alternative approach to the L-spectrum is developed in [48, 49, 50].

Example 4.3.4 (fourth class of examples—fuzzy real lines from probability distributions)**.** We now outline the fuzzy real lines and fuzzy unit intervals from the standpoint of L-probability distributions [26, 17, 64]. Fix L a DeMorgan frame—a frame equipped with an order-reversing involution $'$. We begin with a series of notations and definitions, where $\mathbb{R}$ denotes the traditional real line. Letting $\lambda : \mathbb{R} \to L$ be an antitone map and $t \in \mathbb{R}$, we adopt this series of notations:

$$\lambda(t-) = \bigwedge_{s<t} \lambda(s), \quad \lambda(t+) = \bigvee_{s>t} \lambda(s), \quad \lambda((-\infty)+) = \bigvee_{s \in \mathbb{R}} \lambda(s), \quad \lambda((+\infty)-) = \bigwedge_{s \in \mathbb{R}} \lambda(s).$$

These notations allow us to state the following series of definitions:

$$Real(L) := \{\lambda : \mathbb{R} \to L \mid \lambda \text{ antitone}, \lambda((+\infty)-) = \bot, \lambda((-\infty)+) = \top\},$$

$$\lambda \sim \mu \Leftrightarrow [\forall t \in \mathbb{R}, \lambda(t+) = \mu(t+)] \Leftrightarrow [\forall t \in \mathbb{R}, \lambda(t-) = \mu(t-)],$$

$$[\lambda] := \{\mu \in Real(L) : \lambda \sim \mu\},$$

$$\mathbb{R}(L) := Real(L) / \sim,$$

$$\forall t \in \mathbb{R}, L_t, R_t : \mathbb{R}(L) \to L \quad \text{by} \quad L_t[\lambda] = (\lambda(t-))', \; R_t[\lambda] = \lambda(t+),$$

$$\tau(L) := \langle\langle\{L_t, R_t : t \in \mathbb{R}\}\rangle\rangle.$$

From all these notions come the following comments and results:

(1) $(\mathbb{R}(L), \tau(L))$ is an L-topological space, called the *L-(fuzzy) real line* and also denoted simply $\mathbb{R}(L)$.

(2) $(\mathbb{R}(\mathbf{2}), \tau(\mathbf{2}))$ is **2**-homeomorphic to $G_\chi(\mathbb{R}, \mathfrak{T})$, where $\mathbb{R}$ has the usual topology $\mathfrak{T}$.

(3) For $r \in \mathbb{R}$, define $\lambda_r : \mathbb{R} \to L$ by $\lambda_r(t) = \begin{cases} \top, & t < r \\ \bot, & t > r \end{cases}$. Then $r \mapsto [\lambda_r]$ is an L-embedding of $G_\chi(\mathbb{R}, \mathfrak{T})$ into $(\mathbb{R}(L), \tau(L))$ if L is consistent, in which case it also follows that $\mathbb{R}(\mathbf{2}) = \{[\lambda_r] : r \in \mathbb{R}\}$ and that

$$L_t[\lambda_r] = \chi_{(-\infty, t)}(r), \quad R_t[\lambda_r] = \chi_{(t, +\infty)}(r).$$

Thus, for L consistent, and identifying $\mathbb{R}(\mathbf{2})$ with $\mathbb{R}$, L_t extends the left-handed, subbasic open interval $(-\infty, t)$ and R_t extends the right-handed, subbasic open interval $(t, +\infty)$: this justifies the notation "L_t" and "R_t" for the subbasic L-open subsets of the L-topology $\tau(L)$ on $\mathbb{R}(L)$.

(4) For L a complete DeMorgan chain, there are jointly L-continuous addition $\oplus$ and multiplication $\otimes$ extending the usual addition and multiplication. For example, $(\mathbb{R}(L), \oplus, \tau(L))$ is an abelian, cancellation, L-topological semigroup.

(5) $\mathbb{R}(L)$ is L-T_0 (Example 3.3.3–3.3.4); $\mathbb{R}(L)$ is L-sober (3.3.3–3.3.4)) if and only if L is a complete Boolean algebra; and, for L completely distributive, $\mathbb{R}(L)$ is Hutton-uniformizable, metrizable via $d : L^{\mathbb{R}(L)} \times L^{\mathbb{R}(L)} \to [0, +\infty)$ extending the Euclidean metric in the sense that $d\left(\chi_{\{[\lambda_r]\}}, \chi_{\{[\lambda_s]\}}\right) = |r - s|$, and hence possesses all Hutton-Reilly separation axioms [27, 52, 35, 59, 37, 63].

(6) The subspace of $\mathbb{R}(L)$ resulting from restricting $Real(L)$ to

$$\{\lambda \in Real(L) \mid \lambda(t) = \bot \text{ for } t > 1,\ \lambda(t) = \top \text{ for } t < 0\}$$

is called the *L-(fuzzy) unit interval* and denoted $\mathbb{I}(L)$; and analogues of (1–3,5) above hold for $\mathbb{I}(L)$.

The fuzzy real lines and fuzzy unit intervals are among the most important examples of many-valued topology and possess an extensive literature, a brief sample of which comprises [26, 17, 52, 44, 36, 64, 38]. With these examples, lattice valued topology has Urysohn Lemmas, Tietze Extension Theorems, Tihonov cubes, etc.

Example 4.3.5 (fifth class of examples—fuzzy real lines from L-spectra)**.** Fix L a frame. Given the usual real line $(\mathbb{R}, \mathfrak{T})$ and unit interval $(\mathbb{I}, \mathfrak{T}(\mathbb{I}))$, recall the functor LPt from Example 4.3.3 and put

$$\mathbb{R}^*(L) := LPt(\mathfrak{T}) = \left(Lpt(\mathfrak{T}), (\Phi_L)^{\rightarrow}(\mathfrak{T})\right),$$

$$\mathbb{I}^*(L) := LPt(\mathfrak{T}(\mathbb{I})) = \left(Lpt(\mathfrak{T}(\mathbb{I})), (\Phi_L)^{\rightarrow}(\mathfrak{T}(\mathbb{I}))\right).$$

Then the following hold [63, 64]:

(1) For $L = \mathbf{2}$, $\mathbb{R}^*(\mathbf{2})$ is homeomorphic to $(\mathbb{R}, \mathfrak{T})$ and $\mathbb{I}^*(\mathbf{2})$ is homeomorphic to $(\mathbb{I}, \mathfrak{T}(\mathbb{I}))$.

(2) $G_\chi(\mathbb{R}, \mathfrak{T})$ L-embeds into $\mathbb{R}^*(L)$; $G_\chi(\mathbb{I}, \mathfrak{T}(\mathbb{I}))$ L-embeds into $\mathbb{I}^*(L)$.

(3) There are jointly L-continuous addition $\boxplus$ and multiplication $\boxtimes$ extending the usual addition and multiplication.

(4) Assume L a DeMorgan frame. Then the following are equivalent:
  (a) $\mathbb{R}(L)$ and $\mathbb{R}^*(L)$ are L-homeomorphic.
  (b) $\mathbb{I}(L)$ and $\mathbb{I}^*(L)$ are L-homeomorphic.
  (c) L is a complete Boolean algebra.

(5) $\mathbb{I}^*(L)$ is the L-Čech-Stone compactification of $\mathrm{G}_\chi([0,1])$; and for L a complete Boolean algebra, $\mathbb{I}(L)$ is the L-Čech-Stone compactification of $\mathrm{G}_\chi([0,1])$.

This class of examples gives the frame maps and prime principal ideal approach to fuzzy numbers. The results of (4) seem rather astounding: for L a complete Boolean algebra, the probability distribution approach to fuzzy numbers is equivalent to the frame maps approach to fuzzy numbers. And (5) is also surprising: the traditional unit interval is not as compact as it could be if L is bigger than $\mathbf{2}$; and, for example, for the Boolean algebra $\mathbf{4}$, $\mathbb{I}(\mathbf{4})$ has extra points of "density" and "closure", is more compact than $[0,1]$, and $[0,1]$ in fact $\mathbf{4}$-embeds into $\mathbb{I}(\mathbf{4})$ as a "$\mathbf{4}$-dense" subset.

Example 4.3.6 (sixth class of examples—dual L-topologies on $\mathbb{R}$)**.** Fix L a DeMorgan frame and recall the subbasis $\{L_t, R_t : t \in \mathbb{R}\}$ for the L-topology $\tau(L)$ on $\mathbb{R}(L)$ from Example 4.3.4 above. Put [54, 55]

$$L_{[\lambda]}, R_{[\lambda]} : \mathbb{R} \to L \quad \text{by} \quad L_{[\lambda]}(t) = L_t[\lambda], \; R_{[\lambda]}(t) = R_t[\lambda],$$
$$\tau[L] = \left\langle\left\langle \left\{ L_{[\lambda]}, R_{[\lambda]} : [\lambda] \in \mathbb{R}(L) \right\} \right\rangle\right\rangle,$$
$$\mathbb{R}_L := (\mathbb{R}, \tau[L]).$$

The following hold:

(1) $\mathbb{R}_L$ is an L-topological space, called the *dual L-real line*.
(2) $\mathbb{R}_L$ is L-homeomorphic to $G_\chi(\mathbb{R}, \mathfrak{T})$ if $L = \mathbf{2}$.
(3) $\mathbb{R}_L$ is L-homeomorphic to $\omega_L(\mathbb{R}, \mathfrak{T})$ if L is completely distributive (where ω_L is given in Example 4.3.2).
(4) The usual addition and multiplication are both jointly L-continuous in $\mathbb{R}_L$.
(5) $\mathbb{R}_L$ is L-T_0; and, for L completely distributive, $\mathbb{R}_L$ is Hutton-uniformizable, metrizable via $d : L^{\mathbb{R}} \times L^{\mathbb{R}} \to [0, +\infty)$ extending the Euclidean metric in the sense that $d\left(\chi_{\{r\}}, \chi_{\{s\}}\right) = |r - s|$, and hence possesses all Hutton-Reilly separation axioms.
(6) $\mathbb{I}_L$, as the subspace of $\mathbb{R}_L$ on $[0,1]$, has analogues of (1–3,5).

4.4. **Embeddings of L-Top's into TopSys.** This subsection begins the process of connecting topological systems and L-topological spaces, a process continuing into the next section—see Subsection 5.4 below, and constructs a class of embeddings of the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$ into **TopSys** [7], embeddings which place all the example classes of Subsection 4.3 into **TopSys**.

Fix a frame L, and put

$$L^\bullet = L - \{\top\}, \quad \Pr(L^\bullet) = \{\alpha \in L^\bullet : \alpha \text{ prime}\}.$$

Lemma 4.4.1. *For each $\alpha \in \Pr(L^\bullet)$, there is a functorial embedding $F_\alpha : L\text{-}\mathbf{Top} \rightarrowtail \mathbf{TopSys}$ constructed as follows:*

$$F_\alpha(X, \tau) = (X, \tau, \vDash_\alpha) \quad \textit{by} \quad x \vDash_\alpha u \Leftrightarrow u(x) \nleq \alpha,$$
$$F_\alpha[f : (X, \tau) \to (Y, \sigma)] = \left(f, \left(\left(f_L^{\leftarrow}\right)_{|\sigma}\right)^{op}\right) : (X, \tau, \vDash_\alpha) \to (Y, \sigma, \vDash_\alpha).$$

This lemma gives many-valued extensions of the Vickers embedding E_V : **Top** $\rightarrowtail$ **TopSys** discussed in Section 2. Viewing satisfaction as a kind of generalized membership relation and $u(x)$ as a degree of membership, then the satisfaction relation constructed here is the restriction that the degree of membership cannot be less than a prechosen prime α in the frame of membership values.

The question arises: what if L has no primes? For examples: L could be the family of all regular open subsets of $\mathbb{R}$ (with the usual topology) ordered by inclusion; L could be the localic product of the subspace topology of the rational numbers with itself; or L could be the family of all Lebésgue measurable subsets of $[0,1]$ with the measure 0 subsets identified and the measure 1 subsets identified. For such "atomless" frames L, how is L-**Top** to be embedded into **TopSys**?

We proceed as follows.

(1) For any given frame L, adjoin a new bottom $\bot^*$ to L which is required to be strictly below the bottom $\bot$ of L. This yields a new frame $L_{\bot^*}$ having the property that
$$\bot^* \in \Pr\left(L^{\bullet}_{\bot^*}\right).$$
(2) By Lemma 4.4.1, there exists a functorial embedding $F_{\bot^*}$: $L_{\bot^*}$-**Top** $\rightarrowtail$ **TopSys**.

(3) Construct the concrete functorial embedding $\hookrightarrow$: L-**Top** $\to$ $L_{\bot^*}$-**Top** by
$$(X,\tau) \mapsto (X,\tau_\bot) \quad \text{with} \quad \tau_\bot = \tau \cup \{\underline{\bot^*}\}, \quad f \mapsto f,$$
where $\underline{\bot^*} : X \to L$ is the constant subset given by $\underline{\bot^*}(x) = \bot^*$.

Theorem 4.4.2. *For each frame L, there is a functorial embedding $E_\bot$: L-**Top** $\rightarrowtail$ **TopSys** given by*
$$E_\bot = F_{\bot^*} \circ \hookrightarrow .$$

Corollary 4.4.3. *The embedding E_V : **Top** $\rightarrowtail$ **TopSys** is recovered from Lemma 4.4.1 by choosing $L = \mathbf{2}$ and noting that*
$$E_V = F_\bot \circ G_\chi,$$
where G_χ is from Example 4.3.1.

Corollary 4.4.4. **TopSys** *is a supercategory, up to categorical isomorphisms, of* **Top**, **Loc**, *and the schema* $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$.

The Corollary assures us that the rich inventories of examples in Subsection 4.3 are included in **TopSys**.

5. Topological systems and variable-basis lattice-valued topology

Heuristically, "variable-basis" topology means spaces with different lattice-theoretic bases are all accommodated within the same category; morphisms change both underlying carrier sets and underlying lattices (or bases) of truth/membership values. See a history of variable-basis thinking in many-valued mathematics in Section 1 of [62], a history that begins in a veiled form in 1967 [18] and becomes more explicit in 1981 and 1983 [51]; and a fairly recent update packaged in the language of powerset theories and topological theories may be found in [65]. From a more formal point of view, variable-basis topology is a schema of categories **C-Top**, each with ground category $\mathbf{Set} \times \mathbf{C}$, where $\mathbf{C}^{op}$ is a concrete category of lattice-theoretic structures. By convention, this schema is of the form

$$\{\mathbf{C\text{-}Top} : \mathbf{C} \hookrightarrow \mathbf{SQuant}^{op}\},$$

where **SQuant** [65] is the category of semiquantales (complete lattices with a binary operation $\otimes$) and semiquantale morphisms (mappings preserving arbitrary $\bigvee$ and $\otimes$).

To both simplify this section and facilitate relationships with **TopSys**, which has ground category $\mathbf{Set} \times \mathbf{Loc}$, we choose $\mathbf{C} = \mathbf{Loc}$ in the above schema and focus on the category **Loc-Top** in the sequel; but we recognize that most of what follows works with other, more general choices of $\mathbf{C}$ [62, 65].

5.1. Motivations and powerset monad for Loc-Top.

Discussion 5.1.1. It would be desirable to have a topological category satisfying the following conditions:

(1) it is a supercategory up to isomorphism of both **Top** and **Loc**;
(2) it is a supercategory of the fixed-basis schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$, thus allowing for "internalized" change of basis *vis-a-vis* the "externalized" change of basis given in [25]—whereby conditions are given under which there is a functor from L-**Top** to M-**Top** for different L and M; and
(3) it is a framework for asking and answering the following "comparison" questions concerning several of the example classes given in Subsection 4.3 above:
   (a) how do $\mathbb{R}(L)$ and $\mathbb{R}(M)$ compare and when are they homeomorphic;
   (b) how do $LPt(A)$ and $Mpt(A)$ compare and when are they homeomorphic;
   (c) how do $\mathbb{R}^*(L)$ and $\mathbb{R}^*(M)$ compare and when are they homeomorphic; and

(d) how do $\omega_L(X,\mathfrak{T})$ and $\omega_M(X,\mathfrak{T})$ compare and when are they homeomorphic?

We note that **TopSys** satisfies 5.1.1(1,2); however, **TopSys** is not a topological category. Also, it is unexplored in what sense (3) could be asked and answered in **TopSys**.

Definition 5.1.2 [53, 60, 61, 65]**.** The powerset monad for **Loc-Top** has ground category **Set** $\times$ **Loc**. Let $(f,\varphi):(X,L)\to(Y,M)$ be in **Set**$\times$**Loc**.

(1) Put $(f,\varphi)^{\leftarrow}:L^X\leftarrow M^Y$ by

$$(f,\varphi)^{\leftarrow}(b)=\varphi^{op}\circ b\circ f,\quad \text{i.e.,}\quad \varphi^{op}\circ f_L^{\leftarrow}(b).$$

(2) Put $(f,\varphi)^{\rightarrow}:L^X\to M^Y$ by

$$(f,\varphi)^{\rightarrow}(a)=\bigwedge_{a\,\leq\,(f,\varphi)^{\leftarrow}(b)} b.$$

In the language of [65], $(f,\varphi)^{\rightarrow}$ is the *pseudo-left adjoint* of $(f,\varphi)^{\leftarrow}$.

(3) Put $(f,\varphi)_{\rightarrow}:L^X\to M^Y$ by

$$(f,\varphi)_{\rightarrow}(a)=\bigvee_{(f,\varphi)^{\leftarrow}(b)\,\leq\,a} b.$$

For terminology, $(f,\varphi)^{\rightarrow},(f,\varphi)^{\leftarrow},(f,\varphi)_{\rightarrow}$ are respectively called the *image, preimage,* and *lower image operators* of ground morphism (f,φ).

Theorem 5.1.3. *The following hold:*

(1) $(f,\varphi)^{\rightarrow}$ *is isotone.*
(2) $(f,\varphi)^{\leftarrow}$ *is a frame map and* $(f,\varphi)^{\leftarrow}\dashv(f,\varphi)_{\rightarrow}$.
(3) $(f,\varphi)^{\rightarrow}\dashv(f,\varphi)^{\leftarrow}\dashv(f,\varphi)_{\rightarrow}$ *if* φ^{op} *preserves arbitrary* $\bigwedge$ *(which is the case if* L,M *are DeMorgan).*

5.2. **Loc-Top and its basic categorical properties.**

Definition 5.2.1 (variable-basis topology)**.** The category **Loc-Top** of *topological spaces* and *continuous morphisms/mappings* has ground category **Set** $\times$ **Loc** and comprises data subject to axioms as follows:

(1) *Objects:* (X,L,τ), where $(X,\tau)\in|L\text{-}\mathbf{Top}|$, i.e., $\tau\subset L^X$ is closed under arbitrary $\bigvee$ and finite $\wedge$. In this case, τ is a *topology* on the ground object (X,L).
(2) *Morphisms:* $(f,\varphi):(X,L,\tau)\to(Y,M,\sigma)$ satisfies

$$[(f,\varphi)^{\leftarrow}]^{\rightarrow}(\sigma)\subset\tau,\ \text{i.e.,}\ \forall v\in\sigma,\ (f,\varphi)^{\leftarrow}(v)\in\tau.$$

(3) *Composition, identities:* from **Set** $\times$ **Loc**.

Homeomorphisms are the categorical isomorphisms of **Loc-Top** and are ground morphisms (f, φ) such that each of f, φ^{op} is a bijection (and so φ^{op} is an order isomorphism), (f, φ) is continuous, and $\left(f^{-1}, \left((\varphi^{op})^{-1}\right)^{op}\right)$ is continuous.

Comparison Theorems 5.2.2. *The following statements hold:*

(1) *Let L, M be DeMorgan frames. The following are equivalent:*
  (a) $\mathbb{R}(L)$ *is homeomorphic in* **Loc-Top** *to* $\mathbb{R}(M)$.
  (b) $\mathbb{R}^*(L)$ *is homeomorphic in* **Loc-Top** *to* $\mathbb{R}^*(M)$.
  (c) L *is order isomorphic to* M.

(2) *Let L, M be frames. The following are equivalent:*
  (a) $\forall (X, \mathfrak{T}) \in |\mathbf{Top}|$, $(X, L, \omega_L(\mathfrak{T}))$ *is homeomorphic in* **Loc-Top** *to* $(X, M, \omega_M(\mathfrak{T}))$.
  (b) $\forall (X, \mathfrak{T}) \in |\mathbf{Top}|$, $(X, L, G_\omega(\mathfrak{T}))$ *is homeomorphic in* **Loc-Top** *to* $(X, M, G_\omega(\mathfrak{T}))$.
  (c) $\forall A \in |\mathbf{Loc}|$, $LPt(A)$ *is homeomorphic to* $MPt(A)$.
  (d) L *is order isomorphic to* M.

Proof. The equivalence of (1)(c) with each of (1)(a) and (1)(b) follows respectively from Corollary 7.1.7.1 and Corollary 7.3.7.1 of [62]; and the equivalence of (2)(d) with (2)(c) follows from Corollary 7.4.6.1(2), of [62]. Further, it is immediate that each of (2)(a) and (2)(b) implies (2)(d) using the definition of homeomorphism. We now show that (2)(d) implies each of (2)(a) and (2)(b). To this end, assume that L is order isomorphic to M via some $\varphi^{op} : L \leftarrow M$, and let $(X, \mathfrak{T})$ be a topological space. It follows that L^X is bijective with M^X via the "left-hand" and "right-hand" correspondences

$$a \in L^X \mapsto (\varphi^{op})^{-1} \circ a \in M^X, \quad b \in M^X \mapsto \varphi^{op} \circ b \in L^X,$$

and each of these correspondence is isotone; and hence L^X and M^X are order-isomorphic.

To show (2)(a), it is claimed that

$$(id_X, \varphi) : (X, L, \omega_L(\mathfrak{T})) \to (X, M, \omega_M(\mathfrak{T}))$$

is a homeomorphism. Recall that the subbases of the two topologies are respectively as follows:

$$\{u : X \to L \mid \forall \alpha \in L, [u \not\leq \alpha] \in \mathfrak{T}\}, \ \{v : X \to M \mid \forall \beta \in M, [u \not\leq \beta] \in \mathfrak{T}\}.$$

Let u be a member of the left-hand subbasis, and let $x \in [u \not\leq \alpha]$. Then $u(x) \not\leq \alpha$, and hence

$$(\varphi^{op})^{-1}(u(x)) \not\leq (\varphi^{op})^{-1}(\alpha)$$

by the right-hand correspondence being order-preserving. It follows that $x \in \left[(\varphi^{op})^{-1} \circ u \not\leq (\varphi^{op})^{-1}(\alpha)\right]$, so that

$$[u \not\leq \alpha] \subset \left[(\varphi^{op})^{-1} \circ u \not\leq (\varphi^{op})^{-1}(\alpha)\right].$$

A symmetric argument using the isotonicity of the left-hand correspondence above establishes that

$$[u \not\leq \alpha] \supset \left[(\varphi^{op})^{-1} \circ u \not\leq (\varphi^{op})^{-1}(\alpha)\right],$$

and hence that

$$[u \not\leq \alpha] = \left[(\varphi^{op})^{-1} \circ u \not\leq (\varphi^{op})^{-1}(\alpha)\right].$$

The right-hand set, by the left-hand correspondence, instantiates the predicate of the right-hand subbasis, and this implies that u is a member of the right-hand subbasis. Hence the left-hand subbasis is a subfamily of the right-hand subbasis. There is again a symmetric argument establishing the reverse inclusion, so that

$$\{u : X \to L \mid \forall \alpha \in L,\ [u \not\leq \alpha] \in \mathfrak{T}\} = \{v : X \to M \mid \forall \beta \in M,\ [u \not\leq \beta] \in \mathfrak{T}\}.$$

It follows that

$$\omega_L(\mathfrak{T}) = \omega_M(\mathfrak{T}).$$

Now the action of the preimage operator $(id_X, \varphi)^{\leftarrow}$ on the right-hand subbase is exactly that of the left-hand correspondence above, and so (id_X, φ) is subbasic continuous; and the action of the preimage operator $\left(id_X, \left((\varphi^{op})^{-1}\right)^{op}\right)^{\leftarrow}$ on the left-hand subbase is exactly that of the right-hand correspondence above, and so $\left(id_X, \left((\varphi^{op})^{-1}\right)^{op}\right)$ is subbasic continuous. But Theorem 3.2.6 of [62] now implies that each of (id_X, φ) and $\left(id_X, \left((\varphi^{op})^{-1}\right)^{op}\right)$ is continuous. Hence (id_X, φ) is a homeomorphism and (2)(a) holds.

Now to show (2)(b), it is claimed that $(id_X, \varphi) : (X, L, G_\omega(\mathfrak{T})) \to (X, M, G_\omega(\mathfrak{T}))$ is a homeomorphism. Recall that the subbases of the two topologies are respectively as follows:

$$\{\chi_U : U \in \mathfrak{T}\} \cup \{\underline{\alpha} : \alpha \in L\}, \quad \{\chi_U : U \in \mathfrak{T}\} \cup \{\underline{\beta} : \beta \in M\}.$$

It is straightforward to show that $\forall U \in \mathfrak{T}$, $\forall \alpha \in L$, $\forall \beta \in M$, the following preimages are obtained:

$$(id_X, \varphi)^{\leftarrow}(\chi_U) = \chi_U = \left(id_X, \left((\varphi^{op})^{-1}\right)^{op}\right)^{\leftarrow}(\chi_U),$$

$$(id_X, \varphi)^{\leftarrow}(\underline{\beta}) = \underline{\varphi^{op}(\beta)} \in \{\underline{\alpha} : \alpha \in L\},$$

$$\left(id_X, \left((\varphi^{op})^{-1}\right)^{op}\right)^{\leftarrow}(\underline{\alpha}) = \underline{\varphi^{op}(\alpha)} \in \left\{\underline{\beta} : \beta \in M\right\}.$$

All of this shows that each of (id_X, φ) and $\left(id_X, \left((\varphi^{op})^{-1}\right)^{op}\right)$ is subbasic continuous and hence continuous (Theorem 3.2.6 of [62]). It follows that (id_X, φ) is a homeomorphism and (2)(b) holds. □

It is striking that (2)(a) and (2)(b) above are equivalent, even without the assumption of any lattices being completely distributive, though this does not address the open question stated just above Example 4.3.3. Also, given the fact that stratification is incompatible with L-sobriety for most locales [57, 59, 63, 66], that both ω_L and G_ω produce only stratified spaces—this follows from the definition of "stratified" in Example 4.3.2(4) above, and that LPt produces only L-sober spaces [53, 57, 58, 59, 63], it is even more striking that each of (2)(a) or (2)(b) is equivalent with (2)(c).

Now **Loc-Top** contains many more morphisms "across" different bases than just homeomorphisms. This richness of morphisms is detailed at length in Section 7 of [62], including classes of non-homeomorphisms between objects in each of the example classes of Subsection 4.3 above having different bases; e.g., conditions are given under which **Loc-Top**$(\mathbb{R}(L), \mathbb{R}(M))$ contains a class of non-homeomorphisms. It should be pointed out that **Loc-Top** has far more morphisms than all the morphisms of the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$ put together. Restated, "variable-basis" is much richer with respect to morphisms than "external" change of basis, even when the external changes are brought "inside". For now we content ourselves with the following general proposition:

Proposition 5.2.3. *Fix frame L and a localic non-isomorphism $\varphi \in$ **Loc** (L, L), let (X, L, σ) be a topological space (in **Loc-Top**), and put*

$$\tau := \{\varphi^{op} \circ v : v \in \sigma\}.$$

*Then $(id_X, \varphi) : (X, L, \tau) \to (X, L, \sigma)$ is continuous, not a homeomorphism, and not in any M-**Top**.*

The *forgetful functor* V : **Loc-Top** $\to$ **Set** $\times$ **Loc** is given by

$$V(X, L, \tau) = (X, L), \quad V(f, \varphi) = (f, \varphi).$$

Theorem 5.2.4. *Each V-structured source [sink] in* **Set** $\times$ **Loc** *has a unique initial [final] lift to* **Loc**-**Top**. *Hence,* **Loc**-**Top** *is topological over* **Set** $\times$ **Loc** *w.r.t. V.*

Corollary 5.2.5. Loc-**Top** *is complete and cocomplete.*

5.3. **Loc-Top as supercategory of Top, Loc, L-Top's.** The embeddings of **Top**, **Loc**, and the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$ into **Loc-Top** come from [53, 62] and are now described.

Fix L frame and put L-**Top** $\rightarrowtail$ **Loc-Top** by

$$(X, \tau) \mapsto (X, L, \tau),$$
$$[f : (X, \tau) \to (Y, \sigma)] \mapsto [(f, (id_L)^{op}) : (X, L, \tau) \to (Y, L, \sigma)].$$

This reinforces the richness of morphisms in **Loc-Top** *vis-a-vis* those morphisms occurring in the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$: even for fixed basis L, if $|\mathbf{Frm}(L, L)| > 1$, i.e., L admits a frame endomorphism other than id_L, then **Loc-Top** will have continuous morphisms between spaces, both having lattice-theoretic basis L, which cannot come from morphisms in L-**Top**. Cf. Proposition 5.2.3 above.

It should also be noted that **Top** $\approx$ **2-Top** $\rightarrowtail$ **Loc-Top**. Restated, **Top** embeds into **Loc-Top** by G_χ of Example 4.3.1 followed by the embedding above for $L = \mathbf{2}$.

We consider two functorial embeddings of **Loc** into **Loc-Top**:

(1) For the *empty embedding*, put **Loc** $\rightarrowtail$ **Loc-Top** by

$$A \mapsto \left(\varnothing, A, A^{\varnothing} \equiv 1\right),$$
$$[\varphi : A \to B] \mapsto [(id_{\varnothing} \equiv \varnothing, \varphi) : (\varnothing, A, A^{\varnothing}) \to (\varnothing, B, B^{\varnothing})].$$

Since **Loc-Top** is variable-basis topology by Theorem 5.2.4 above, this embedding justifies thinking of **Loc** as "point-free" or "pointless" topology.

(2) For the *singleton embedding*, put **Loc** $\rightarrowtail$ **Loc-Top** by

$$A \mapsto \left(\mathbf{1}, A, A^{\mathbf{1}}\right),$$
$$[\varphi : A \to B] \mapsto \left[(id_{\mathbf{1}}, \varphi) : \left(\mathbf{1}, A, A^{\mathbf{1}}\right) \to \left(\mathbf{1}, B, B^{\mathbf{1}}\right)\right].$$

Again, given that **Loc-Top** is variable-basis topology by 5.2.4, this justifies thinking of **Loc** as the "topology of singleton spaces".

Taking the embeddings of **Top** and the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$ into **Loc-Top** on one hand, and the embeddings of **Loc** into **Loc-Top** on the other hand, it emerges that **Top** and the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$ represent "variable carrier set and fixed lattice-theoretic basis" topology, while **Loc** represents "fixed carrier set and variable lattice-theoretic basis" topology; and hence, neither **Top** nor **Loc** is more general. All these embeddings also justify speaking of **Loc-Top** (and the other **C-Top**'s of [62]) as "point-set lattice-theoretic" or "poslat" topology—see [56].

Finally, it is from the standpoint of **Loc-Top** that it may be fairly said that **Loc** is part of "point-set" topology, namely as part of point-set lattice-theoretic topology.

5.4. **Loc-Top as supercategory of TopSys.** This subsection constructs two embeddings of **TopSys** into **Loc-Top** [7] and thereby continues the theme, begun in Subsection 4.4 above, that topological systems and many-valued topological spaces are intertwined, in multiple ways, and that this intertwining is essential to understanding both spaces and systems. Embedding variable-basis spaces into a systems context is more delicate, needs more ideas, and is discussed later in this paper.

Developing relationships between **TopSys** and **Loc-Top** is mandated by these considerations:

- **TopSys** and **Loc-Top** are *both* supercategories up to isomorphism of **Top**, **Loc**, and the schema $\{L\text{-}\mathbf{Top} : L \in |\mathbf{Frm}|\}$;
- **TopSys** is essentially algebraic and **Loc-Top** is topological; and
- both **TopSys** and **Loc-Top** have the *same* ground category $\mathbf{Set} \times \mathbf{Loc}$.

It is helpful to note *ab initio* what relationships *cannot* exist between **TopSys** and **Loc-Top**, as seen in our first result.

Theorem 5.4.1. *There cannot exist a concrete isomorphism* J: **TopSys** $\to$ **Loc-Top**.

Proof. Recall the forgetful functors $T : \mathbf{TopSys} \to \mathbf{Set} \times \mathbf{Loc}$ and $V : \mathbf{Loc\text{-}Top} \to \mathbf{Set} \times \mathbf{Loc}$ given earlier (above Proposition 3.1.1 and Theorem 5.2.4, respectively). Now suppose a concrete isomorphism $J : \mathbf{TopSys} \to \mathbf{Loc\text{-}Top}$ exists. Then it is the case that

$$T = V \circ J, \quad V = T \circ J^{-1}.$$

Now let

$$(T\,(f_\gamma, \varphi_\gamma) : (X, A) \to T\,(X_\gamma, A_\gamma, \models))_{\gamma \in \Gamma}$$

be a T-structured source in $\mathbf{Set} \times \mathbf{Loc}$. Then it follows

$$(VJ\,(f_\gamma, \varphi_\gamma) : (X, A) \to VJ\,(X_\gamma, A_\gamma, \models))_{\gamma \in \Gamma}$$

may be regarded as a V-structured source in $\mathbf{Set} \times \mathbf{Loc}$. Now the topologicity (Theorem 5.2.4) of **Loc-Top** implies that this second source in $\mathbf{Set} \times \mathbf{Loc}$ has a unique, initial lift

$$(J\,(f_\gamma, \varphi_\gamma) : (X, A, \tau) \to J\,(X_\gamma, A_\gamma, \models))_{\gamma \in \Gamma}\,,$$

and it follows from $J : \mathbf{TopSys} \to \mathbf{Loc\text{-}Top}$ being a concrete isomorphism that

$$\left((f_\gamma, \varphi_\gamma) : J^{-1}\,(X, A, \tau) \to (X_\gamma, A_\gamma, \models)\right)_{\gamma \in \Gamma}$$

is a unique, initial lift of the first source in **Set** $\times$ **Loc**. This implies that **TopSys** is topological over **Set** $\times$ **Loc** w.r.t. T, a contradiction (Theorem 3.1.6 above). □

Even though concrete isomorphisms between **TopSys** and **Loc**-**Top** do not exist, other functorial relationships do in fact exist. Now constructing a space from a system or a system from a space over the same ground object from **Set** $\times$ **Loc** requires a fundamental paradigm shift in the interpretation of that ground object:

- Given $(X, A, \models) \in |\mathbf{TopSys}|$, the ground object (X, A) in some examples may be interpreted as follows: X is a set of bitstrings and A is a locale of predicates; and in this case $\models$ is a satisfaction relation on ground object $(X, A) \in |\mathbf{Set} \times \mathbf{Loc}|$.
- Given $(X, L, \tau) \in |\mathbf{Loc\text{-}Top}|$, the ground object (X, L) in some examples may be interpreted as follows: X is a set of points and L is a frame of membership or truth values; and in this case τ is a topology on ground object $(X, L) \in |\mathbf{Set} \times \mathbf{Loc}|$.

Theorem 5.4.2 (satisfaction embedding $F_{\models}$)**.** *Construct* $F_{\models} : \mathbf{TopSys} \rightarrowtail \mathbf{Loc\text{-}Top}$ *by*

$$F_{\models}(X, A, \models) = (X, A, \tau_{\models}),$$

where

$$\tau_{\models} = \left\{ u \in A^X : u = \bot \ \textit{ or } \ [\forall x \in X,\ x \models u(x)] \right\},$$

$$F_{\models}(f, \varphi) = (f, \varphi).$$

Then $F_{\models}$ *concretely embeds* **TopSys** *into* **Loc**-**Top**.

As shown above (5.4.1), $F_{\models}$ cannot be an isomorphism. On the other hand, being an embedding insures that $F_{\models}^{\rightarrow}(\mathbf{TopSys})$ is a subcategory of **Loc-Top**, albeit a proper subcategory. Given the reinterpretation issues noted above for making spaces from systems, the subcategory $F_{\models}^{\rightarrow}(\mathbf{TopSys})$ is of interest and it is important to better understand this subcategory. The following characterization theorem resolves this issue.

Theorem 5.4.3 ($F_{\models}$ characterization theorem)**.** *For* $(X, L, \tau) \in |\mathbf{Loc\text{-}Top}|$, *let*

$$\tau^* = \{u \in \tau : u \neq \bot\}.$$

Assume $X \neq \varnothing$ *and* L *to be consistent. Then* $(X, L, \tau) \in F_{\models}^{\rightarrow}(\mathbf{TopSys})$ *if and only if all of the following hold:*

(1) τ^* *is a filter;*
(2) $\forall x \in X, \forall u \in \tau^*, u(x) > \bot$;
(3) $\forall x \in X, \forall u \in \tau^*, \forall \{\alpha_\gamma : \gamma \in \Gamma\} \subset L$ *with* $u(x) = \bigvee_{\gamma \in \Gamma} \alpha_\gamma$, $\exists \gamma_0 \in \Gamma$, $\exists v \in \tau^*$, $v(x) = \alpha_{\gamma_0}$;
(4) $\forall \{u_x : x \in X\} \subset \tau^*, \exists u \in \tau^*, \forall x \in X, u(x) = u_x(x)$.

Since spaces can be easily constructed which do not satisfy all the conditions of Theorem 5.4.2, it necessarily follows that $F_{\models}$ cannot be an isomorphism, confirming Theorem 5.4.1 above.

A second and strikingly different functor embeds **TopSys** into **Loc-Top**, and this second functor is closely tied to the example classes catalogued in Subsection 4.3 above.

Theorem 5.4.4 (truncation embedding F_k)**.** *Construct* $F_k : \mathbf{TopSys} \rightarrowtail \mathbf{Loc}\text{-}\mathbf{Top}$ *by*

$$F_k(X, A, \models) = (X, A, \tau_k),$$

where

$$\tau_k = \left\langle\left\langle\left\{\underline{a} \wedge \chi_{ext(a)} : a \in A\right\}\right\rangle\right\rangle$$

and the notion of extent is from Theorem 3.2.2 above, and

$$F_k(f, \varphi) = (f, \varphi).$$

Then F_k *concretely embeds* **TopSys** *into* **Loc-Top***.*

For the same reasons given above for $F_{\models}$, $F_k^{\rightarrow}(\mathbf{TopSys})$ is a proper subcategory of **Loc-Top**, and it is therefore important to better understand this subcategory. The following characterization theorem resolves this corresponding issue for F_k.

Theorem 5.4.5 (F_k characterization theorem)**.** *Let* $(X, L, \tau) \in |\mathbf{Loc}\text{-}\mathbf{Top}|$. *Then* $(X, L, \tau) \in F_k^{\rightarrow}(\mathbf{TopSys})$ *if and only if* $\exists \{U_\alpha : \alpha \in L\} \subset \wp(X)$ *satisfying all of the following:*

(1) $\forall \{\alpha_\gamma : \gamma \in \Gamma\} \subset L$, $\bigcup_{\gamma \in \Gamma} U_{\alpha_\gamma} = U_{\bigvee_{\gamma \in \Gamma} \alpha_\gamma}$;

(2) $\forall \{\alpha_\gamma : \gamma \in \Gamma\} \subset L$ *(Γ finite)*, $\bigcap_{\gamma \in \Gamma} U_{\alpha_\gamma} = U_{\bigwedge_{\gamma \in \Gamma} \alpha_\gamma}$;

(3) $\tau = \langle\{\underline{\alpha} \wedge \chi_{U_\alpha} : \alpha \in L\}\rangle$, *where the single brackets indicate a topology generated from a basis.*

As with $F_{\models}$, Theorem 5.4.5 shows that F_k is not an isomorphism since spaces can be constructed not satisfying all the conditions of 5.4.5, thus confirming Theorem 5.4.1 above.

With respect to objects, the truncation functor F_k is essentially built from the *Ext* functor of Theorem 3.2.2(2) followed by the schema of the G_ω functors introduced in the paragraph between Example 4.3.2 and Example 4.3.3. To see this more precisely, fix $(X, A, \models)$ in **TopSys**. Applying $Ext : \mathbf{TopSys} \to \mathbf{Top}$ yields the topological space $(X, ext^{\rightarrow}(A))$. Now applying $G_\omega : \mathbf{Top} \to L\text{-}\mathbf{Top}$ yields the L-topological space $(X, G_\omega(ext^{\rightarrow}(A)))$, where

$$G_\omega(ext^{\rightarrow}(A)) = \left\langle\left\langle\{\underline{a} : a \in A\} \cup \left\{\chi_{ext(b)} : b \in A\right\}\right\rangle\right\rangle.$$

But the first infinite distributive law implies the subbasis $\{\underline{a} : a \in A\} \cup \left\{\chi_{ext(b)} : b \in A\right\}$ generates a *basis*

$$\left\{\underline{a} \wedge \chi_{ext(b)} : (a, b) \in A \times A\right\}$$

for the L-topology $G_\omega\left(ext^{\rightarrow}(A)\right)$. This basis is actually finer than what is needed for the A-topology τ_k. To address this problem, put

$\langle G_\omega \circ ext^{\rightarrow}\rangle : A \times A \to G_\omega\left(ext^{\rightarrow}(A)\right)$ by $\langle G_\omega \circ ext^{\rightarrow}\rangle (a, b) = \underline{a} \wedge \chi_{ext(b)}$,

recall the inclusion map

$$\hookrightarrow_\Delta : \Delta(A \times A) \to A \times A,$$

of the diagonal $\Delta(A \times A)$ into $A \times A$, and note the usual bijection

$$j : A \to \Delta(A \times A).$$

Then

$$\left(\langle G_\omega \circ ext^{\rightarrow}\rangle \circ \hookrightarrow_\Delta \circ j\right)^{\rightarrow}(A) = \left\{\underline{a} \wedge \chi_{ext(a)} : a \in A\right\}, \qquad (\mathbf{K})$$

which can be shown using the first infirnite distributive law to be a basis for τ_k, actually improving the statement of Theorem 5.4.4 above which is in terms of subbases. Now put

$$\left\lfloor \langle G_\omega \circ ext^{\rightarrow}\rangle \circ \hookrightarrow_\Delta \circ j \right\rfloor (A) := \left(X, \left\langle\left(\langle G_\omega \circ ext^{\rightarrow}\rangle \circ \hookrightarrow_\Delta \circ j\right)^{\rightarrow}(A)\right\rangle\right),$$

an A-topological space, where "$\lfloor\ \rfloor$" denotes that a space using X as the underlying carrier set is being created. The last step in constructing $F_k(X, A, \models)$ applies the embedding A-**Top** $\rightarrowtail$ **Loc-Top** recorded in Subsection 5.3 above. So the sequence of actions in forming $F_k(X, A, \models)$ may be seen as

$$[(\ )\text{-}\mathbf{Top} \rightarrowtail \mathbf{Loc\text{-}Top}] \circ \left\lfloor \langle G_\omega \circ ext^{\rightarrow}\rangle \circ \hookrightarrow_\Delta \circ j \right\rfloor.$$

It is remarkable that this composite action is injective on objects and is compatible with the action $F_k(f, \varphi) = (f, \varphi)$ on morphisms to produce F_k as a concrete embedding. Line (**K**) above explains the "k" in F_k as standing for "truncation" or "cuts", since the basic open sets of the topology τ_k are truncations or cuts of characteristics by constant maps.

We observe that the preceding paragraph adds to what is known in [7] and proves the following new factorization of F_k w.r.t. objects:

Theorem 5.4.6 (factorization of F_k)**.** *As an object level mapping,* $F_k : |\mathbf{TopSys}| \rightarrowtail |\mathbf{Loc}\text{-}\mathbf{Top}|$ *factors as follows:*

$$F_k = [(\ )\text{-}\mathbf{Top} \rightarrowtail \mathbf{Loc}\text{-}\mathbf{Top}] \circ \left\lfloor \langle G_\omega \circ ext^{\rightarrow}\rangle \circ \hookrightarrow_\Delta \circ j \right\rfloor.$$

We note in the above discussion and proposition that ω_L may be substituted for G_ω when the locale A of predicates is completely distributive (Example 4.3.2), bringing us back almost full circle to the beginning of the many-valued topology literature. Finally, this discussion suggests a possibly new construction of spaces from systems, namely—for fixed $(X, A, \models)$—that given by

$$[A\text{-}\mathbf{Top} \rightarrowtail \mathbf{Loc}\text{-}\mathbf{Top}] \circ \lfloor \langle \omega_A \circ ext^{\rightarrow} \rangle \circ \hookrightarrow_{\Delta} \circ j \rfloor ,$$

and possibly a new bi-level mapping, say, $F_\omega : \mathbf{TopSys} \rightarrowtail \mathbf{Loc}\text{-}\mathbf{Top}$ by

$$F_\omega (X, A, \models) = (X, A, \tau_\omega) ,$$

where

$$\tau_\omega \ = \ \langle (\langle \omega_L \circ ext^{\rightarrow} \rangle \circ \hookrightarrow_{\Delta} \circ j)^{\rightarrow} (A) \rangle$$

and the notion of extent is from Theorem 3.2.2 above, along with

$$F\omega (f, \varphi) = (f, \varphi) .$$

If F_ω is restricted to the full subcategory of **TopSys** in which all sets of predicates are completely distributive, then F_ω coincides with F_k given above and is in that instance a concrete embedding.

Yet other ways exist to generate spaces from systems. There is the concrete functor F^k already given in [7], constructed using the (sub)basis

$$\left\{ \underline{a} \wedge \chi_{ext(b)} : (a, b) \in A \times A \right\} ,$$

and which can be factored at the object level as

$$F^k = [(\ \ )\text{-}\mathbf{Top} \rightarrowtail \mathbf{Loc}\text{-}\mathbf{Top}] \circ G_\omega \circ Ext$$

Also one might define yet another concrete functor F^ω by choosing

$$\tau^\omega \ = \ \langle \langle \{ u : X \to A \mid \forall a \in A,\, [u \not\leq a] \in ext^{\rightarrow} (A) \} \rangle \rangle ,$$

which can be factored at the object level as

$$F^\omega = [(\ \ )\text{-}\mathbf{Top} \rightarrowtail \mathbf{Loc}\text{-}\mathbf{Top}] \circ \omega_{(\ \ )} \circ Ext.$$

Then F^ω coincides with F^k when restricted to the full subcategory of **TopSys** in which the sets of predicates are completely distributive. The behaviors of F_ω and F^ω are open questions perhaps related to the open question stated immediately above Example 4.3.3.

Discussion 5.4.7 (possible applications)**.** There could be potential applicability of $F_\models$, F_k to topological systems. It is known that **TopSys** lacks initial and final structures (Lemma 3.1.5, Theorem 3.1.6), and these embeddings may help mitigate this lack, given that **Loc-Top** has all initial and final structures (Theorem 5.2.4). The case for initial structures is now discussed; the final structures case is obverse and left to the reader. Recall the forgetful functors T : **TopSys** $\to$ **Set** $\times$ **Loc** and V : **Loc-Top** $\to$ **Set** $\times$ **Loc**, and take a T-structured source

$$((f_\gamma, \varphi_\gamma) : (X, A) \to T (X_\gamma, A_\gamma, \models_\gamma))_{\gamma \in \Gamma}$$

in **Set** $\times$ **Loc** lacking a unique initial lift in **TopSys**. Because both $F_{\models}$, F_k are concrete—

$$T = F_{\models} \circ V, \quad T = F_k \circ V,$$

then this T-structured source may be interpreted as both of these V-structured sources:

$$\left((f_\gamma, \varphi_\gamma) : (X, A) \to V\left(X_\gamma, A_\gamma, \tau_{\models_\gamma}\right)\right)_{\gamma \in \Gamma},$$

$$\left((f_\gamma, \varphi_\gamma) : (X, A) \to V\left(X_\gamma, A_\gamma, \tau_{k,\gamma}\right)\right)_{\gamma \in \Gamma}.$$

In each case there is, respectively, a unique, initial lift in **Loc-Top**:

$$\left((f_\gamma, \varphi_\gamma) : (X, A, \tau_s) \to \left(X_\gamma, A_\gamma, \tau_{\models_\gamma}\right)\right)_{\gamma \in \Gamma},$$

$$\left((f_\gamma, \varphi_\gamma) : (X, A, \tau_t) \to \left(X_\gamma, A_\gamma, \tau_{k,\gamma}\right)\right)_{\gamma \in \Gamma}.$$

It remains to be seen if programmers find these initial structures in **Loc-Top**—and **Loc-Top** filling in these structural "gaps" of **TopSys**, to be useful.

6. Generalizations and future directions

This section briefly outlines some generalizations of the ideas previously considered in this paper as well as indicating new directions of current research. These are considered in three subsections as separate themes for the sake of clarity; but we emphasize that ideas from these three subsections can and should be combined, and to some degree that is currently taking place.

6.1. Lattice-valued satisfaction relations, Loc-TopSys, Loc-F^{2}Top.

For programming flexibility, it is appropriate to consider satisfaction relations for which a given bitstring satisfies a given predicate to a certain degree. Staying within the traditional finite observational logic of frames, this means introducing an additional frame as the lattice of "satisfaction degrees". One question to be considered is whether this additional frame is to be common to all the systems under consideration—essentially a fixed-basis approach to degrees of satisfaction, or whether this additional frame should vary from system to system—essentially a variable-basis approach to degrees of satisfaction. In the interests of brevity, we take the second approach and proceed straightaway to the variable-basis approach, an approach which has some advantages regarding the potential applications of Discussion 5.4.7 above and blends results from [9, 71, 72, 73].

The variable-basis approach to degrees of satisfaction necessitates a fundamental change in the ground category for systems as well as in the overlying category for topological systems.

Definition 6.1.1. The category $\mathbf{Set} \times \mathbf{Loc}^2$ comprises the following data:

(1) *Objects:* (X, L, A), where X is a set, L is a frame, A is a locale.
(2) *Morphisms:* $(f, \varphi, \psi) : (X, L, A) \to (Y, M, B)$, where $f : X \to Y$ is in **Set** and $\varphi : L \to M$, $\psi : A \to B$ are in **Loc**, i.e., $\varphi^{op} : L \leftarrow M$, $\psi^{op} : A \leftarrow B$ are in **Frm**.
(3) *Composition, identities:* component-wise from **Set** and **Loc**.

It can be shown that $\mathbf{Set} \times \mathbf{Loc}^2$ is both complete and cocomplete.

Definition 6.1.2 (variable-basis topological systems)**.** The category **Loc-TopSys** of *topological systems* and *continuous mappings* has ground category $\mathbf{Set} \times \mathbf{Loc}^2$ and comprises data subject to axioms as follows:

(1) *Objects:* $(X, L, A, \vDash)$, where $(X, L, A) \in \left|\mathbf{Set} \times \mathbf{Loc}^2\right|$ and $\vDash \ : X \times A \to L$ is an (*L-valued*) *satisfaction relation* possessing the arbitrary $\bigvee$ and finite $\wedge$ *interchange laws*:

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \ \vDash \left(x, \bigvee_{\gamma \in \Gamma} a_\gamma\right) = \bigvee_{\gamma \in \Gamma} \vDash (x, a_\gamma);$$

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \vDash \left(x, \bigwedge_{\gamma \in \Gamma} a_\gamma\right) = \bigwedge_{\gamma \in \Gamma} \vDash (x, a_\gamma) \quad (\Gamma \text{ finite}).$$

The set X in some examples could be interpreted as bitstrings; L may be interpreted as a frame of *satisfaction* values; A may be interpreted as a locale of (open) predicates; and $\vDash (x, a)$ may be said to be the *degree to which* (bitstring) x *satisfies* (predicate) a.
(2) *Morphisms:* $(f, \varphi, \psi) : (X, L, A, \vDash_1) \to (Y, M, B, \vDash_2)$, where $(f, \varphi, \psi) : (X, L, A) \to (Y, M, B)$ in $\mathbf{Set} \times \mathbf{Loc}^2$ and (f, φ, ψ) satisfies *adjointness:*

$$\forall b \in B, \ \forall x \in X, \ \vDash_1 (x, \psi^{op}(b)) = \varphi^{op}[\vDash_2 (f(x), b)].$$

(3) *Composition, identities:* from $\mathbf{Set} \times \mathbf{Loc}^2$.

The adjointness condition in this new setting is saying that the degree $\vDash_1 (x, \psi^{op}(b))$ to which an input satisfies the pullback of a postcondition predicate is the same as the shift by φ^{op} of the degree $\vDash_2 (f(x), b)$ to which the corresponding output satisfies the postcondition predicate. If it were the case that $L = M$ and $\varphi^{op} = id_L$, then it would be the case that adjointness would be insisting that the two degrees of satisfaction be the same.

To describe the behavior of **Loc-TopSys**, the appropriate forgetful functor needs to be recorded. Put $W : \mathbf{Loc\text{-}TopSys} \to \mathbf{Set} \times \mathbf{Loc}^2$ by

$$W(X, L, A, \vDash) = (X, L, A), \quad W(f, \varphi, \psi) = (f, \varphi, \psi).$$

6.1.3 Theorem (categorical properties of **Loc-TopSys**) [9]**.**

(1) *The functor W reflects isomorphisms and is transportable and (generating, mono-source)-factorizable.*
(2) **Loc-TopSys** *is essentially algebraic over* $\mathbf{Set} \times \mathbf{Loc}^2$ *w.r.t.* W.
(3) **Loc-TopSys** *is complete and cocomplete.*
(4) *W-structured sources [sinks] need not have unique initial [final] lifts, not even singleton W-structured sources.*
(5) **Loc-TopSys** *is not topological over* $\mathbf{Set} \times \mathbf{Loc}^2$ *w.r.t.* W*. In fact,* **Loc-TopSys** *is neither mono-, nor epi-, nor (small) existentially, nor (small) essentially topological over* $\mathbf{Set} \times \mathbf{Loc}^2$ *w.r.t.* W.

So as with **TopSys**, **Loc-TopSys** is algebraic and non-topological in nature. However, it is closely related to topology in various ways, as we see in the sequel.

Theorem 6.1.4. **Loc-TopSys** *is a supercategory up to isomorphism of both* **TopSys** *and* **Loc-Top***. More precisely, the following hold:*

(1) $E_{\mathbf{TopSys}} : \mathbf{TopSys} \to \mathbf{Loc\text{-}TopSys}$, *defined by*

$$E_{\mathbf{TopSys}}(X, A, \models) = (X, \mathbf{2}, A, \models_{\mathbf{2}}),$$

where

$$\models_{\mathbf{2}}(x, a) \;=\; \begin{cases} \top, & x \models a \\ \bot, & x \not\models a \end{cases},$$

and

$$E_{\mathbf{TopSys}}(f, \varphi) = (f, id_{\mathbf{2}}, \varphi),$$

is a functorial embedding.

(2) $E_{\mathbf{Loc\text{-}Top}} : \mathbf{Loc\text{-}Top} \to \mathbf{Loc\text{-}TopSys}$, *defined by*

$$E_{\mathbf{Loc\text{-}Top}}(X, L, \tau) = (X, L, \tau, \models),$$

where

$$\models \,: X \times \tau \to L \quad by \quad \models(x, u) = u(x),$$

and

$$E_{\mathbf{Loc\text{-}Top}}((f, \varphi) : (X, L, \tau) \to (Y, M, \sigma)) =$$
$$\left[\left(f, \varphi, \left((f, \varphi)^{\leftarrow}_{|\sigma}\right)^{op}\right) : (X, L, \tau, \models) \to (Y, M, \sigma, \models)\right],$$

where $(f, \varphi)^{\leftarrow}$ is given in Definition 5.1.2(1), is a functorial embedding.

The first embedding interprets each traditional satisfaction relation as a "crisp" relation; and the second embedding is exactly a many-valued version of E_V in which degrees of memberships in an open set are reinterpreted as degrees of satisfaction with respect to that open set viewed as a predicate. Both of these embeddings are part of adjoint relationships which parallel and extend those discussed in Theorems 3.2.3 and 3.2.6 above; see [9] for more details—the adjoint of $E_{\mathbf{Loc\text{-}Top}}$ involves a lattice-valued version of extent parallel to that used in Discussion 6.2.6 below.

Discussion 6.1.5 (Discussion 5.4.7 revisited)**.** The potential utility of the embedding $E_{\mathbf{Loc\text{-}Top}}$ involves giving a systems solution to the problem posed in Discussion 5.4.7 above. Recall that T-structured sources [sinks] in $\mathbf{Set} \times \mathbf{Loc}$ lacking unique initial [final] lifts in **TopSys** can be furnished these lifts in **Loc-Top** as spaces via both $F_{\models}$, F_k. Note these lifts are space solutions to a systems problem. Now these space solutions can be moved into **Loc-TopSys** as systems via $E_{\mathbf{Loc\text{-}Top}}$. More precisely, these latter systems are solutions of the original T-lifting problem with respect to the functor $P : \mathbf{Loc\text{-}TopSys} \to \mathbf{Set} \times \mathbf{Loc}$, given by

$$P(X, L, A, \models) = (X, L), \quad P(f, \varphi, \psi) = (f, \varphi),$$

and the following commutivities:

$$T = V \circ F_{\models}, \quad T = V \circ F_k, \quad P \circ E_{\mathbf{L\text{-}T}} = V.$$

We now shift our attention to Kubiak-Šostak topologies [34, 39, 74] and their variable-basis generalizations first given in [62], one such generalization being the category **Loc-FTop**. Our purpose here is to give a category **Loc-F^{2}Top** [9] which is a variation of **Loc-FTop**, but which is more explicitly and conveniently tied to topological systems.

The category **Loc-F^{2}Top** has ground category $\mathbf{Set} \times \mathbf{Loc}^2$; but in order to motivate **Loc-F^{2}Top**, each ground object (X, L, A) from $\mathbf{Set} \times \mathbf{Loc}^2$ needs reinterpretation:

- For systems, X in some examples may be interpreted as bitstrings, L as satisfaction values, and A as (open) predicates.
- For spaces, X should be interpreted as points, L as degrees of openness, A as degrees of membership—the latter two comprising variable bases (hence the exponent "2" in **Loc-F^{2}Top**).

Definition 6.1.6 (topologies as openness operators)**.** The category **Loc-F^{2}Top** of *fuzzy topological spaces* and *fuzzy continuous mappings* has ground category $\mathbf{Set} \times \mathbf{Loc}^2$ and comprises data subject to axioms as follows:

(1) *Objects:* $(X, L, A, \mathcal{T})$, where $\mathcal{T} : A^X \to L$ is an (A, L)-topology, cf. [21, 39, 62]; i.e., $\mathcal{T}$ satisfies:
   - $\forall \{u_\gamma\}_{\gamma \in \Gamma} \subset A^X,\ \mathcal{T}\left(\bigvee_{\gamma\in\Gamma} u_\gamma\right) \geq \bigwedge_{\gamma\in\Gamma} \mathcal{T}(u_\gamma)$;
   - $\forall u, v \in A^X$, $\mathcal{T}(u \wedge v) \geq \mathcal{T}(u) \wedge \mathcal{T}(v)$; and

- $\mathcal{T}(\underline{\top}) = \top$.
  It is a fact that $\mathcal{T}(\underline{\bot}) = \top$. $\mathcal{T}$ called an (A, L)-*fuzzy topology* on (X, L, A) and an (A, L)-fuzzy topological space. $\mathcal{T}(u)$ is the *degree of openness* of u in $\mathcal{T}$; and $\mathcal{T}$ is an *openness predicate/operator*.

(2) *Morphisms:* $(f, \varphi, \psi) : (X, L, A, \mathcal{T}) \to (Y, M, B, \mathcal{S})$ satisfies
$$\mathcal{T} \circ (f, \psi)^{\leftarrow} \ \geq \ \varphi^{op} \circ \mathcal{S};$$
i.e., $\forall v \in B^Y$,
$$\mathcal{T}\left[(f, \psi)^{\leftarrow}(v)\right] \geq \varphi^{op}(\mathcal{S}(v)),$$
cf., [62, 9]. It is a fact that $\mathcal{T}\left[(f, \psi)^{\leftarrow}\left(\underline{\top_B}\right)\right] = \mathcal{T}\left[(f, \psi)^{\leftarrow}\left(\underline{\bot_B}\right)\right] = \top_L$

(3) *Composition, identities:* from **Set** $\times$ **Loc**^{2}.

Note the following: openness of a union (or binary intersection) is no less than that of the least open set; "whole carrier set" and "empty set" are fully open; $\mathcal{T} \in L^{\left(A^X\right)}$, consistent with the exponent in **Loc-F**^{2}**Top**; and the preimage of a subset from codomain is at least as open in domain as the original subset was in codomain, *modulo* the shift by φ^{op}.

The forgetful functor F : **Loc-F**^{2}**Top** $\to$ **Set** $\times$ **Loc**^{2}, given by
$$F(X, L, A, \tau) = (X, L, A), \quad F(f, \varphi, \psi) = (f, \varphi, \psi),$$
is used to describe the categorical behavior of **Loc-F**^{2}**Top**.

Theorem 6.1.7 (categorical properties of **Loc-F**^{2}**Top**)[9]**.**

(1) *Each F-structured source [sink] in* **Set** $\times$ **Loc**^{2} *has a unique initial [final] lift to* **Loc-F**^{2}**Top**.
(2) **Loc-F**^{2}**Top** *is topological over* **Set** $\times$ **Loc**^{2} *w.r.t. F.*
(3) **Loc-F**^{2}**Top** *is complete and cocomplete.*

We now come to the climax of this subsection: embedding **Loc-TopSys** into **Loc-F**^{2}**Top**; it will then follow that **Loc-Top** embeds into **Loc-F**^{2}**Top**.

Theorem 6.1.8. **Loc-F**^{2}**Top** *is a supercategory up to isomorphism of* **Loc-TopSys**. *Specifically,* $F^*_{\models}$: **Loc-TopSys** $\to$ **Loc-F**^{2}**Top**, *defined by*
$$F^*_{\models}(X, L, A, \models) = (X, L, A, \mathcal{T}_{\models}),$$
where $\mathcal{T}_{\models} : A^X \to L$ *by*
$$\mathcal{T}_{\models}(u) = \begin{cases} \bigwedge_{x \in X} \models (x, u(x)), & u \neq \underline{\bot} \\ \top, & u = \underline{\bot} \end{cases},$$
and
$$F^*_{\models}(f, \varphi, \psi) = (f, \varphi, \psi),$$
is a concrete functorial embedding.

Note the degree of openness of u is the least degree to which every point in X satisfies its degree of membership in u, showing that $\mathcal{T}_{\models}$ is an extension of the $\tau_{\models}$ constructed by the embedding $F_{\models}$ in Theorem 5.4.2 above, and therefore that $F_{\models}^{*}$ is an extension of $F_{\models}$, explaining the notation "$F_{\models}^{*}$". Thus we have in this situation a new satisfaction embedding. The issue of $F_{\models}^{*}$ extending $F_{\models}$ can be made more precise—namely it is the case, using Theorems 6.1.4 and 6.1.8, that

$$F_{\models}^{*} \circ E_{\mathbf{TopSys}} = E_{\mathbf{Loc\text{-}Top}}^{*} \circ F_{\models},$$

where $E_{\mathbf{Loc\text{-}Top}}^{*} : \mathbf{Loc\text{-}Top} \rightarrowtail \mathbf{Loc\text{-}TopSys}$ by

$$(X, L, \tau) \mapsto (X, L, 2, \chi_{\tau}), \quad (f, \varphi) \mapsto (f, id_2, \varphi).$$

Is there a corresponding extension of the embedding F_k to a new embedding of **Loc-TopSys** into **Loc-F^{2}Top**? This is a question currently under investigation by the authors.

Discussion 6.1.9 (cf. Discussion 5.4.7)**.** Since **Loc-TopSys** and **Loc-F^{2}Top** share the same ground category $\mathbf{Set} \times \mathbf{Loc}^2$, and since $F_{\models}^{*}$ factors forgetful functor W through forgetful functor F, so in the same manner discussed in Discussion 5.4.7, the initial/final structures missing in **Loc-TopSys** can be given in **Loc-F^{2}Top**.

Discussion 6.1.10 (interweaving of algebra and topology)**.** Hermann Weyl is claimed to have said, "The house of mathematics is built upon the twin pillars of algebra and topology." Systems and spaces as considered in this paper illustrate how these two pillars interweave and reinforce each other. Categorically, we have from above:

$$\mathbf{Top}, L\text{-}\mathbf{Top}'\text{s} \rightarrowtail \mathbf{TopSys} \rightarrowtail \mathbf{Loc\text{-}Top} \rightarrowtail \mathbf{Loc\text{-}TopSys} \rightarrowtail \mathbf{Loc\text{-}F^2Top}.$$

Using the categorical properties catalogued above, we metamathematically have

$$\text{topology} \rightarrowtail \text{algebra} \rightarrowtail \text{topology} \rightarrowtail \text{algebra} \rightarrowtail \text{topology}.$$

It is conjectured that this pattern continues indefinitely to the right.

6.2. **Non-commutivity and generalized structures.** In programming applications, the conjunction of predicates is frequently not commutative. This section briefly explores this issue. We begin with a motivating example.

Example 6.2.1. Suppose there are two numerical variables x and y in use. Traditional truth values are used, and the program in question operates as follows:

- The variable y tells how many times a website has been looked at, i.e., y holds a copy of the value in the website's counter.

- Whenever y is used in an expression, the website (which y counts) is accessed or looked at *before* the value in y is used or read so that y is always kept updated.
- Each time y is read, its value increases by one.
- The website is not accessed when conjunctions of predicates are formed.
- Conjunctions of predicates are read in the order left-to-right; i.e., the conjunction $P \wedge Q$ is read in the order P, Q.

Now assume that the current value in x is 9 and in y is 8; i.e., $x = 9$, $y = 8$. Now form the predicates P and Q as follows:

$$P : [\, x = y \,] ; \quad Q : [\, y \geq 10 \,] .$$

(1) What is the truth value of $P \wedge Q$? When P is read, the value of y changes from 8 to 9; when Q is read, the value of y changes from 9 to 10; so that when $P \wedge Q$ is read, both P and Q are true, and hence $P \wedge Q$ is **true**.
(2) Now what is the truth value of $Q \wedge P$, again assuming that $x = 9$, $y = 8$? When Q is read, the value of y changes from 8 to 9; when P is read, the value of y changes from 9 to 10; so that when $Q \wedge P$ is read, Q (and P) is false, so that $Q \wedge P$ is **false**.
(3) So in this example, $P \wedge Q$ and $Q \wedge P$ are not equivalent—the conjunction is sensitive to the order in which the predicates are read.

In the preceding sections the "Lindenbaum algebra" of finite observational logic has been taken to be a frame; and in a frame, the conjunction is modeled by the binary $\wedge$, which is commutative. The above example shows there is justification in having a structure in which the operation modeling conjunction is not commutative; and hence, there is justification in having a structure allowing for non-commutative cases.

There are several non-commutative approaches to systems: topological systems as motivated by attachment relations and in which predicates form residuated lattices [19, 20]; topological systems with predicates forming algebraic varieties [71, 72, 73]; and Chu systems with predicates forming "flat" sets [11].

The approach to topological systems outlined below is based on the notion of a unital quantale [68, 24]; but it is only one of several generalizations. Associated with such systems are "extent" topological spaces with non-commutative topologies—topologies in which the intersection of open sets need not be commutative. What recommends unital quantales is that they need not have commutative tensor products to express conjunctions, but they retain the infinite distributive laws which are part of finite observational logic, and hence are appropriate models of non-commutative finite observational logic.

Definition 6.2.2 (unital quantales)**.** A *unital quantale* $(L, \leq, \otimes, e)$ comprises data subject to axioms as follows:

(1) $(L, \leq)$ is a complete lattice.

(2) $\otimes : L \times L \to L$ is an associative binary operation, called *tensor product*, satisfying both *left* and *right infinite distributive laws*: $\forall a \in L$, $\forall \{b_\gamma\}_{\gamma \in \Gamma} \subset L$,

$$a \otimes \left(\bigvee_{\gamma \in \Gamma} b_\gamma \right) = \bigvee_{\gamma \in \Gamma} (a \otimes b_\gamma),$$

$$\left(\bigvee_{\gamma \in \Gamma} b_\gamma \right) \otimes a = \bigvee_{\gamma \in \Gamma} (b_\gamma \otimes a).$$

(3) e is a two-sided identity or unit for $\otimes$: $\forall a \in L$, $a \otimes e = a = e \otimes a$.

It should be noted that $\bot$ is a two-sided *annihilator* or *zero* for $\otimes$: $\forall a \in L$, $a \otimes \bot = \bot = \bot \otimes a$.

Definition 6.2.3 (category of unital quantales)**.** The category **UQuant** of unital quantales and *unital quantalic* mappings comprises data subject to axioms as follows:

(1) *Objects:* Unital quantales as defined in 6.2.2 above.

(2) *Morphisms:* $f : (L, \leq, \otimes_1, e_1) \to (M, \leq, \otimes_2, e_2)$, where $f : L \to M$ is a mapping which preserves arbitrary $\bigvee$ and the tensors—$f \circ \otimes_1 = \otimes_2 \circ (f \times f)$ and the units—$f(e_1) = e_2$.

(3) *Composition, identities:* from **Set**.

For convenience below, the opposite category $\mathbf{UQuant}^{op}$ is denoted **LoUQuant**, the prefix "Lo" motivated by **Loc** being the opposite category of **Frm**.

Examples 6.2.4. Some example classes of unital quantales include the following:

(1) L any frame with $\otimes$ the binary meet and $e = \top$. These are commutative, unital quantales.

(2) $L = [0, 1]$ with $\otimes$ any t-norm (e.g., $\otimes = \wedge$, multiplication, or Łukasiewicz conjunction) and $e = 1$. These are also commutative, unital quantales.

(3) Let L be any (join-)complete lattice, put

$$S(L) = \left\{ g : L \to L \mid g \text{ preserves arbitrary } \bigvee \right\},$$

and equip $S(L)$ with the point-wise order, with $\otimes$ as functional composition $\circ$, and with e as id_L. It should be noted that e is not the top element $\mathbf{1}$ of $S(L)$ defined as

$$\mathbf{1}(a) = \begin{cases} \top, & a > \bot \\ \bot, & a = \bot \end{cases}.$$

It can be shown that each $(S(L), \leq, \circ, id_L)$ is a unital quantale, which is usually *non-commutative*. It is also the case that L order-embeds into $S(L)$ via $\eta_L : L \to S(L)$ defined by

$$\eta_L(a)(b) = \begin{cases} a, & |L| = 1 \\ \begin{cases} a, & b > \bot \\ \bot, & b = \bot \end{cases}, & |L| \geq 1 \end{cases}.$$

Detailed analysis of η_L and its role in relating **UQuant** to **CSLat**($\bigvee$) are given in [65].

A category of topological systems based upon unital quantales can now be given.

Definition 6.2.5 (quantale based topological systems)**.** The category **LoUQuant-TopSys** has ground category $\mathbf{Set} \times \mathbf{LoUQuant}^2$ and comprises data subject to axioms as follows:

(1) *Objects:* $(X, L, A, \vDash)$, where $(X, L, A) \in |\mathbf{Set} \times \mathbf{LoUQuant}^2|$ and $\vDash : X \times A \to L$ is an (L-*valued*) *satisfaction relation* possessing the arbitrary $\bigvee$ and $\otimes$ and unitary *interchange laws*:

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A, \vDash \left(x, \bigvee_{\gamma \in \Gamma} a_\gamma \right) = \bigvee_{\gamma \in \Gamma} \vDash (x, a_\gamma);$$

$$\forall x \in X, \forall a, b \in A, \vDash (x, a \otimes b) = \vDash (x, a) \otimes \vDash (x, b),$$

$$\forall x \in X, \vDash (x, e_A) = \top,$$

where $\otimes$ stands for the tensors on A and L, and e_A is the unit for the tensor on A.

(2) *Morphisms:* $(f, \varphi, \psi) : (X, L, A, \vDash_1) \to (Y, M, B, \vDash_2)$, where $(f, \varphi, \psi) : (X, L, A) \to (Y, M, B)$ in $\mathbf{Set} \times \mathbf{LoUQuant}^2$ and (f, φ, ψ) satisfies *adjointness:*

$$\forall b \in B, \forall x \in X, \vDash_1 (x, \psi^{op}(b)) = \varphi^{op}[\vDash_2 (f(x), b)].$$

(3) *Composition, identities:* from $\mathbf{Set} \times \mathbf{LoUQuant}^2$.

Discussion 6.2.6. The type of topology associated with topological systems based on unital quantales can be explored by constructing the appropriate notion of "extent" to mimic the relationship between **TopSys** and **Top**. Such a notion of extent for systems from **Loc-TopSys** is already available [9] which constructs the (adjoint) relationship between

Loc-TopSys and **Loc-Top**. Let $(X, L, A, \vDash) \in |\mathbf{LoUQuant\text{-}TopSys}|$, and put

$$ext : A \to L^X \quad \text{by} \quad ext\,(a) : X \to L \quad \text{by} \quad ext\,(a)\,(x) = \,\vDash (x, a)\,.$$

Then it can be shown that $ext^{\to}(A)$ is a subset of L^X closed under arbitrary $\bigvee$—a "union" condition and closed under tensor products (as lifted to L^X from L)—an "intersection condition"; it is also the case that $\underline{e_L} \in ext^{\to}(A)$, where

$$\underline{e_L} : X \to L \quad \text{by} \quad \underline{e_L}\,(x) = e_L.$$

It is also a consequence of the union condition that $\underline{\perp} \in ext^{\to}(A)$. To summarize, $ext^{\to}(A)$ may be viewed as closed under arbitrary "unions" and binary "intersections" and containing the "whole space" and "empty set", and hence it should viewed as a kind of topology and $(X, L, ext^{\to}(A))$ as a kind of topological space. But these are topologies and spaces in which the intersection *need not be commutative*. Such spaces have been seen before, e.g., in [25, 62], in which are found spaces over complete quasi-monoidal lattices, structures which need not have commutative tensor products; but such structures need not be residuated and hence lack the structure appropriate for topological systems.

Definition 6.2.7 (quantale based topologies)**.** The category **LoUQuant-Top** has ground category **Set** $\times$ **LoUQuant** and comprises data subject to axioms as follows:

(1) *Objects:* (X, L, τ), where $(X, L) \in |\mathbf{Set} \times \mathbf{LoUQuant}|$ and $\tau \subset L^X$ is closed under arbitrary $\bigvee$ and tensor products and contains $\underline{e_L}$.
(2) *Morphisms:* $(f, \varphi) : (X, L, \tau) \to (Y, M, \sigma)$, where $(f, \varphi) : (X, L) \to (Y, M)$ in **Set** $\times$ **LoUQuant** and $\left((f, \varphi)^{\leftarrow}\right)^{\to}(\sigma) \subset \tau$, i.e.,

$$\forall v \in \sigma,\ (f, \varphi)^{\leftarrow}(v) \in \tau.$$

(3) *Composition, identities:* from **Set** $\times$ **LoUQuant**.

The proofs of [62] can be adapted to show that **LoUQuant-Top** is topological over **Set** $\times$ **LoUQuant** w.r.t. the expected forgetful functor. Next, **LoUQuant-Top** embeds into **LoUQuant-TopSys** by a functor analogous to $E_{\mathbf{Loc\text{-}Top}}$ given in Theorem 6.1.4(2) above; and an adjoint functor can be given which is based on the extent topologies and spaces of Discussion 6.2.6 above. Finally, it is possible to again follow the pattern begun in [62] of building variable-basis frameworks for Kubiak-Šostak topologies and define a category **LoUQuant-F²Top**, analogous to **Loc-F²Top** in Definition 6.1.6 above, into which **LoUQuant-TopSys** concretely embeds.

This subsection has focused on unital quantales; but much work remains to be done to pin down the generalizations with greatest potential for applications of topological systems.

6.3. **Lattice-valued preorders and enriched/preordered topological systems and topological spaces.** A question from programming arises: if bitstring x compares with bitstring y to some degree α, and if bitstring y satisfies predicate a to some degree β, then how should the possibility be mathematically modeled that bitstring x satisfies predicate a to at least some degree related to both α and β? Many applications of (partial) responses to this question might exist in data-mining, a field in which pattern-matching is an important and commonly used method.

As discussed in [8, 12, 13], ideas from enriched categories over monoidal categories address this question and enable pattern-matching techniques to be extended to many-valued contexts. In particular, notions of enriched category theory naturally lead to the notion of L-valued preorders, and then to topological systems enriched with frame-valued preorders and associated extent spaces as enriched (or preordered) many-valued topological spaces—the compatibility axioms for such systems and spaces allow us to answer the programming question posed above. We summarize these notions below and give an extensive inventory of example classes, including programming examples, and an extensive discussion of examples based on the L-spectrum of a locale outlined in Example 4.3.3 above.

A partially ordered set in which each finite subset has a greatest lower bound, or meet, is a *meet semilattice*—it follows that such a poset has a greatest or top element $\top$.

Definition 6.3.1. Let L be a meet semilattice. Then a set X has an L-*enrichment relation* or L-*(valued) preorder* P on X if P satisfies:

P1. $P : X \times X \to L$ is a mapping (*degrees of comparison*).
P2. $\forall x \in X,\ P(x, x) = \top$ (*total existence* or *reflexivity*).
P3. $\forall x, y, z \in X,\ P(x, y) \wedge P(y, z) \le P(x, z)$ (*transitivity*).

We may speak of (X, P) as an L-*enriched set*—since it is an enriched category over the moniodal category L—or more often as an L-*preordered set*; and (X, L, P) is an *enriched* or *preordered set*, setting the stage for subsequent variable-basis settings in which the base L may change from set to set.

Definition 6.3.2 (category of frame-valued preordered sets)**.** The category **Loc-PreSet** has **Set** $\times$ **Loc** as a ground category and comprises data subject to axioms as follows:

(1) *Objects:* (X, L, P), where L is a frame and (X, L, P) is a preordered set.

(2) *Morphisms:* $(f, \varphi) : (X, L, P) \to (Y, M, Q)$, where $f : X \to Y$ is a mapping, $\varphi^{op} : L \leftarrow M$ is a frame morphism, and $\forall x, y \in X$,

$$P(x, y) \leq \varphi^{op} [Q(f(x), f(y))].$$

Such morphisms are said to be *enriched* or *order-preserving* or *isotone*.

(3) *Compositions, identities:* from **Set** $\times$ **Loc**.

Denote by **Set** $\times$ **Loc** $(\bigwedge)$ the subcategory of **Set** $\times$ **Loc** in which, for each morphism (f, φ), φ^{op} preserves arbitrary $\bigwedge$ as well as arbitrary $\bigvee$, and denote by **Loc** $(\bigwedge)$**-PreSet** the subcategory of **Loc-PreSet** having ground category **Set** $\times$ **Loc** $(\bigwedge)$ and in which, for each morphism (f, φ), φ^{op} preserves arbitrary $\bigwedge$ as well as arbitrary $\bigvee$. It is shown in [12] that **Loc** $(\bigwedge)$**-PreSet** is topological over **Set** $\times$ **Loc** $(\bigwedge)$ with respect to the expected forgetful functor, a result implying that **Set** $\times$ **Loc** $(\bigwedge)$ has no known degree of algebraicity over **Set** $\times$ **Loc** as well as generalizing the fact [1] that the traditional category **PreSet** for preordered sets is topological over **Set** with respect to the expected forgetful functor.

The next definition combines the notion of frame-valued preorders with the variable-basis notion of topological systems embodied in **Loc-TopSys** discussed in Subsection 6.1 above.

Definition 6.3.3 (enriched/preordered topological systems)**. PreTopSys** has ground category **Loc-PreSet** $\times$ **Loc** and comprises the following data and axioms:

(1) *Objects:* $((X, L, P), A, \models)$, or $(X, L, P, A, \models)$, called *enriched* or *preordered topological systems*, where:
  (a) (X, L, P) is a preordered set, A is a locale (*ground condition*);
  (b) $(X, L, A, \models)$ is a topological system in **Loc-TopSys**, i.e., $\models$ is an L-satisfaction relation on (X, A) satisfying both arbitrary $\bigvee$ and finite $\wedge$ interchange laws (*topological system condition*);
  (c) P and $\models$ are *compatible*, i.e., $\forall x, y \in X$, $\forall a \in A$,

$$P(x, y) \wedge \models (y, a) \ \leq \ \models (x, a)$$

  (*compatibility condition*).

(2) *Morphisms:* $(f, \varphi, \psi) : ((X, L, P), A, \models) \to ((Y, M, Q), B, \models)$, called *isotone continuous functions*, where:
  (a) $(f, \varphi) : (X, L, P) \to (Y, M, Q)$ is an isotone mapping, $\psi : A \to B$ is a localic morphism (*ground condition*);
  (b) $(f, \varphi, \psi) : (X, L, A, \models) \to (Y, M, B, \models)$ is a **Loc-TopSys** morphism (*continuity condition*).

(3) *Composition, identities:* from **Loc-PreSet** $\times$ **Loc**.

It is the compatibility condition in the above definition which addresses the programming question posed at the beginning of this subsection.

Now let U : **PreTopSys** $\to$ **Loc-PreSet** $\times$ **Loc** be the forgetful functor given by

$$U(X, L, P, A, \models) = (X, L, P, A),$$

$$U[(f, \varphi, \psi) : (X, L, P, A, \models) \to (Y, M, Q, B, \models)] = (f, \varphi, \psi) : (X, L, P, A) \to (Y, M, Q, B).$$

Theorem 6.3.4. **PreTopSys** *is neither quasi-algebraic* [12] *nor essentially topological nor existentially topological in the sense of* [9] *over* **Loc-PreSet** $\times$ **Loc** *w.r.t.* U*; and hence* **PreTopSys** *is neither essentially algebraic nor topological over* **Loc-PreSet** $\times$ **Loc** *w.r.t.* U.

This theorem suggests a comparison of **PreTopSys** with **TopGrp**, which is neither algebraic nor topological over **Set**; but it is known that **TopGrp** is essentially algebraic over **Top** and topological over **Grp**. It is ongoing work of the authors to resolve the question of whether **PreTopSys** can have a degree of algebraicity over one ground and a degree of topologicity over another ground and identifying pairs of such grounds; e.g., the authors are studying the behavior of **PreTopSys** over **Loc-PreSet** and over **Loc-TopSys**. With regard to the latter category, **PreTopSys** has an adjoint relationship given by the expected concrete forgetful functor G : **PreTopSys** $\to$ **Loc-TopSys** and its left adjoint and embedding H : **Loc-TopSys** $\rightarrowtail$ **PreTopSys** constructed using the crisp equality relation

$$E(x, y) = \begin{cases} \top, & x = y \\ \bot, & x \neq y \end{cases}$$

as the needed lattice-valued preorders; and $H \dashv G$ is an isoreflection.

Discussion 6.3.5 (motivation of lattice-valued preordered topologies)**.** As seen in Theorem 3.2.2 and Discussion 6.2.6, a type of topological system is generally matched with a corresponding type of topological space via the notion of extent. Recalling the notion of extent used in 6.2.6 for unital quantales, we use this notion instantiated for frames and $\otimes = \wedge$, the binary meet, to discover the kind of topological spaces associated with preordered topological systems. Let $(X, L, P, A, \models)$ be a preordered topological system, and recall $ext : A \to L^X$ given by

$$ext(a) : X \to L \quad \text{by} \quad ext(a)(x) = \models(x, a).$$

Now let $x, y \in X$. Then the compatibility axiom for $(X, L, P, A, \models)$ states that

$$P(x, y) \wedge \models(y, a) \leq \models(x, a),$$

and hence

$$P(x,y) \wedge ext(a)(y) \leq ext(a)(x).$$

Finally, it is noted that $ext^{\rightarrow}(A)$ is an L-topology on X and that its images $ext(a)$ are open sets. This leads to the next definition.

Definition 6.3.6 (category of preordered topological spaces)**.** The category **PreTop** of *preordered topological spaces* and *isotone continuous mappings* has ground category **Loc-PreSet** and comprises the following data and axioms:

(1) *Objects:* (X, L, P, τ), where (X, L, P) is a preordered set, (X, L, τ) is a topological space in **Loc-Top** (5.2.1), and τ satisfies the *compatibility axiom*:

$$\forall x, y \in X,\ \forall u \in \tau,\ P(x,y) \wedge u(y) \leq u(x).$$

(2) *Morphisms:* $(f, \varphi) : (X, L, P, \tau) \to (Y, M, Q, \sigma)$, where $(f, \varphi) : (X, L, P) \to (Y, M, Q)$ is a **Loc-PreSet** morphism and $(f, \varphi) : (X, L, \tau) \to (Y, M, \sigma)$ is a **Loc-Top** morphism.

(3) *Composition, identities:* from **Loc-PreSet**.

Theorem 6.3.7 [12]**. PreTop** *is a topological category over* **Loc-PreSet** *w.r.t. the expected forgetful functor.*

In [12], many-valued specializations are considered in the context of preordered topological spaces and it is shown that these preorders satisfy a certain antisymmetry axiom if and only if the space is L-T_0, a separation axiom intrinsic to L-sobriety and the study of L-spectra (4.3.3 above) and is the many-valued generalization of the traditional T_0 axiom.

An inventory of example classes of preordered topological systems and topological spaces closes out this subsection, preceded by an inventory of enriched or preordered sets.

Example 6.3.8 (classes of many-valued preordered sets)**.**

(1) Indiscrete Preordered Sets. Let X be a set and L be a meet semilattice. Put $P : X \times X \to L$ by $P(x,y) = \top$. Then (X, L, P) is a preordered set which we call an *indiscrete* preordered set.

(2) Discrete Preordered Sets. Let X be a set and L be a meet semilattice with $|L| \geq 2$. Choose $\alpha \in L - \{\top\}$ and put $P : X \times X \to L$ by

$$P(x,y) = \begin{cases} \alpha, & x \neq y \\ \top, & x = y \end{cases}.$$

Then (X, L, P) is a preordered set which we call the α-*discrete* preordered set.

(3) Ultrametric Spaces. Let (X,d) be an ultrametric space (with strengthened triangle inequality $d(x,z) \le d(x,y) \vee d(y,z)$) bounded by 1, and put $P : X \times X \to [0,1]$ by

$$P(x,y) = 1 - d(x,y).$$

Then (X,L,P) is a preordered set.

(4) Bitstring Based Examples. Consider two (countably) infinite binary bitstrings σ_1, σ_2 (of 0's, 1's), where, for $n \in \mathbb{N}$,

$$\sigma_i(n) = \text{bit in } n^{th} \text{ place},$$

and define a *comparison bitstring* $P(\sigma_1, \sigma_2)$ as follows:

$$P(\sigma_1,\sigma_2)(n) = \begin{cases} 1, & \sigma_1(n) = \sigma_2(n) \\ 0, & \sigma_1(n) \neq \sigma_2(n) \end{cases}.$$

Let $X = \mathbf{B} = \mathbf{2}^\omega = \{\sigma : \sigma \text{ is a countably infinite bitstring}\}$ equipped with the pointwise ordering. Then $\mathbf{B}$ is a complete Boolean algebra, and $P : X \times X \to \mathbf{B}$ as constructed above is a well-defined mapping. Also, noting that $\top$ in $\mathbf{B}$ is the bitstring with all 1's, reflexivity follows since

$$\forall \sigma \in X,\ P(\sigma,\sigma) = \top.$$

Further, letting $\sigma_1, \sigma_2, \sigma_3 \in X$ and $n \in \mathbb{N}$, assume

$$P(\sigma_1,\sigma_2)(n) \wedge P(\sigma_2,\sigma_3)(n) = 1.$$

Then

$$\sigma_1(n) = \sigma_2(n) = \sigma_3(n).$$

Hence

$$P(\sigma_1,\sigma_3)(n) = 1.$$

Therefore transitivity follows since

$$P(\sigma_1, \sigma_2) \wedge P(\sigma_2, \sigma_3) \le P(\sigma_1, \sigma_3).$$

And so $(X, \mathbf{B}, P)$ is a preordered set.

(a) There are a number of variations on the construction of the preceding class which each yield preordered sets. Keeping $\mathbf{B}$ as before, X could be the set $\mathbf{2}^k$ of all finite length strings of some specified length k, or the set $\mathbf{2}^*$ of all finite length strings, or the set $\mathbf{2}^{*\omega}$ of all strings which are countable (finite or infinite). To illustrate, suppose X is the set of all k-length bitstrings, for some fixed $k \in \mathbb{N}$; and put $P : X \times X \to \mathbf{B}$ by

$$P(\sigma_1, \sigma_2)(n) = \begin{cases} 1, \ \sigma_1(n) = \sigma_2(n) \ \text{ and } n \le k \\ 0, \ \sigma_1(n) \neq \sigma_2(n) \ \text{ and } n \le k \\ \quad 1, \ n > k \end{cases}.$$

Then $(X, \mathbf{B}, P)$ is a preordered set.

(b) The previous two classes can be generalized to non-binary string induced examples. Let Σ be an alphabet with $|\Sigma| \ge 2$, and let $\Sigma^{*\omega}$ be the set of all countable strings (both finite and infinite) on Σ. Now let $\mathbf{B}$ be appropriately generalized from above and put $P : \Sigma^{*\omega} \times \Sigma^{*\omega} \to \mathbf{B}$ as above. Then $(\Sigma^{*\omega}, \mathbf{B}, P)$ is a preordered set.

(5) Locale Based Examples. Let L be a frame. For each locale A, put

$$P_A : Lpt(A) \times Lpt(A) \to L \ \text{ by } P_A(p, q) = \bigwedge_{a \in A} (q(a) \to p(a)),$$

where we recall the definition of the carrier set of the L-spectrum from Example 4.3.3 above, namely, that

$$Lpt(A) = \left\{ p : A \to L \mid p \text{ preserves arbitrary } \bigvee \text{ and finite } \wedge \right\},$$

and where $\to$ refers to Heyting residuation ($\alpha \to \beta \ge \gamma \Leftrightarrow \alpha \wedge \gamma \le \beta$).

(a) $(Lpt(A), L, P)$ is a preordered set. Reflexivity follows since

$$P_A(p, p) = \bigwedge_{a \in A} (p(a) \to p(a)) = \bigwedge_{a \in A} \top = \top$$

and transitivity follows since

$$\begin{aligned}
P_A(p,q) \wedge P_A(q,r) &= \\
\bigwedge_{a\in A}(q(a)\to p(a)) \wedge \bigwedge_{b\in A}(r(b)\to q(b)) &= \\
\bigwedge_{(a,b)\,\in\, A\times A} [(q(a)\to p(a)) \wedge (r(b)\to q(b))] &\leq \\
\bigwedge_{a\in A} [(q(a)\to p(a)) \wedge (r(a)\to q(a))] &= \\
\bigwedge_{a\in A} [(r(a)\to q(a)) \wedge (q(a)\to p(a))] &\leq \\
\bigwedge_{a\in A} (r(a)\to p(a)) &= P_A(p,r),
\end{aligned}$$

where the transitivity of $\to$ is used in the next to last line.

(b) The question arises as to why the order of implication $q(a)\to p(a)$ was chosen in the definition of P_A and not $p(a)\to q(a)$. The chosen order is forced by the "compatibility condition" imposed in subsequent sections on preordered topological systems and preordered topological spaces, a condition which specifically answers the programming question stated at the beginning of this subsection.

(c) This example is of particular interest since it is fundamentally related to specialization orders related to L-spectra of locales.

(d) This example is also of particular interest when L is spatial and A is non-spatial, since it is an L-preordered set not generated from (preordered) topological spaces.

Example 6.3.9 (preordered topological systems and topological spaces)**.**

(1) Fibres of Preordered Topologies. Let (X,L,P) be a preordered set with L a frame and consider the fibre $\mathbb{T}$ of all preordered topologies on (X,L,P) ordred by inclusion. Further, consider the following families of L-valued subsets of X:

$$\begin{aligned}
\tau_{\max} &= \left\{u\in L^X : \forall x,y\in X,\ P(x,y)\wedge u(y)\leq u(x)\right\}, \\
\tau_l &= \left\{u\in L^X : \forall x,y\in X,\ P(x,y)\wedge u(y) = P(x,y)\wedge u(x)\right\}, \\
\tau_r &= \left\{u\in L^X : \forall x,y\in X,\ P(x,y)\wedge u(y) = P(y,x)\wedge u(x)\right\}, \\
\tau_{const} &= \left\{u\in L^X : \forall x,y\in X,\ P(x,y)\wedge u(y) = u(x)\right\}, \\
\tau_{\min} &= \{\underline{\perp},\underline{\top}\}.
\end{aligned}$$

Then the following hold:
(a) $\mathbb{T}$ is a complete lattice ordered by inclusion.
(b) $\tau_{\max}$ is the largest member of $\mathbb{T}$ and $\tau_{\min}$ is the smallest member of $\mathbb{T}$.
(c) $\forall y \in X$, put $P_y : X \to L$ by $P_y(x) = P(x,y)$. Then $\langle\langle\{P_y : y \in X\}\rangle\rangle \subset \tau_{\max}$.
(d) $\tau_{const} = \{\underline{\alpha} : \alpha \in L\}$; τ_{const}, τ_l, $\tau_r \in \mathbb{T}$; $\tau_{const} \subset \tau_l$, $\tau_{const} \subset \tau_r$, so that each of τ_{const}, τ_l, τ_r, $\tau_{\max}$ is a stratified L-topology.
(e) Put $\tau = \left\{u \in L^X : \forall x, y \in X,\ P(u,x,y,R)\right\}$, where $P(u,x,y,R)$ is of the form

$$P(x,y) \wedge u(y) \;\; R \; rhs,$$

where the binary relation R on L is either $=$ or $\leq$, and rhs is a string comprising any combination of $u(x), P(x,y), P(y,x), \wedge$. Then τ is one of τ_{const}, τ_l, τ_r, $\tau_{\max}$.

(2) Preordered Spaces to Preordered Systems. Let (X, L, P, τ) be a preordered topological space. Then $(X, L, P, \tau, \models_\tau)$ is a preordered topological system, where

$$\models_\tau (x, u) = u(x).$$

(3) Preordered Systems from L-Spectra—see Example 6.3.8(5) above. Let L be a frame and A be a locale. Then

$$(Lpt(A), L, P_A, A, \models_A)$$

is a preordered topological system, where $Lpt(A) = \mathbf{Frm}(A, L)$, $P_A : Lpt(A) \times Lpt(A) \to L$ by $P_A(p,q) = \bigwedge_{a \in A}(q(a) \to p(a))$, and $\models_A : Lpt(A) \times A \to L$ is defined by

$$\models_A (p, a) = p(a).$$

(4) Bitstring Based Preordered Topologies and Topological Systems Not in (1) and (2) Above. It is possible for preordered sets $(\Sigma^{*\omega}, P, \mathbf{B})$ in Example 6.3.8(4) to construct preordered topologies not listed in (1) above. For $\alpha \in \Sigma$, put $p^\alpha : \Sigma^{*\omega} \to \mathbf{B}$ by

$$p^\alpha(\sigma)(n) = \begin{cases} 1, & \sigma(n) = \alpha \\ 0, & \sigma(n) \neq \alpha \end{cases}.$$

Let $\mathcal{Q} \subset \mathbf{B}^{\Sigma^{*\omega}}$ by

$$\mathcal{Q} = \left\{p^{\alpha_1} \wedge \ldots \wedge p^{\alpha_k} : k \in \mathbb{N},\ \{\alpha_i\}_{i=1}^k \subset \Sigma\right\},$$

let

$$\mathfrak{Q} = \left\{\bigvee \hat{\mathcal{Q}} : \hat{\mathcal{Q}} \subset \mathcal{Q}\right\}.$$

Then $\mathfrak{Q}$ is a subframe of $\mathbf{B}^{\Sigma^{*\omega}}$, i.e., $\mathfrak{Q}$ is a $\mathbf{B}$-topology on $\Sigma^{*\omega}$; in fact,

$$\mathfrak{Q} = \left\langle\left\langle \left\{ p^{\alpha_1}, ..., p^{\alpha_k} : k \in \mathbb{N},\ \{\alpha_i\}_{i=1}^{k} \subset \Sigma \right\} \right\rangle\right\rangle.$$

Finally, put $\vDash_{\mathfrak{Q}} : \Sigma^{*\omega} \times \mathfrak{Q} \to \mathbf{B}$ by

$$\vDash_{\mathfrak{Q}} (\sigma, q) = q(\sigma).$$

It follows that $(\Sigma^{*\omega}, P, \mathbf{B}, \mathfrak{Q}, \vDash)$ is a preordered topological system by Theorem 7.4 of [12].

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References

[1] J. Adámek, H. Herrlich, G. E. Strecker, Abstract and Concrete Categories, second edition, Dover Publications (New York, 2009).

[2] M. Barr, ∗-Autonomous Categories, Lecture Notes in Mathematics **752** (1979), Springer-Verlag (Berlin/Heidelberg/New York).

[3] T. S. Blyth, *Sur certaines images homomorphes des demi-groupes ordonnés*, Bull. Soc. Math. France **94** (1970), 101–111.

[4] F. Bayoumi, S. E. Rodabaugh, *Overview and comparison of localic and fixed-basis topological products*, Fuzzy Sets and Systems **161** (2010), 2397–2439 (Elsevier B.V., doi:10.1016/j.fss.2010.05.013).

[5] N. Bourbaki, Topologie Générale, Actualités Sci. Ind. **1084** (Paris, 1949) / Hermann Press (Paris, 1965).

[6] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

[7] J. T. Denniston, S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, Quaestiones Mathematicae **32:2** (2009), 139–186.

[8] J. T. Denniston, A. Melton, S. E. Rodabaugh, *Enriched topological systems and variable-basis enriched functors*, in U. Höhle, L. N. Stout, E. P. Klement, Enriched Category Theory and Related Topics: Abstracts of 33rd Linz Seminar (14–18 February 2012), Universitätsdirecktion Johannes Kepler Universität (Linz, Austria), 16–20.

[9] ———, *Interweaving algebra and topology: Lattice-valued topological systems*, Fuzzy Sets and Systems **192** (2012), 58–103.
[10] ———, *Formal concept analysis and lattice-valued Chu systems*, Fuzzy Sets and Systems, **216** (2013), 52–90.
[11] ———, *Lattice-valued predicate transformers, Chu games, and lattice-valued transformer systems,* in submission.
[12] ———, *Enriched topological systems, enriched topologies, and variable-basis enriched functors*, in submission.
[13] J. T. Denniston, A. Melton, S. E. Rodabaugh, S. A. Solovjovs, *Lattice-valued preordered sets as lattice-valued topological systems*, in Radko Mesiar, Endre Pap, E. P. Klement, Non-Classical Measures and Integrals, Abstracts of 34rd Linz Seminar (26 February – 2 March 2013), Universitätsdirecktion Johannes Kepler Universität (Linz, Austria), 28–34.
[14] E. W. Dijkstra, *A Discipline of Programming*, Prentice-Hall: Engelwood (Cliffs, New Jersey, 1976).
[15] C. H. Dowker, D. Papert, *On Urysohn's lemma*, General Topology and Its Relation to Modern Analysis and Algebra II (1967), 111–114, Academia (Prague).
[16] M. Fourman, D. S. Scott, *Sheaves and logic*, Applications of Sheaves: Lecture Notes in Mathematics **753** (1979), 302–401, Springer-Verlag (Berlin, Heidelberg, New York).
[17] T. E. Gantner, R. C. Steinlage, and R. H. Warren, *Compactness in fuzzy topological spaces*, J. Math. Anal. Appl. **62** (1978), 547–562.
[18] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–167.
[19] C. Guido, *Attachment between fuzzy points and fuzzy sets*, in U. Bodenhofer, B. De Baets, E. P. Klement, Abstracts of the 30th Linz Seminar, Universitätsdirektion Johannes Kepler Universität, Linz, Austria, 3–7 February 2009, pp. 52–54.
[20] ———, *Fuzzy points and attachment*, Fuzzy Sets and Systems **161:16** (2010), 2150–2165.
[21] U. Höhle, *Uppersemicontinuous fuzzy sets and applications*, J. Math. Anal. Appl. **78** (1980), 659–673.
[22] ———, *Fuzzy topologies and topological space objects in a topos*, Fuzzy Sets and Systems **19** (1986), 299–304.
[23] ———, *Presheaves over GL-monoids*, in U. Höhle, E. P. Klement, Non-Classical Logics and their Applications to Fuzzy Subsets: Theory and Decision Library: Series B: Mathematical and Statistical Methods **32** (1995), Kluwer Academic Publishers (Boston, Dordrecht, London), pp. 127–158.
[24] U. Höhle, T. Kubiak, *A non-commutative and non-idempotent theory of quantale sets*, Fuzzy Sets and Systems **166** (2011), 1–43.
[25] U. Höhle and A. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, in: U. Höhle, S. E. Rodabaugh, *Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory,* The Handbooks of Fuzzy Sets Series **3**(1999), 123–272 (Chapter 3), Springer Verlag / Kluwer Academic Publishers.
[26] B. Hutton, *Normality in fuzzy topological spaces*, J. Math. Anal. Appl. **50** (1975), 74–79.
[27] B. Hutton and I. Reilly, *Separation axioms in fuzzy topological spaces,* Fuzzy Sets and Systems **3** (1980), 93–104.
[28] J. R. Isbell, *Atomless parts of spaces*, Math. Scand.**31** (1972), 5–32.
[29] P. T. Johnstone, Stone Spaces, Cambridge University Press (Cambridge, 1982).
[30] J. L. Kelley, General Topology, Van Nostrand (New York, 1955).

[31] G. M. Kelly, Basic Concepts of Enriched Category Theory, Reprints in Theory and Applications of Categories **10** (2005).

[32] W. Kotzé, *Lattice morphisms, sobriety, and Urysohn Lemmas*, in: S. E. Rodabaugh, E. P. Klement, and U. Höhle, Applications of Category Theory to Fuzzy Subsets: Series B: Mathematical and Statistical Methods **14** (1992), 257–274 (Chapter 10), Kluwer Academic Publishers (Boston/Dordrecht/London).

[33] ———, *Lifting of sobriety concepts with particular reference to* (L, M)*-topological spaces*, in: S. E. Rodabaugh, E. P. Klement, Topological And Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic **20** (2003), 415–426 (Chapter 16), Kluwer Academic Publishers (Boston, Dordrecht, London).

[34] T. Kubiak, On Fuzzy Topologies, Ph.D. dissertation, Adam Mickiewicz University, Poznan (Poland), 1985.

[35] ———, *L-fuzzy normal spaces and Tiezte extension theorem*, J. Math. Anal. Appl. **125** (1987), 141–153.

[36] ———, *The topological modification of the L-fuzzy unit interval*, in: S. E. Rodabaugh, E. P. Klement, and U. Höhle, Applications of Category Theory to Fuzzy Subsets: Series B: Mathematical and Statistical Methods **14** (1992), 275–305 (Chapter 11), Kluwer Academic Publishers (Boston/Dordrecht/London).

[37] ———, *Separation axioms: extension of mappings and embedding of spaces,* in: U. Höhle, S. E. Rodabaugh, Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory: The Handbooks of Fuzzy Sets Series **3** (1999), 433–480 (Chapter 6), Springer Verlag / Kluwer Academic Publishers.

[38] ———, *Fuzzy reals: topological results surveyed, Brouwer fixed point theorem, open questions*, in: S. E. Rodabaugh, E. P. Klement, Topological And Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic **20** (2003), 137–151 (Chapter 5), Kluwer Academic Publishers (Boston, Dordrecht, London).

[39] T. Kubiak, A. Šostak, *Lower set-valued fuzzy topologies*, Quaestiones Mathematicae **20:3** (1997), 423–429.

[40] H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems **157** (2006), 1865–1885.

[41] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), 621–633.

[42] S. Mac Lane, Categories for the Working Mathematician, second edition, Graduate Texts in Mathematics **5** (1998), Springer Verlag (Berlin, Heidelberg, New York).

[43] H. W. Martin, *Weakly induced fuzzy topological spaces*, J. Math. Anal. Appl. **78** (1980), 634–639.

[44] G. H. J. Meßner, *Sobriety of* $\mathbb{R}(L)$, chapter in draft of Ph.D. thesis, 1987, Johannes Kepler Universität (Linz, Austria).

[45] C. J. Mulvey, *On the geometry of choice*, in: S. E. Rodabaugh, E. P. Klement, Topological And Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic **20** (2003), 309–336 (Chapter 11), Kluwer Academic Publishers (Boston, Dordrecht, London).

[46] D. Papert and S. Papert, *Sur les treillis des ouverts et les paratopologiqies*, Séminaire Ehresmann (topologie et géometrie differentielle), Ire anneé (1957–1958), exposé 1.

[47] Q. Pu, D. Zhang, *Preordered sets valued in a GL-monoid*, Fuzzy Sets and Systems **187** (2012), 1–32.

[48] A. Pultr, S. E. Rodabaugh, *Lattice-valued frames, functor categories, and classes of sober spaces*, in: S. E. Rodabaugh, E. P. Klement, Topological And Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic **20** (2003), 153–187 (Chapter 6), Kluwer Academic Publishers (Boston, Dordrecht, London).

[49] ______, *Examples for different sobrieties in fixed-basis topology*, in: S. E. Rodabaugh, E. P. Klement, Topological And Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Trends in Logic **20** (2003), 427–440 (Chapter 17), Kluwer Academic Publishers (Boston, Dordrecht, London).

[50] ______, *Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations / Part II: Applications to lattice-valued topology,* Fuzzy Sets and Systems **159:5** (2008), 501–528 / 529–558.

[51] S. E. Rodabaugh, *A categorical accommodation of various notions of fuzzy topology*, Fuzzy Sets and Systems **9** (1983), 241–265. Preliminary report, in: E. P. Klement, Proceedings of the Third International Seminar on Fuzzy Set Theory **3** (1981), 119–152, Johannes Kepler Universitätsdirektion (Linz, Austria).

[52] ______, *Separation axioms and the L-fuzzy real lines*, Fuzzy Sets and Systems **11** (1983), 163–183.

[53] ______, *A point-set lattice-theoretic framework* $\mathbb{T}$ *for topology which contains* **Loc** *as a subcategory of singleton subspaces and in which there are general classes of Stone Representation and Compactification Theorems*, First printing February 1986, Second printing April 1987, Youngstown State University Printing Office (Youngstown, Ohio(USA)).

[54] ______, *Dynamic topologies and their applications to crisp topologies, fuzzification of crisp topologies, and fuzzy topologies on the crisp real line*, J. Math. Anal. Appl. **131** (1988), 25–66.

[55] ______, *Lowen, para-Lowen, and* α*-level functors and fuzzy topologies on the crisp real line*, J. Math. Anal. Appl. **131** (1988), 157–169.

[56] ______, *Point-set lattice-theoretic topology*, Fuzzy Sets and Systems **40** (1991), 297–345.

[57] ______, *Necessity of Chang-Goguen topologies*, Rend. Circolo Mat. Palermo (Suppl: Ser. II) **29** (1992), 299–314.

[58] ______, *Categorical frameworks for Stone representation theories*, in: S. E. Rodabaugh, E. P. Klement, and U. Höhle, Applications of Category Theory to Fuzzy Subsets: Series B: Mathematical and Statistical Methods **14** (1992), 177–231 (Chapter 7), Kluwer Academic Publishers (Boston/Dordrecht/London).

[59] ______, *Applications of localic separation axioms, compactness axioms, representations, and compactifications to poslat topological spaces*, Fuzzy Sets and Systems **73** (1995), 55–87.

[60] ______, *Powerset operator based foundation for point-set lattice-theoretic (poslat) fuzzy set theories and topologies*, Quaestiones Mathematicae **20:3** (1997), 463–530.

[61] ______, *Powerset operator foundations for poslat fuzzy set theories and topologies*, in: U. Höhle, S. E. Rodabaugh, *Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory,* The Handbooks of Fuzzy Sets Series **3** (1999), 91–116 (Chapter 2), Springer Verlag / Kluwer Academic Publishers.

[62] ______, *Categorical foundations of variable-basis fuzzy topology*, in: U. Höhle, S. E. Rodabaugh, Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory: The Handbooks of Fuzzy Sets Series **3** (1999), 273–388 (Chapter 4), Springer Verlag / Kluwer Academic Publishers.

[63] ______, *Separation axioms: representation theorems, compactness, compactifications* in: U. Höhle, S. E. Rodabaugh, Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory: The Handbooks of Fuzzy Sets Series **3** (1999), 481–552 (Chapter 7), Springer Verlag / Kluwer Academic Publishers.

[64] ______, *Fuzzy real lines and dual real lines as poslat topological, uniform, and metric ordered semirings with unity,* in: U. Höhle, S. E. Rodabaugh, Mathematics Of Fuzzy Sets: Logic, Topology, And Measure Theory: The Handbooks of Fuzzy Sets Series **3** (1999), 607–632 (Chapter 10), Springer Verlag / Kluwer Academic Publishers.

[65] ______, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, International Journal of Mathematics and Mathematical Sciences **2007:3**, Article ID 43645, 71 pp., doi: 10.1155/2007/43645, ⟨http://www.hindwai.com/gearticle.aspx?⟩.

[66] ______, *Necessity of non-stratified and anti-stratified spaces in lattice-valued topology*, Fuzzy Sets and Systems **161** (2010), 1253–1269.

[67] ______, *Programming semantics, topological systems, and lattice-valued topology*, 27th Summer Conference on General Topology and Its Applications, 25–28 July 2012, Minnesota State University (Mankato, Minnesota).

[68] K. I. Rosenthal, Quantales and Their Applications, Pitman Research Notes in Mathematics **234** (1990), Pitman (Longman/Burnt Mill/Harlow).

[69] S. Shenoi, A. Melton, *Proximity relations in the fuzzy relational database model*, Fuzzy Sets and Systems **31** (1989), 285–296.

[70] M. Smyth, *Powerdomains and predicate transformers: a topological view*, Automata, Languages, and Programming, Lecture Notes in Computer Science, **154** (1983) 662–675, Springer-Verlag (Berlin/Heidelberg/New York).

[71] S. A. Solovjovs, *Embedding topology into algebra*, in U. Bodenhofer, B, De Baets, E. P. Klement, Abstracts of the 30th Linz Seminar (37 February 2009), Universitätsdirecktion Johannes Kepler Universität (Linz, Austria), 106–110.

[72] ______, *Variable-basis topological systems versus variable-basis topological spaces*, Soft Computing **14:10** (2010), 1059–1068.

[73] ______, *Categorical foundations of variety-based topology and topological systems*, Fuzzy Sets and Systems, to appear.

[74] A. P. Šostak, *On a fuzzy topological structure*, Rendiconti Circolo Matematico Palermo (Suppl: Serie II) **11** (1985) 89–103.

[75] S. J. Vickers, Topology Via Logic, Cambridge University Press (Cambridge, 1989).

[76] W. Yao, F.-G. Shi, *A note on specialization L-preorder of L-topological spaces, L-fuzzifying topological spaces, and L-fuzzy topological spaces*, Fuzzy Sets and Systems **159** (2008), 2586–2595.

[77] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

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