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FUNCTION MEASURABLE SPACES

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Abstract. Let $\mathcal{M}(Y, Z)$ be the set of all measurable maps from a topological space Y into a topological space Z and $\sigma_Z(\tau_Y)$ be the set consisting of all subsets $f^{-1}(B)$ of Y, where $f \in \mathcal{M}(Y, Z)$ and B is a measurable subset of Z. We introduce and study the notions of coordinately measurable and measurable $\mathcal{A}$-splitting and $\mathcal{A}$-admissible topologies on the set $\mathcal{M}(Y, Z)$, where $\mathcal{A}$ is an arbitrary family of topological spaces. In the set $\mathcal{M}(Y, Z)$ generally there does not exist the greatest coordinately measurable $\mathcal{A}$-splitting topology. This fact gives a different result from the classical theory of function topological spaces. Also, we present and research relations between the topologies on the set $\mathcal{M}(Y, Z)$ and the topologies on the set $\sigma_Z(\tau_Y)$, concerning the notions of coordinately measurable $\mathcal{A}$-splitting and measurable $\mathcal{A}$-admissible topologies. Finally, we present these notions for the set of all Baire measurable maps.

1. Preliminaries

Let X be a set. A *σ-algebra* on X is a collection of subsets of X, containing $\emptyset$ and is closed under complements and countable unions (so also under countable intersections). If $\mathcal{E}$ is a collection of subsets of X, then there is the smallest σ-algebra containing $\mathcal{E}$, called the *σ-algebra generated by* $\mathcal{E}$ and denoted by $\sigma(\mathcal{E})$. Also, $\mathcal{E}$ is called a set of *generators* for $\sigma(\mathcal{E})$.

In this paper we shall use the following notions from [2], [3], [8], [9], and [11].

Let X be a set and $\mathcal{S}$ a σ-algebra on X. The pair $(X, \mathcal{S})$ is called *measurable* (or *Borel*) *space* and the members of $\mathcal{S}$ are called *measurable* (or *Borel*) sets.

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Let $(Y, \mathcal{S}_1)$, $(Z, \mathcal{S}_2)$ be two measurable spaces. A map $f : Y \to Z$ is called *measurable* (or *Borel*) if $f^{-1}(A) \in \mathcal{S}_1$ for every $A \in \mathcal{S}_2$. We observe that it is enough to require this for every $A \in \mathcal{E}$, if $\mathcal{E}$ generates $\mathcal{S}_2$.

Let $(X, \mathcal{S}_1)$ and $(Y, \mathcal{S}_2)$ be two measurable spaces. The *product measurable space* $(X \times Y, \mathcal{S})$ is the one generated by the subsets $A \times B$, where $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$.

Let X, Y, and Z be measurable spaces and $F : X \times Y \to Z$ a measurable map. Then for each $x \in X$ and $y \in Y$ the maps $F_x : Y \to Z$ and $F^y : X \to Z$, defined by $F_x(y) = F(x, y)$ for every $y \in Y$ and by $F^y(x) = F(x, y)$, are measurable.

Let X be a topological space. By τ_X we denote the topology of the space X. The σ-algebra that is generated by the open sets of X is called *measurable* and we denote it by $\sigma(\tau_X)$. We call the pair $(X, \sigma(\tau_X))$ as the *measurable space* of X. We observe that $\sigma(\tau_X)$ contains all open, closed, F_σ, and G_δ sets in X.

Let Y and Z be two topological spaces. A map $f : Y \to Z$ is called *measurable* if it is measurable with respect to the measurable spaces $(Y, \sigma(\tau_Y))$ and $(Z, \sigma(\tau_Z))$. The set of all measurable maps of Y into Z is denoted by $\mathcal{M}(Y, Z)$.

In this paper we try to generalize the results presented in [1], [5], [6], and [7] for measurable and Baire spaces. More precisely, in section 1 the paper preliminaries are given. In sections 2 and 3 we introduce and study the notions of coordinately measurable and measurable $\mathcal{A}$-splitting and $\mathcal{A}$-admissible topologies on the set $\mathcal{M}(Y, Z)$, where $\mathcal{A}$ is an arbitrary family of topological spaces. In section 4 we define and study relations between the topologies on the set $\mathcal{M}(Y, Z)$ and the topologies on the set $\sigma_Z(\tau_Y)$, concerning the notions of coordinately measurable $\mathcal{A}$-splitting and measurable $\mathcal{A}$-admissible topologies. In section 5 we present and research topologies on the set of all Baire measurable maps. In section 6 we examine dual topologies on the set of Baire sets. Finally, in section 7 we present some questions concerning function measurable spaces.

2. Coordinately measurable $\mathcal{A}$-splitting and $\mathcal{A}$-admissible topologies on $\mathcal{M}(Y, Z)$

In what follows, if t is a topology on the set $\mathcal{M}(Y, Z)$, then the corresponding topological space is denoted by $\mathcal{M}_t(Y, Z)$.

Notations. Let X be an arbitrary topological space and F a map of $X \times Y$ into Z. We denote by F_x the map of Y into Z, defined by $F_x(y) = F(x, y)$ for every $y \in Y$ and by F^y the map of X into Z, defined by $F^y(x) = F(x, y)$ for every $x \in X$.

Also, by e we denote the map of $\mathcal{M}(Y,Z)\times Y$ into Z, defined by $e(f,y) = f(y)$ for every $(f,y)\in\mathcal{M}(Y,Z)\times Y$. The map e is called the $\mathcal{M}$-*evaluation map.*

Definition 2.1. Let X be an arbitrary topological space. A map $F : X\times Y\to Z$ is called *coordinately measurable* if for every $x\in X$ and $y\in Y$ the maps $F_x : Y\to Z$ and $F^y : X\to Z$ are measurable.

We observe that if the map $F : X\times Y\to Z$ is measurable, then this map is also coordinately measurable.

Notations. (1) Let $F : X\times Y\to Z$ be a coordinately measurable map. Then, by $\widehat{F}$ we denote the map of X into the set $\mathcal{M}(Y,Z)$ defined by $\widehat{F}(x) = F_x$ for every $x\in X$.
(2) Let G be a map of X into $\mathcal{M}(Y,Z)$. Then, by $\widetilde{G}$ we denote the map of $X\times Y$ into Z defined by $\widetilde{G}(x,y) = G(x)(y)$ for every $(x,y)\in X\times Y$.

Definition 2.2. Let $\mathcal{A}$ be a family of topological spaces.
(1) A topology t on $\mathcal{M}(Y,Z)$ is called *coordinately measurable $\mathcal{A}$-splitting* if for every space $X\in\mathcal{A}$ the following implication holds: if the map $F : X\times Y\to Z$ is coordinately measurable, then the map $\widehat{F} : X\to\mathcal{M}_t(Y,Z)$ is measurable.
(2) A topology t on $\mathcal{M}(Y,Z)$ is called *coordinately measurable $\mathcal{A}$-admissible* if for every space $X\in\mathcal{A}$ the following implication holds: if the map $G : X\to\mathcal{M}_t(Y,Z)$ is measurable, then the map $\widetilde{G} : X\times Y\to Z$ is coordinately measurable.

Remark. If $\mathcal{A}$ is the family of all topological spaces, then we use the notions of coordinately measurable splitting and coordinately measurable admissible topologies instead of the notions of coordinately measurable $\mathcal{A}$-splitting and coordinately measurable $\mathcal{A}$-admissible topologies, respectively.

Proposition 2.3. *Let t be a topology on $\mathcal{M}(Y,Z)$ such that $\mathcal{M}_t(Y,Z)\in\mathcal{A}$. Then, the topology t on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-admissible if and only if the $\mathcal{M}$-evaluation map $e : \mathcal{M}_t(Y,Z)\times Y\to Z$ is coordinately measurable.*

Proof. Let t be a coordinately measurable $\mathcal{A}$-admissible topology on $\mathcal{M}(Y,Z)$ such that $\mathcal{M}_t(Y,Z)\in\mathcal{A}$. We consider as X the space $\mathcal{M}_t(Y,Z)$ and as G the identity map $id : \mathcal{M}_t(Y,Z)\to\mathcal{M}_t(Y,Z)$. Since the map G is measurable and the topology t is coordinately measurable $\mathcal{A}$-admissible, we have that the map

$$\widetilde{G} : \mathcal{M}_t(Y,Z)\times Y\to Z$$

is coordinately measurable. We observe that $\widetilde{G} = e$. Thus, the map e is coordinately measurable.

Conversely, let t be a topology on $\mathcal{M}(Y,Z)$ such that the $\mathcal{M}$-evaluation map e is coordinately measurable, $X \in \mathcal{A}$, and $G : X \to \mathcal{M}_t(Y,Z)$ a measurable map. It's sufficient to prove that the map $\widetilde{G} : X \times Y \to Z$ is coordinately measurable. Let $x \in X$. Then, for every $y \in Y$ we have

$$\widetilde{G}(x,y) = G(x)(y) = e(G(x),y).$$

Thus, $\widetilde{G}_x(y) = e_{G(x)}(y)$ for every $y \in Y$. Therefore, $\widetilde{G}_x = e_{G(x)}$.
Since the map e is coordinately measurable, the map $e_{G(x)}$ and, therefore, the map $\widetilde{G}_x$ is measurable.
Now, let $y \in Y$. Then, for every $x \in X$ we have

$$\widetilde{G}^y(x) = \widetilde{G}(x,y) = G(x)(y) = e(G(x),y) = e^y(G(x)) = (e^y \circ G)(x).$$

Since the map e is coordinately measurable, the map e^y is measurable. Therefore, since the map G is measurable, the composition $e^y \circ G$ is measurable. Thus, the map $\widetilde{G}^y$ is measurable and, therefore, the map $\widetilde{G}$ is coordinately measurable. □

Corollary 2.4. *A topology t on $\mathcal{M}(Y,Z)$ is coordinately measurable admissible if and only if the $\mathcal{M}$-evaluation map e is coordinately measurable.*

Proposition 2.5. *Let t be a topology on $\mathcal{M}(Y,Z)$.*
(1) If t is smaller than a coordinately measurable $\mathcal{A}$-splitting topology u on $\mathcal{M}(Y,Z)$, then t is also coordinately measurable $\mathcal{A}$-splitting.
(2) If t is larger than a coordinately measurable $\mathcal{A}$-admissible topology u on $\mathcal{M}(Y,Z)$ and $\mathcal{M}_t(Y,Z) \in \mathcal{A}$, then t is also coordinately measurable $\mathcal{A}$-admissible.

Proof. (1) Let $X \in \mathcal{A}$ and $F : X \times Y \to Z$ be a coordinately measurable map. Since the topology u is coordinately measurable $\mathcal{A}$-splitting, the map $\widehat{F} : X \to \mathcal{M}_u(Y,Z)$ is measurable. Since $t \subseteq u$, we have $\sigma(t) \subseteq \sigma(u)$ and, therefore, the identity map $id : \mathcal{M}_u(Y,Z) \to \mathcal{M}_t(Y,Z)$ is measurable. Thus, the map $id \circ \widehat{F} : X \to \mathcal{M}_t(Y,Z)$ is measurable and, therefore t is coordinately measurable $\mathcal{A}$-splitting.
(2) Since $u \subseteq t$, we have $\sigma(u) \subseteq \sigma(t)$ and, therefore, the identity map $G = id : \mathcal{M}_t(Y,Z) \to \mathcal{M}_u(Y,Z)$ is measurable. Since the topology u is coordinately measurable $\mathcal{A}$-admissible and $\mathcal{M}_t(Y,Z) \in \mathcal{A}$, the map

$$\widetilde{G} : \mathcal{M}_t(Y,Z) \times Y \to Z$$

is coordinately measurable. We observe that $\widetilde{G} = e$.
Thus, by Proposition 2.3 the topology t is coordinately measurable $\mathcal{A}$-admissible. □

Corollary 2.6. *The following statements are true:*
(1) A topology on $\mathcal{M}(Y,Z)$ smaller than a coordinately measurable splitting topology on $\mathcal{M}(Y,Z)$ is also coordinately measurable splitting.
(2) A topology on $\mathcal{M}(Y,Z)$ greater than a coordinately measurable admissible topology on $\mathcal{M}(Y,Z)$ is also coordinately measurable admissible.

Proposition 2.7. *Let t be a coordinately measurable $\mathcal{A}$-admissible topology on $\mathcal{M}(Y,Z)$ such that $\mathcal{M}_t(Y,Z) \in \mathcal{A}$. Then, for every coordinately measurable $\mathcal{A}$-splitting topology u on $\mathcal{M}(Y,Z)$ we have*

$$\sigma(u) \subseteq \sigma(t).$$

Proof. By Proposition 2.3 we have that the $\mathcal{M}$-evaluation map

$$e : \mathcal{M}_t(Y,Z) \times Y \to Z$$

is coordinately measurable. Also, since the topology u is coordinately measurable $\mathcal{A}$-splitting and $\mathcal{M}_t(Y,Z) \in \mathcal{A}$, we have that the map

$$\widehat{e} : \mathcal{M}_t(Y,Z) \to \mathcal{M}_u(Y,Z)$$

is measurable. We observe that $\widehat{e} = id$. Thus, $\sigma(u) \subseteq \sigma(t)$. □

Corollary 2.8. *Let t be a coordinately measurable admissible topology on $\mathcal{M}(Y,Z)$. Then, for every coordinately measurable splitting topology u on $\mathcal{M}(Y,Z)$ we have $\sigma(u) \subseteq \sigma(t)$.*

Remark. Let $\mathcal{A}$ be a family of topological spaces. It is clear that in the set $\mathcal{M}(Y,Z)$ there does not generally exist the greatest coordinately measurable $\mathcal{A}$-splitting topology. This fact gives a different result from the classical theory of function topological spaces (see [1] and [6]).

Easily we can prove the following two propositions.

Proposition 2.9. *The antidiscrete topology t_{tr} on $\mathcal{M}(Y,Z)$ is the smallest coordinately measurable $\mathcal{A}$-splitting topology.*

Proposition 2.10. *The discrete topology t_d on $\mathcal{M}(Y,Z)$ is the greatest coordinately measurable $\mathcal{A}$-admissible topology.*

Definition 2.11. Let $y \in Y$. The t_y topology on $\mathcal{M}(Y,Z)$ is the one having all sets

$$(\{y\}, U) = \{f \in \mathcal{M}(Y,Z) : f(y) \in U\}$$

as basis, where U is an open subset of Z. The topology t_y is called *y-topology.*

Proposition 2.12. *The y-topology t_y on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-splitting topology.*

Proof. Let $X \in \mathcal{A}$ and $F : X \times Y \to Z$ be a coordinately measurable map. It's sufficient to prove that the map $\widehat{F} : X \to \mathcal{M}_{t_y}(Y, Z)$ is measurable. Let $O \in t_y$. Then, there exist

$$U_1^i, \ldots, U_{k(i)}^i \text{ open subsets of } Z,\ i \in I$$

such that

$$O = \cup\{(\{y\}, U_1^i) \cap \ldots \cap (\{y\}, U_{k(i)}^i) : i \in I\}.$$

We have

$$\begin{aligned} \widehat{F}^{-1}(O) &= \widehat{F}^{-1}(\cup\{(\{y\}, U_1^i) \cap \ldots \cap (\{y\}, U_{k(i)}^i) : i \in I\}) \\ &= \widehat{F}^{-1}((\{y\}, \cup\{U_1^i \cap \ldots \cap U_{k(i)}^i : i \in I\})) \\ &= \{x \in X : \widehat{F}(x)(y) = F^y(x) \in \cup\{U_1^i \cap \ldots \cap U_{k(i)}^i : i \in I\}\} \\ &= (F^y)^{-1}(\cup\{U_1^i \cap \ldots \cap U_{k(i)}^i : i \in I\}). \end{aligned}$$

Since F^y is measurable and $\cup\{U_1^i \cap \cdots \cap U_{k(i)}^i : i \in I\}$ is an open subset of Z,

$$(F^y)^{-1}(\cup\{U_1^i \cap \ldots \cap U_{k(i)}^i : i \in I\}) \in \sigma(\tau_X).$$

Thus, the map $\widehat{F}$ is measurable. □

Definition 2.13. The t_p topology on $\mathcal{M}(Y, Z)$ is the one having all sets

$$(\{y\}, U) = \{f \in \mathcal{M}(Y, Z) : f(y) \in U\}$$

as subbasis, where $y \in Y$ and U an open subset of Z. The topology t_p is called *point-open topology.*

Proposition 2.14. *The point-open topology t_p on $\mathcal{M}(Y, Z)$ is coordinately measurable $\mathcal{A}$-admissible topology.*

Proof. Let $X \in \mathcal{A}$ and $G : X \to \mathcal{M}_{t_p}(Y, Z)$ be a measurable map. It's sufficient to prove that the map $\widetilde{G} : X \times Y \to Z$ is coordinately measurable.
Let $x \in X$. We consider the map $\widetilde{G}_x : Y \to Z$. Then, we have

$$\widetilde{G}_x(y) = \widetilde{G}(x, y) = G(x)(y),$$

for every $y \in Y$. Since $G(x) \in \mathcal{M}_{t_p}(Y, Z)$, the map $\widetilde{G}_x = G(x)$ is measurable.
Now, let $y \in Y$. Consider the map $\widetilde{G}^y : X \to Z$. Then,

$$\widetilde{G}^y(x) = \widetilde{G}(x, y),$$

for every $x \in X$. We prove that the map $\widetilde{G}^y$ is measurable. Let W be an open subset of Z. Then,

$$\begin{aligned} (\widetilde{G}^y)^{-1}(W) &= \{x \in X : \widetilde{G}^y(x) = \widetilde{G}(x, y) = G(x)(y) \in W\} \\ &= G^{-1}((\{y\}, W)). \end{aligned}$$

Since G is measurable, $G^{-1}((\{y\}, W)) \in \sigma(\tau_X)$ and, therefore, $(\widetilde{G}^y)^{-1}(W) \in \sigma(\tau_X)$. Thus, the map $\widetilde{G}^y$ is measurable. □

Proposition 2.15. *If the subbasis mentioned in Definition 2.13 is countable, then the point-open topology t_p on $\mathcal{M}(Y, Z)$ is coordinately measurable $\mathcal{A}$-splitting topology.*

Proof. Let $X \in \mathcal{A}$ and $F : X \times Y \to Z$ be a coordinately measurable map. We prove that the map $\widehat{F} : X \to \mathcal{M}_{t_p}(Y, Z)$ is measurable. Let

$$\bigcup_{i \in I} (\bigcap_{j \in J_i} (\{y_j\}, W_j)) \in t_p,$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, $y_j \in Y$ and $W_j \in \tau_Z$ for every $j \in J_i$. Then,

$$\widehat{F}^{-1}(\bigcup_{i \in I} (\bigcap_{j \in J_i} (\{y_j\}, W_j))) = \bigcup_{i \in I} (\bigcap_{j \in J_i} \widehat{F}^{-1}(\{y_j\}, W_j))$$

$$= \bigcup_{i \in I} (\bigcap_{j \in J_i} \{x \in X : \widehat{F}(x)(y_j) = F(x, y_j) = F^{y_j}(x) \in W_j\})$$

$$= \bigcup_{i \in I} (\bigcap_{j \in J_i} (F^{y_j})^{-1}(W_j)).$$

Since F^{y_i} is measurable, $(F^{y_j})^{-1}(W_j) \in \sigma(\tau_X)$ and, therefore,

$$\bigcap_{j \in J_i} (F^{y_j})^{-1}(W_j), \; \bigcup_{i \in I} (\bigcap_{j \in J_i} (F^{y_j})^{-1}(W_j)) \in \sigma(\tau_X).$$

Thus, the map $\widehat{F}$ is measurable. □

Definition 2.16. The t_{pm} topology on $\mathcal{M}(Y, Z)$ is the one having all sets

$$(\{y\}, B) = \{f \in \mathcal{M}(Y, Z) : f(y) \in B\}$$

as subbasis, where $y \in Y$ and $B \in \sigma(\tau_Z)$. The topology t_{pm} is called *point-measurable topology.*

We observe that $t_p \subseteq t_{pm}$.

Proposition 2.17. *The point-measurable topology t_{pm} on $\mathcal{M}(Y, Z)$ is coordinately measurable $\mathcal{A}$-admissible topology.*

Proof. The proof of this follows by the fact that $t_p \subseteq t_{pm}$ and by Propositions 2.5 (2) and 2.14. □

The proof of the following proposition is similar to the proof of Proposition 2.15.

Proposition 2.18. *If the subbasis mentioned in Definition 2.16 is countable, then the point-measurable topology t_{pm} on $\mathcal{M}(Y, Z)$ is coordinately measurable $\mathcal{A}$-splitting topology.*

Definition 2.19. The t_{pG_δ} topology on $\mathcal{M}(Y,Z)$ is that having as subbasis all sets

$$(\{y\},A)=\{f\in\mathcal{M}(Y,Z):f(y)\in A\},$$

where $y\in Y$ and A is a G_δ-set of Z. The topology t_{pG_δ} is called *point-G_δ topology.*

We observe that $t_p\subseteq t_{pG_\delta}\subseteq t_{pm}$.

Proposition 2.20. *The point-G_δ topology t_{pG_δ} on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-admissible topology.*

Proof. The proof of this follows by the fact $t_p\subseteq t_{pG_\delta}$ and by Propositions 2.5 (2) and 2.14. □

Proposition 2.21. *If the subbasis mentioned in Definition 2.19 is countable, then the point-G_δ topology t_{pG_δ} on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-splitting topology.*

Proof. The proof of this proposition is similar to the proof of Proposition 2.15. □

3. Measurable $\mathcal{A}$-splitting and $\mathcal{A}$-admissible topologies on $\mathcal{M}(Y,Z)$

The proofs in the present section are similar to those of section 2 and so are omitted. We show only the proof of Proposition 3.10.

Definition 3.1. Let $\mathcal{A}$ be a family of spaces.
(1) A topology t on $\mathcal{M}(Y,Z)$ is called *measurable $\mathcal{A}$-splitting* if for every space $X\in\mathcal{A}$ the following implication holds: if the map $F:X\times Y\to Z$ is measurable, then the map $\widehat{F}:X\to\mathcal{M}_t(Y,Z)$ is measurable.
(2) A topology t on $\mathcal{M}(Y,Z)$ is called *measurable $\mathcal{A}$-admissible* if for every space $X\in\mathcal{A}$ the following implication holds: if the map $G:X\to\mathcal{M}_t(Y,Z)$ is measurable, then the map $\widetilde{G}:X\times Y\to Z$ is measurable.

Remark. If $\mathcal{A}$ is the family of all topological spaces, then we use the notions of measurable splitting and measurable admissible topologies instead of the notions of measurable $\mathcal{A}$-splitting and measurable $\mathcal{A}$-admissible topologies, respectively.

Proposition 3.2. *Let t be a topology on $\mathcal{M}(Y,Z)$.*
(1) If t is a coordinately measurable $\mathcal{A}$-splitting topology, then t is also a measurable $\mathcal{A}$-splitting.
(2) If t is a measurable $\mathcal{A}$-admissible topology, then t is also a coordinately measurable $\mathcal{A}$-admissible topology.

Proposition 3.3. *Let t be a topology on $\mathcal{M}(Y,Z)$ such that $\mathcal{M}_t(Y,Z) \in \mathcal{A}$. Then, the topology t on $\mathcal{M}(Y,Z)$ is measurable $\mathcal{A}$-admissible if and only if the $\mathcal{M}$-evaluation map*

$$e : \mathcal{M}_t(Y,Z) \times Y \to Z$$

is measurable.

Corollary 3.4. *A topology t on $\mathcal{M}(Y,Z)$ is measurable admissible if and only if the $\mathcal{M}$-evaluation map $e : \mathcal{M}_t(Y,Z) \times Y \to Z$ is measurable.*

Proposition 3.5. *Let t be a topology on $\mathcal{M}(Y,Z)$.*
(1) If u is a measurable $\mathcal{A}$-splitting topology on $\mathcal{M}(Y,Z)$ and $t \subseteq u$, then t is measurable $\mathcal{A}$-splitting.
(2) If u is a measurable $\mathcal{A}$-admissible topology on $\mathcal{M}(Y,Z)$, $\mathcal{M}_t(Y,Z) \in \mathcal{A}$, and $u \subseteq t$, then t is measurable $\mathcal{A}$-admissible.

Corollary 3.6. *The following statements are true:*
(1) A topology on $\mathcal{M}(Y,Z)$ smaller than a measurable splitting topology on $\mathcal{M}(Y,Z)$ is also measurable splitting.
(2) A topology on $\mathcal{M}(Y,Z)$ greater than a measurable admissible topology on $\mathcal{M}(Y,Z)$ is also measurable admissible.

Proposition 3.7. *Let t be a measurable $\mathcal{A}$-admissible topology on $\mathcal{M}(Y,Z)$ such that $\mathcal{M}_t(Y,Z) \in \mathcal{A}$. Then, for every measurable $\mathcal{A}$-splitting topology u on $\mathcal{M}(Y,Z)$ we have $\sigma(u) \subseteq \sigma(t)$.*

Corollary 3.8. *Let t be a measurable admissible topology on $\mathcal{M}(Y,Z)$. Then, for every measurable splitting topology u on $\mathcal{M}(Y,Z)$ we have $\sigma(u) \subseteq \sigma(t)$.*

Definition 3.9. The t_{count} topology on $\mathcal{M}(Y,Z)$ is the one having all sets

$$(M,U) = \{f \in \mathcal{M}(Y,Z) : f(M) \subseteq U\}$$

as subbasis, where M is a countable subset of Y and U is an open subset of Z. The topology t_{count} is called *countable-open topology.*

Proposition 3.10. *If the subbasis mentioned in Definition 3.9 is countable, then the countable-open topology t_{count} on $\mathcal{M}(Y,Z)$ is measurable $\mathcal{A}$-splitting topology.*

Proof. Let $X \in \mathcal{A}$ and $F : X \times Y \to Z$ be a measurable map. It's sufficient to prove that the map $\widehat{F} : X \to \mathcal{M}_{t_{count}}(Y,Z)$ is measurable. Indeed, let

$$\bigcup_{i\in I}\Big(\bigcap_{j\in J_i}(M_j, W_j)\Big) \in t_{count},$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, M_j is a countable subset of Y and W_j an open subset of Z. Then,

$$\begin{aligned}
\widehat{F}^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(M_j, W_j))) &= \bigcup_{i\in I}(\bigcap_{j\in J_i}\widehat{F}^{-1}(M_j, W_j)) \\
&= \bigcup_{i\in I}(\bigcap_{j\in J_i}\widehat{F}^{-1}(\bigcup_{y\in M_j}\{y\}, W_j)) \\
&= \bigcup_{i\in I}(\bigcap_{j\in J_i}\widehat{F}^{-1}(\bigcap_{y\in M_j}(\{y\}, W_j))) \\
&= \bigcup_{i\in I}(\bigcap_{j\in J_i}(\bigcap_{y\in M_j}(\widehat{F}^{-1}(\{y\}, W_j)))).
\end{aligned}$$

Also, for every $y \in M_j$ we have

$$\begin{aligned}
\widehat{F}^{-1}((\{y\}, W_j)) &= \{x \in X : \widehat{F}(x)(y) = F(x, y) = F^y(x) \in W_j\} \\
&= (F^y)^{-1}(W_j).
\end{aligned}$$

Since the map F is measurable, this map is also coordinately measurable. So, the map F^y is measurable. This means that

$$(F^y)^{-1}(W_j) \in \sigma(\tau_X), \text{ for every } y \in M_j.$$

By the above we have

$$\widehat{F}^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(M_j, W_j))) = \bigcup_{i\in I}(\bigcap_{j\in J_i}(\bigcap_{y\in M_j}(F^y)^{-1}(W_j))) \in \sigma(\tau_X).$$

Thus, the map $\widehat{F}$ is measurable and, therefore, the topology t_{count} is measurable $\mathcal{A}$-splitting. □

4. Dual topologies on the set of measurable sets

In what follows, for two fixed topological spaces Y and Z we consider the set $\mathcal{M}(Y, Z)$ of all measurable maps from Y into Z and the set

$$\sigma_Z(\tau_Y) \equiv \{f^{-1}(B) : f \in \mathcal{M}(Y, Z),\ B \in \sigma(\tau_Z)\}.$$

We define and study some relations between the topologies on the set $\mathcal{M}(Y, Z)$ and the topologies on the set $\sigma_Z(\tau_Y)$ concerning the notions of coordinately measurable $\mathcal{A}$-splitting and coordinately measurable $\mathcal{A}$-admissible topologies.

Notations. Let $\mathbb{H} \subseteq \sigma_Z(\tau_Y)$, $\mathcal{H} \subseteq \mathcal{M}(Y, Z)$, and $B \in \sigma(\tau_Z)$. We set

$$(\mathbb{H}, B) = \{f \in \mathcal{M}(Y, Z) : f^{-1}(B) \in \mathbb{H}\}$$

and

$$(\mathcal{H}, B) = \{f^{-1}(B) : f \in \mathcal{H}\}.$$

Definition 4.1. Let τ be a topology on $\sigma_Z(\tau_Y)$. The $t(\tau)$ topology on $\mathcal{M}(Y,Z)$ is the one having all the sets $(\mathbb{H}, B)$ as subbasis, where $\mathbb{H} \in \tau$ and $B \in \sigma(\tau_Z)$. The $t(\tau)$ topology is called *dual to* τ.

In what follows by $s(\tau)$ we denote the family

$$\{(\mathbb{H}, B) : \mathbb{H} \in \tau,\ B \in \sigma(\tau_Z)\}.$$

Definition 4.2. Let t be a topology on $\mathcal{M}(Y,Z)$. The $\tau(t)$ topology on $\sigma_Z(\tau_Y)$, is the one having all the sets $(\mathcal{H}, B)$ as subbasis, where $\mathcal{H} \in t$ and $B \in \sigma(\tau_Z)$. The $\tau(t)$ topology is called *dual to* t.

In what follows by $r(t)$ we denote the family

$$\{(\mathcal{H}, B) : \mathcal{H} \in t,\ B \in \sigma(\tau_Z)\}.$$

Notations. (1) Suppose that $F : X \times Y \to Z$ is a coordinately measurable map. By $\overline{F}$ we denote the map of $X \times \sigma(\tau_Z)$ into the set $\sigma_Z(\tau_Y)$, for which $\overline{F}(x, B) = F_x^{-1}(B)$ for every $x \in X$ and $B \in \sigma(\tau_Z)$.
(2) Let G be a map of X into $\mathcal{M}(Y,Z)$. By $\overline{G}$ we denote the map of $X \times \sigma(\tau_Z)$ into $\sigma_Z(\tau_Y)$, for which $\overline{G}(x, B) = (G(x))^{-1}(B)$ for every $x \in X$ and $B \in \sigma(\tau_Z)$.

Definition 4.3. Let τ be a topology on $\sigma_Z(\tau_Y)$. We say that a map M of $X \times \sigma(\tau_Z)$ into $\sigma_Z(\tau_Y)$ is *measurable with respect to the first variable* if for every fixed element B of $\sigma(\tau_Z)$, the map $M_B : X \to (\sigma_Z(\tau_Y), \tau)$ is measurable, where $M_B(x) = M(x, B)$ for every $x \in X$.

Definition 4.4. Let $\mathcal{A}$ be a family of topological spaces.
(1) A topology τ on $\sigma_Z(\tau_Y)$ is called *coordinately measurable $\mathcal{A}$-splitting* if for every space $X \in \mathcal{A}$ the following implication holds: if the map $F : X \times Y \to Z$ is coordinately measurable, then the map

$$\overline{F} : X \times \sigma(\tau_Z) \to (\sigma_Z(\tau_Y), \tau)$$

is measurable with respect to the first variable.
(2) A topology τ on $\sigma_Z(\tau_Y)$ is called *coordinately measurable $\mathcal{A}$-admissible* if for every space $X \in \mathcal{A}$ and for every map

$$G : X \to \mathcal{M}(Y,Z)$$

the following implication holds: if the map

$$\overline{G} : X \times \sigma(\tau_Z) \to (\sigma_Z(\tau_Y), \tau)$$

is measurable with respect to the first variable, then the map

$$\widetilde{G} : X \times Y \to Z$$

is coordinately measurable.

Remark. If $\mathcal{A}$ is the family of all topological spaces, then we use the notions of coordinately measurable splitting and coordinately measurable admissible instead of the notions of coordinately measurable $\mathcal{A}$-splitting and coordinately measurable $\mathcal{A}$-admissible, respectively.

Proposition 4.5. *Let τ be a topology on $\sigma_Z(\tau_Y)$. If the topology $t(\tau)$ on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-splitting, then the topology τ is coordinately measurable $\mathcal{A}$-splitting.*

Proof. Suppose that the topology $t(\tau)$ on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-splitting, $X \in \mathcal{A}$, and $F : X \times Y \to Z$ is a coordinately measurable map. It's sufficient to prove that the map

$$\overline{F} : X \times \sigma(\tau_Z) \to (\sigma_Z(\tau_Y), \tau)$$

is measurable with respect to the first variable. Indeed, let $B \in \sigma(\tau_Z)$ and $\mathbb{H} \in \tau$. We need to prove that $\overline{F}_B{}^{-1}(\mathbb{H}) \in \sigma(\tau_X)$. We have

$$\begin{aligned} \overline{F}_B{}^{-1}(\mathbb{H}) &= \{x \in X : \overline{F}_B(x) = F_x{}^{-1}(B) = \widehat{F}(x)^{-1}(B) \in \mathbb{H}\} \\ &= \widehat{F}^{-1}((\mathbb{H}, B)). \end{aligned}$$

Since $F : X \times Y \to Z$ is coordinately measurable and $t(\tau)$ is coordinately measurable $\mathcal{A}$-splitting, the map

$$\widehat{F} : X \to \mathcal{M}_{t(\tau)}(Y,Z)$$

is measurable. Thus, $\overline{F}_B{}^{-1}(\mathbb{H}) \in \sigma(\tau_X)$. □

Proposition 4.6. *Let τ be a topology on $\sigma_Z(\tau_Y)$. If $|s(\tau)| \leq \aleph_0$ and the topology τ is coordinately measurable $\mathcal{A}$-splitting, then the topology $t(\tau)$ on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-splitting.*

Proof. Suppose that the topology τ on $\sigma_Z(\tau_Y)$ is coordinately measurable $\mathcal{A}$-splitting, $X \in \mathcal{A}$, and $F : X \times Y \to Z$ is a coordinately measurable map. It's sufficient to prove that the map $\widehat{F} : X \to \mathcal{M}_{t(\tau)}(Y,Z)$ is measurable. Let

$$\bigcup_{i \in I} (\bigcap_{j \in J_i} (\mathbb{H}_j, B_j)) \in t(\tau),$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, $\mathbb{H}_j \in \tau$, and $B_j \in \sigma(\tau_Z)$ for every $j \in J_i$ and $i \in I$. Then, we have

$$\widehat{F}^{-1}(\bigcup_{i \in I}(\bigcap_{j \in J_i}(\mathbb{H}_j, B_j))) = \bigcup_{i \in I}(\bigcap_{j \in J_i} \widehat{F}^{-1}(\mathbb{H}_j, B_j)) = \bigcup_{i \in I}(\bigcap_{j \in J_i} \overline{F}_{B_j}^{-1}(\mathbb{H})).$$

Since the map $\overline{F}_{B_j}$ is measurable and $|I| \leq \aleph_0$, we have that

$$\widehat{F}^{-1}(\bigcup_{i \in I}(\bigcap_{j \in J_i}(\mathbb{H}_j, B_j))) \in \sigma(\tau_X).$$

Thus, the map $\widehat{F}$ is measurable. □

Corollary 4.7. *Let τ be a topology on $\sigma_Z(\tau_Y)$. Then, the following propositions are true:*
(1) If the topology $t(\tau)$ on $\mathcal{M}(Y,Z)$ is coordinately measurable splitting, then the topology τ is coordinately measurable splitting.
(2) If $|s(\tau)| \leq \aleph_0$ and the topology τ is coordinately measurable splitting, then the topology $t(\tau)$ on $\mathcal{M}(Y,Z)$ is coordinately measurable splitting.

Proposition 4.8. *Let t be a topology on $\mathcal{M}(Y,Z)$. If the topology $\tau(t)$ on $\sigma_Z(\tau_Y)$ is coordinately measurable $\mathcal{A}$-splitting, then the topology t on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-splitting.*

Proof. The proof is similar to the proof of Proposition 4.5. □

Proposition 4.9. *Let t be a topology on $\mathcal{M}(Y,Z)$. If $|r(t)| \leq \aleph_0$ and the topology t is coordinately measurable $\mathcal{A}$-splitting, then the topology $\tau(t)$ is coordinately measurable $\mathcal{A}$-splitting.*

Proof. The proof is similar to the proof of Proposition 4.6. □

Corollary 4.10. *Let t be a topology on $\mathcal{M}(Y,Z)$. Then, the following propositions are true:*
(1) If the topology $\tau(t)$ on $\sigma_Z(\tau_Y)$ is coordinately measurable splitting, then the topology t on $\mathcal{M}(Y,Z)$ is coordinately measurable splitting.
(2) If $|r(t)| \leq \aleph_0$ and the topology t is coordinately measurable splitting, then the topology $\tau(t)$ is coordinately measurable splitting.

Proposition 4.11. *Let τ be a topology on $\sigma_Z(\tau_Y)$. If τ is a coordinately measurable $\mathcal{A}$-admissible, then the topology $t(\tau)$ on $\mathcal{M}(Y,Z)$ is coordinately measurable $\mathcal{A}$-admissible.*

Proof. Suppose that the topology τ on $\sigma_Z(\tau_Y)$ is coordinately measurable $\mathcal{A}$-admissible, $X \in \mathcal{A}$, and $G : X \to \mathcal{M}_{t(\tau)}(Y,Z)$ is a measurable map. It's sufficient to prove that the map $\widetilde{G} : X \times Y \to Z$ is coordinately measurable. We need to prove that the map

$$\overline{G} : X \times \sigma(\tau_Z) \to (\sigma_Z(\tau_Y), \tau)$$

is measurable with respect to the first variable. Indeed, let $B \in \sigma(\tau_Z)$ and $\mathbb{H} \in \tau$. Then, we have

$$\begin{aligned} \overline{G}_B^{-1}(\mathbb{H}) &= \{x \in X : \overline{G}_B(x) = G(x)^{-1}(B) \in \mathbb{H}\} \\ &= G^{-1}((\mathbb{H}, B)). \end{aligned}$$

Since G is measurable, $G^{-1}((\mathbb{H}, B)) \in \sigma(\tau_X)$ and, therefore, $\overline{G}_B^{-1}(\mathbb{H}) \in \sigma(\tau_X)$. Thus, the map $\overline{G}$ is measurable with respect to the first variable. □

Proposition 4.12. *Let τ be a topology on $\sigma_Z(\tau_Y)$. If $|s(\tau)| \leq \aleph_0$ and $t(\tau)$ is a coordinately measurable $\mathcal{A}$-admissible topology, then τ is coordinately measurable $\mathcal{A}$-admissible topology.*

Proof. Suppose that the topology $t(\tau)$ on $\mathcal{M}(Y, Z)$ is coordinately measurable $\mathcal{A}$-admissible, $X \in \mathcal{A}$, $G : X \to \mathcal{M}(Y, Z)$, and

$$\overline{G} : X \times \sigma(\tau_Z) \to (\sigma_Z(\tau_Y), \tau)$$

is a measurable map with respect to the first variable. We have to prove that the map $\widetilde{G} : X \times Y \to Z$ is coordinately measurable. It's sufficient to prove that the map

$$G : X \to \mathcal{M}_{t(\tau)}(Y, Z)$$

is measurable. Indeed, let

$$\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}_j, B_j)) \in t(\tau),$$

where $|I| \leq \aleph_0$, $|J_i| < \aleph_0$, $\mathbb{H}_j \in \tau$, and $B_j \in \sigma(\tau_Z)$ for every $j \in J_i$ and $i \in I$. Then, we have

$$G^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}_j, B_j))) = \bigcup_{i\in I}(\bigcap_{j\in J_i} G^{-1}(\mathbb{H}_j, B_j)) = \bigcup_{i\in I}(\bigcap_{j\in J_i} \overline{G}_{B_j}^{-1}(\mathbb{H})).$$

Since the map $\overline{G}_{B_j}$ is measurable and $|I| \leq \aleph_0$, we have that

$$G^{-1}(\bigcup_{i\in I}(\bigcap_{j\in J_i}(\mathbb{H}_j, B_j))) \in \sigma(\tau_X).$$

Thus, the map G is measurable. □

Corollary 4.13. *Let τ be a topology on $\sigma_Z(\tau_Y)$. The following propositions are true:*
(1) If τ is a coordinately measurable admissible, then the topology $t(\tau)$ on $\mathcal{M}(Y, Z)$ is coordinately measurable admissible.
(2) If $|s(\tau)| \leq \aleph_0$ and $t(\tau)$ is a coordinately measurable admissible topology, then τ is coordinately measurable admissible topology.

Proposition 4.14. *Let t be a topology on $\mathcal{M}(Y, Z)$. If t is a coordinately measurable $\mathcal{A}$-admissible, then the topology $\tau(t)$ on $\sigma_Z(\tau_Y)$ is coordinately measurable $\mathcal{A}$-admissible.*

Proof. The proof is similar to the proof of Proposition 4.11. □

Proposition 4.15. *Let t be a topology on $\mathcal{M}(Y, Z)$. If $|r(t)| \leq \aleph_0$ and the topology $\tau(t)$ is a coordinately measurable $\mathcal{A}$-admissible, then the topology t is coordinately measurable $\mathcal{A}$-admissible.*

Proof. The proof is similar to the proof of Proposition 4.12. □

Corollary 4.16. *Let t be a topology on $\sigma_Z(\tau_Y)$. The following propositions are true:*
(1) If t is a coordinately measurable admissible, then the topology $\tau(t)$ on $\sigma_Z(\tau_Y)$ is coordinately measurable admissible.
(2) If $|r(t)| \leq \aleph_0$ and the topology $\tau(t)$ is a coordinately measurable admissible, then the topology t is coordinately measurable admissible.

Definition 4.17. A topology on $\mathcal{M}(Y,Z)$ (respectively, on $\sigma_Z(\tau_Y)$) is said to be *a family-measurable topology* if it is dual to a topology on $\sigma_Z(\tau_Y)$ (respectively, to a topology on $\mathcal{M}(Y,Z)$).

Lemma 4.18. *Let τ be a topology on $\sigma_Z(\tau_Y)$. If the set γ is a subbasis for τ, then the set*

$$s(\gamma) \equiv \{(\mathbb{H}, B) : \mathbb{H} \in \gamma,\ B \in \sigma(\tau_Z)\}$$

is a subbasis for $t(\tau)$.

Proof. Let $\mathbb{H} \in \tau$ and $B \in \sigma(\tau_Z)$. Suppose that $f \in (\mathbb{H}, B)$. Then, there exist finitely many elements $\mathbb{H}_0, \ldots, \mathbb{H}_n$ of γ such that

$$f^{-1}(B) \in \mathbb{H}_0 \cap \ldots \cap \mathbb{H}_n \subseteq \mathbb{H}.$$

Therefore,

$$f \in (\mathbb{H}_0 \cap \ldots \cap \mathbb{H}_n, B) \subseteq (\mathbb{H}, B).$$

Thus, any element of the subbasis $s(\tau)$ of $t(\tau)$ is a union of finite intersections of elements of the set $s(\gamma)$, which means that this set is also a subbasis for $t(\tau)$. □

Lemma 4.19. *Let t be a topology on $\mathcal{M}(Y,Z)$. If the set s is a subbasis for t, then the set*

$$r(s) \equiv \{(\mathcal{H}, B) : \mathcal{H} \in s,\ B \in \sigma(\tau_Z)\}$$

is a subbasis for $\tau(t)$.

Proof. Let $\mathcal{H} \in t$ and $B \in \sigma(\tau_Z)$. Let $A \in (\mathcal{H}, B)$. Then, there exists an element $f \in \mathcal{H}$ such that $A = f^{-1}(B)$. There exist finitely many elements $\mathcal{H}_0, \ldots, \mathcal{H}_n$ of s such that

$$f \in \mathcal{H}_0 \cap \ldots \cap \mathcal{H}_n \subseteq \mathcal{H}.$$

Therefore,

$$A \in (\mathcal{H}_0 \cap \ldots \cap \mathcal{H}_n, B) \subseteq (\mathcal{H}, B).$$

Thus, any element of the subbasis $r(t)$ of $\tau(t)$ is a union of finite intersections of elements of the set $r(s)$, which means that this set is also a subbasis for $\tau(t)$. □

Examples 4.20. Below we give some family-measurable topologies on $\mathcal{M}(Y,Z)$.

(1) For every $y \in Y$ we set

$$\sigma_Z(y) = \{A \in \sigma_Z(\tau_Y) : y \in A\}.$$

The τ topology on $\sigma_Z(\tau_Y)$ is the one having all sets $\sigma_Z(y)$ as subassis, where $y \in Y$.

By Lemma 4.18 the set

$$\{(\sigma_Z(y), B) : y \in Y,\ B \in \sigma(\tau_Z)\}$$

is a subbasis for $t(\tau)$. It is easy to see that

$$(\sigma_Z(y), B) = (\{y\}, B),$$

for every $y \in Y$ and $B \in \sigma(\tau_Z)$. Therefore, $t(\tau) = t_{pm}$, which means that the point-measurable topology is family-measurable.

(2) The t_{cm} topology on $\mathcal{M}(Y,Z)$ is the one having all sets

$$(M, B) = \{f \in \mathcal{M}(Y,Z) : f(M) \subseteq B\}$$

as subbasis, where M is a countable subset of Y and $B \in \sigma(\tau_Z)$. The topology t_{cm} is called *countable-measurable topology.*

For every countable subset M of Y we set

$$\sigma_Z(M) = \{A \in \sigma_Z(\tau_Y) : M \subseteq A\}.$$

The τ topology on $\sigma_Z(\tau_Y)$ is the one having all sets $\sigma_Z(M)$ as subbasis, where M is a countable subset of Y.

By Lemma 4.18 the set

$$\{(\sigma_Z(M), B) : M \text{ countable subset of } Y,\ B \in \sigma(\tau_Z)\}$$

is a subbasis for $t(\tau)$. It is easy to see that

$$(\sigma_Z(M), B) = (M, B)$$

for every countable subset M of Y and $B \in \sigma(\tau_Z)$. Therefore, $t(\tau) = t_{cm}$, which means that the countable-measurable topology is family-measurable.

Lemma 4.21. *Let $\mathbb{H}$ be a subset of $\sigma_Z(\tau_Y)$. Then,*

$$\mathbb{H} = \bigcup \{\big((\mathbb{H}, B), B\big) : B \in \sigma(\tau_Z)\}.$$

Proof. For every $B \in \sigma(\tau_Z)$ we have

$$\big((\mathbb{H}, B), B\big) = \{f^{-1}(B) : f \in (\mathbb{H}, B)\} = \{f^{-1}(B) : f^{-1}(B) \in \mathbb{H}\} \subseteq \mathbb{H}.$$

Therefore,

$$\bigcup \{\big((\mathbb{H}, B), B\big) : B \in \sigma(\tau_Z)\} \subseteq \mathbb{H}.$$

Now, let $A \in \mathbb{H}$. Then, there exist $f \in \mathcal{M}(Y,Z)$ and $B \in \sigma(\tau_Z)$ such that $A = f^{-1}(B)$. Since $f^{-1}(B) \in \big((\mathbb{H},B),B\big)$, we have

$$\mathbb{H} \subseteq \bigcup \big\{\big((\mathbb{H},B),B\big) : B \in \sigma(\tau_Z)\big\}.$$

Thus,

$$\mathbb{H} = \bigcup \big\{\big((\mathbb{H},B),B\big) : B \in \sigma(\tau_Z)\big\}.$$

□

Lemma 4.22. *Let τ_1 and τ_2 be two topologies on the set $\sigma_Z(\tau_Y)$ such that $\tau_1 \subseteq \tau_2$. Then, $t(\tau_1) \subseteq t(\tau_2)$.*

Proof. Follows easily from Definition 4.1. □

Lemma 4.23. *Let t_1 and t_2 be two topologies on the set $\mathcal{M}(Y,Z)$ such that $t_1 \subseteq t_2$. Then, $\tau(t_1) \subseteq \tau(t_2)$.*

Proof. Follows easily from Definition 4.2. □

Proposition 4.24. *Let τ be a topology on the set $\sigma_Z(\tau_Y)$. Then, we have*

$$\tau \subseteq \tau\big(t(\tau)\big) \subseteq \tau\Big(t\big(\tau\big(t(\tau)\big)\big)\Big) \subseteq \dots$$

and

$$t(\tau) \subseteq t\big(\tau\big(t(\tau)\big)\big) \subseteq t\Big(\tau\Big(t\big(\tau\big(t(\tau)\big)\big)\Big)\Big) \subseteq \dots$$

Proof. The including $\tau \subseteq \tau\big(t(\tau)\big)$ follows by Lemma 4.21. All other including follow by Lemmas 4.22 and 4.23. □

Definition 4.25. Let τ be a topology on $\sigma_Z(\tau_Y)$ and t a topology on $\mathcal{M}(Y,Z)$. The pair (τ,t) is called *a pair of mutually dual topologies* if $\tau = \tau(t)$ and $t = t(\tau)$.

Lemma 4.26. *Let t be a topology on $\mathcal{M}(Y,Z)$. Suppose that $\{\tau_i : i \in I\}$ is a set of topologies on $\sigma_Z(\tau_Y)$ such that $t(\tau_i) = t$ for every $i \in I$. Then, $t(\tau) = t$, where $\tau = \bigvee\{\tau_i : i \in I\}$.*

Proof. The set $\gamma = \bigcup\{\tau_i : i \in I\}$ is a subbasis for τ. By Lemma 4.18 the set $s(\gamma)$ is a subbasis for $t(\tau)$. On the other hand, $s(\gamma)$ is a subbasis for the topology t. Thus, $t(\tau) = t$. □

Lemma 4.27. *Let τ be a topology on $\sigma_Z(\tau_Y)$. Suppose that $\{t_i : i \in I\}$ is a set of topologies on $\mathcal{M}(Y,Z)$ such that $\tau(t_i) = \tau$ for every $i \in I$. Then, $\tau(t) = \tau$, where $t = \bigvee\{t_i : i \in I\}$.*

Proof. The proof is similar to the proof of Lemma 4.26. □

Proposition 4.28. *Let t be a family-measurable topology on $\mathcal{M}(Y,Z)$. Then, on the set $\sigma_Z(\tau_Y)$ there exists a topology τ with the properties:*
(a) $t(\tau) = t$ and
(b) if τ' is a topology on $\sigma_Z(\tau_Y)$ such that $t(\tau') = t$, then $\tau' \subseteq \tau$.

Proof. Follows by Lemma 4.26. □

Proposition 4.29. *Let τ be a family-measurable topology on $\sigma_Z(\tau_Y)$. Then, on the set $\mathcal{M}(Y,Z)$ there exists a topology t with the properties:*
(a) $\tau(t)=\tau$ and
(b) if t' is a topology on $\mathcal{M}(Y,Z)$ such that $\tau(t')=\tau$, then $t'\subseteq t$.

Proof. Follows by Lemma 4.27. □

Proposition 4.30. *Let τ_0 be a topology on $\sigma_Z(\tau_Y)$. Then, there exists a pair (τ,t) of mutually dual topologies with the properties:*
(a) $\tau_0\subseteq\tau$ and
(b) if (τ',t') is a pair of mutually dual topologies such that $\tau_0\subseteq\tau'$, then $\tau\subseteq\tau'$ and $t\subseteq t'$.

Proof. We set

$$\tau_1=\tau\big(t(\tau_0)\big),\ \tau_2=\tau\Big(t\big(\tau\big(t(\tau_0)\big)\big)\Big),\ldots$$

and

$$\tau=\bigvee\big\{\tau_i : i\in\{0,1,2,\ldots\}\big\}.$$

Also, we set

$$t_1=t(\tau_0),\ t_2=t\big(\tau\big(t(\tau_0)\big)\big),\ldots$$

and

$$t=\bigvee\big\{t_i : i\in\{1,2,\ldots\}\big\}.$$

Obviously, $\tau_0\subseteq\tau$. The set

$$\gamma=\bigcup\big\{\tau_i : i\in\{0,1,2,\ldots\}\big\}$$

is a subbasis for τ and, therefore, the set

$$s(\gamma)=\bigcup\big\{s(\tau_i) : i\in\{0,1,2,\ldots\}\big\}$$

is a subbasis for $t(\tau)$. On the other hand, $s(\tau_i)$ is a subbasis for $t(\tau_i)=t_{i+1}$, which means that

$$\bigcup\big\{s(\tau_i) : i\in\{0,1,2,\ldots\}\big\}$$

is a subbasis for t. Thus, $t(\tau)=t$. Similarly, we can prove that $\tau(t)=\tau$. Thus, (τ,t) is a pair of mutually dual topologies.
Now, let (τ',t') be a pair of mutually dual topologies such that $\tau_0\subseteq\tau'$. Then,

$$t(\tau_0)\subseteq t(\tau')=t',\ \tau\big(t(\tau_0)\big)\subseteq\tau(t')=\tau',\ t\Big(\tau\big(t(\tau_0)\big)\Big)\subseteq t(\tau')=t',\ldots$$

Therefore, $\tau_i\subseteq\tau'$ for every $i\in\{0,1,2,\ldots\}$ and $t_i\subseteq t'$ for every $i\in\{1,2,\ldots\}$. Thus, $\tau\subseteq\tau'$ and $t\subseteq t'$. □

Proposition 4.31. *Let t_0 be a topology on $\mathcal{M}(Y,Z)$. Then, there exists a pair (τ, t) of mutually dual topologies with the properties:*
(a) $t_0 \subseteq t$ and
(b) if (τ', t') is a pair of mutually dual topologies such that $t_0 \subseteq t'$, then $\tau \subseteq \tau'$ and $t \subseteq t'$.

Proof. The proof is similar to the proof of Proposition 4.30. □

5. Topologies on the set of all Baire measurable maps

Notations. Let X be a completely regular Hausdorff space and A a subset of X. Then, A is a *zero set* in X if there exists a continuous function $f : X \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers with the usual topology, such that $A = f^{-1}(\{0\})$. A *cozero set* in X is a subset whose complement is a zero set.
Let X be a completely regular Hausdorff space. By $\mathcal{Z}(X)$ we denote the family of all zero sets of X.
The σ-algebra that is generated by the zero sets of X is called *Baire* and we denote it by $\sigma(\mathcal{Z}(X))$. We call the elements of $\sigma(\mathcal{Z}(X))$ *Baire sets* and the pair $(X, \sigma(\mathcal{Z}(X)))$ the *Baire space* of X (see, for example, [4]).

We observe that:
(1) $\sigma(\mathcal{Z}(X))$ contains all zero and cozero sets in X.
(2) The pair $(X, \sigma(\mathcal{Z}(X)))$ is a measurable space.
(3) If X is metrizable, then $\sigma(\mathcal{Z}(X)) = \sigma(\tau_X)$.

Let Y and Z be two completely regular Hausdorff spaces. A map $f : Y \to Z$ is called *Baire measurable* if it is measurable with respect to $(Y, \sigma(\mathcal{Z}(Y))), (Z, \sigma(\mathcal{Z}(Z)))$. We observe that the map f is Baire measurable if $f^{-1}(B) \in \sigma(\mathcal{Z}(Y))$, for every zero set B of the space Z.

The set of all Baire measurable maps of Y into Z is denoted by $\mathcal{B}(Y,Z)$. In what follows, if t is a topology on the set $\mathcal{B}(Y,Z)$, then the corresponding topological space is denoted by $\mathcal{B}_t(Y,Z)$. Also, by e we denote the map of $\mathcal{B}(Y,Z) \times Y$ into Z defined by $e(f,y) = f(y)$ for every $(f,y) \in \mathcal{B}(Y,Z) \times Y$. The map e is called the *$\mathcal{B}$-evaluation map.*

Let X be an arbitrary completely regular Hausdorff space. A map $F : X \times Y \to Z$ is called *coordinately Baire measurable* if for every $x \in X$ and $y \in Y$ the maps $F_x : Y \to Z$ and $F^y : X \to Z$ are Baire measurable.

We observe that if the map $F : X \times Y \to Z$ is Baire measurable, then this map is also coordinately Baire measurable.

Definition 5.1. Let $\mathcal{A}$ be a family of completely regular Hausdorff spaces.
(1) A topology t on $\mathcal{B}(Y,Z)$ is called *coordinately Baire measurable $\mathcal{A}$-splitting* if for every space $X \in \mathcal{A}$ the following implication holds: if the map $F : X \times Y \to Z$ is coordinately Baire measurable, then the map $\widehat{F} : X \to \mathcal{B}_t(Y,Z)$ is Baire measurable.

(2) A topology t on $\mathcal{B}(Y,Z)$ is called *coordinately Baire measurable $\mathcal{A}$-admissible* if for every space $X \in \mathcal{A}$ the following implication holds: if the map $G : X \to \mathcal{B}_t(Y,Z)$ is Baire measurable, then the map $\widetilde{G} : X \times Y \to Z$ is coordinately Baire measurable.

Remarks. (1) If $\mathcal{A}$ is the family of all completely regular Hausdorff spaces, then we use the notions of coordinately Baire measurable splitting and coordinately Baire measurable admissible topologies instead of the notions of coordinately Baire measurable $\mathcal{A}$-splitting and coordinately Baire measurable $\mathcal{A}$-admissible topologies, respectively.

(2) We observe that Propositions 2.3, 2.5, 2.7, 2.9, 2.10 and Corollaries 2.4, 2.6, 2.8 remain true if we replace the notions of section 2 by the corresponding notions of this section.

Definition 5.2. The t_{pB} topology on $\mathcal{B}(Y,Z)$ is the one having all sets

$$(\{y\}, B) = \{f \in \mathcal{B}(Y,Z) : f(y) \in B\}$$

as subbasis, where $y \in Y$ and $B \in \sigma(\mathcal{Z}(Z))$. The topology t_{pB} is called *point-Baire topology.*

As in section 2 we can prove the following two propositions.

Proposition 5.3. *Let $\mathcal{A}$ be a family of completely regular Hausdorff spaces. The point-Baire topology t_{pB} on $\mathcal{B}(Y,Z)$ is coordinately measurable $\mathcal{A}$-admissible topology.*

Proposition 5.4. *If the subbasis mentioned in Definition 5.2 is countable, then the point-Baire topology t_{pB} on $\mathcal{B}(Y,Z)$ is coordinately Baire measurable $\mathcal{A}$-splitting topology.*

6. Dual Topologies on the set of Baire sets

In what follows, for two fixed completely regular Hausdorff topological spaces Y and Z we consider the set $\mathcal{B}(Y,Z)$ of all Baire measurable maps of Y into Z and the set

$$\sigma_Z(\mathcal{Z}(Y)) \equiv \{f^{-1}(B) : f \in \mathcal{B}(Y,Z),\ B \in \sigma(\mathcal{Z}(Z))\}.$$

Notations. Let $\mathbb{H} \subseteq \sigma_Z(\mathcal{Z}(Y))$, $\mathcal{H} \subseteq \mathcal{B}(Y,Z)$, and $B \in \sigma(\mathcal{Z}(Z))$. We set

$$(\mathbb{H}, B) = \{f \in \mathcal{B}(Y,Z) : f^{-1}(B) \in \mathbb{H}\}$$

and

$$(\mathcal{H}, B) = \{f^{-1}(B) : f \in \mathcal{H}\}.$$

Definition 6.1. Let τ be a topology on $\sigma_Z(\mathcal{Z}(Y))$. The $t(\tau)$ topology on $\mathcal{B}(Y,Z)$ is the one having all the sets $(\mathbb{H}, B)$ as subbasis, where $\mathbb{H} \in \tau$ and $B \in \sigma(\mathcal{Z}(Z))$. The $t(\tau)$ topology is called *dual to τ*.

In what follows by $s(\tau)$ we denote the family

$$\{(\mathbb{H}, B) : \mathbb{H} \in \tau,\ B \in \sigma(\mathcal{Z}(Z))\}.$$

Definition 6.2. Let t be a topology on $\mathcal{B}(Y, Z)$. The $\tau(t)$ topology on $\sigma_Z(\mathcal{Z}(Y))$ is the one having all the sets $(\mathcal{H}, B)$ as subbasis, where $\mathcal{H} \in t$ and $B \in \sigma(\mathcal{Z}(Z))$. The $\tau(t)$ topology is called *dual to* t.

In what follows by $r(t)$ we denote the family

$$\{(\mathcal{H}, B) : \mathcal{H} \in t,\ B \in \sigma(\mathcal{Z}(Z))\}.$$

Notations. (1) Suppose that $F : X \times Y \to Z$ is a coordinately Baire measurable map. By $\overline{F}$ we denote the map of $X \times \sigma(\mathcal{Z}(Z))$ into the set $\sigma_Z(\mathcal{Z}(Y))$, for which $\overline{F}(x, B) = F_x^{-1}(B)$ for every $x \in X$ and $B \in \sigma(\mathcal{Z}(Z))$.
(2) Let G be a map of X into $\mathcal{B}(Y, Z)$. By $\overline{G}$ we denote the map of $X \times \sigma(\mathcal{Z}(Z))$ into $\sigma_Z(\mathcal{Z}(Y))$, for which $\overline{G}(x, B) = (G(x))^{-1}(B)$ for every $x \in X$ and $B \in \sigma(\mathcal{Z}(Z))$.

Definition 6.3. Let τ be a topology on $\sigma_Z(\mathcal{Z}(Y))$. We say that a map M of $X \times \sigma(\mathcal{Z}(Z))$ into $\sigma_Z(\mathcal{Z}(Y))$ is *Baire measurable with respect to the first variable* if for every fixed element B of $\sigma(\mathcal{Z}(Z))$, the map $M_B : X \to (\sigma_Z(\mathcal{Z}(Y)), \tau)$ is Baire measurable, where $M_B(x) = M(x, B)$ for every $x \in X$.

Definition 6.4. Let $\mathcal{A}$ be a family of completely regular Hausdorff spaces.
(1) A topology τ on $\sigma_Z(\mathcal{Z}(Y))$ is called *coordinately Baire measurable $\mathcal{A}$-splitting* if for every space $X \in \mathcal{A}$ the following implication holds: if the map $F : X \times Y \to Z$ is coordinately Baire measurable, then the map

$$\overline{F} : X \times \sigma(\mathcal{Z}(Z)) \to (\sigma_Z(\mathcal{Z}(Y)), \tau)$$

is Baire measurable with respect to the first variable.
(2) A topology τ on $\sigma_Z(\mathcal{Z}(Y))$ is called *coordinately Baire measurable $\mathcal{A}$-admissible* if for every space $X \in \mathcal{A}$ and for every map

$$G : X \to \mathcal{B}(Y, Z)$$

the following implication holds: if the map

$$\overline{G} : X \times \sigma(\mathcal{Z}(Z)) \to (\sigma_Z(\mathcal{Z}(Y)), \tau)$$

is Baire measurable with respect to the first variable, then the map

$$\widetilde{G} : X \times Y \to Z$$

is coordinately Baire measurable.

Remarks. (1) If $\mathcal{A}$ is the family of all completely regular Hausdorff spaces, then we use the notions of coordinately Baire measurable splitting and coordinately Baire measurable admissible instead of the notions of coordinately measurable $\mathcal{A}$-splitting and coordinately measurable $\mathcal{A}$-admissible, respectively.

(2) We observe that all Propositions 4.5, 4.6, 4.8, 4.9, 4.11, 4.12, 4.14, 4.15 and Corollaries 4.7, 4.10, 4.13, and 4.16 remain true if we replace the notions of section 4 by the corresponding notions of this section.

Definition 6.5. A topology on $\mathcal{B}(Y,Z)$ (respectively, on $\sigma_Z(\mathcal{Z}(Y))$) is said to be *a family-Baire measurable topology* if it is dual to a topology on $\sigma_Z(\mathcal{Z}(Y))$ (respectively, to a topology on $\mathcal{B}(Y,Z)$).

Remark. We observe that all Lemmas 4.18, 4.19, 4.21, 4.22, 4.23 and Proposition 4.24 remain true if we replace the notions of section 4 by the corresponding notions of this section.

Definition 6.6. Let τ be a topology on $\sigma_Z(\mathcal{Z}(Y))$ and t a topology on $\mathcal{B}(Y,Z)$. The pair (τ,t) is called *a pair of mutually dual topologies* if $\tau=\tau(t)$ and $t=t(\tau)$.

Remark. We observe that all Lemmas 4.26, 4.27 and Propositions 4.28, 4.29, 4.30, and 4.31 remain true if we replace the notions of section 4 by the corresponding notions of this section.

7. Some open problems

(1) Let t_0 be a topology on $\mathcal{M}(Y,Z)$. Is there a family-measurable topology t on $\mathcal{M}(Y,Z)$ such that $t_0 \subseteq t$? In particular, is it true that $t_0 \subseteq t(\tau(t_0))$?

(2) Let $\mathcal{A}$ be a family of topological spaces. Does there exist on the set $\sigma_Z(\tau_Y)$ the greatest coordinately measurable $\mathcal{A}$-splitting topology?

(3) Let $\mathcal{A}$ be a family of topological spaces. Does there exist on the set $\sigma_Z(\tau_Y)$ the greatest coordinately measurable $\mathcal{A}$-splitting family-measurable topology?

(4) Similar problems as the above problems 1, 2, and 3 we have for the Baire measurable maps and Baire sets.

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