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ULTRAFILTERS AND PROPERTIES RELATED TO COMPACTNESS

J. ANGOA, Y. F. ORTIZ-CASTILLO, AND Á. TAMARIZ-MASCARÚA

Abstract. In this article we introduce and analyze the following concepts: Let $p \in \mathbb{N}^*$ and let X be a topological space. We say that

(a) X is *strongly p-compact* if X is p-pseudocompact and for each sequence $(x_n)_{n\in\mathbb{N}}$ of points in X, there exists a sequence of open subsets $(U_n)_{n\in\mathbb{N}}$ of X, with $x_n \in U_n$ for each $n \in \mathbb{N}$, such that the set of p-limit points of the sequence $(U_n)_{n\in\mathbb{N}}$ is a non-empty compact subspace of X;

(b) X is *strongly p-pseudocompact* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X, there exist a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X and $x \in X$ such that $x_n \in U_n$ and $x = p - lim\ x_n$;

(c) X is *pseudo-ω-bounded* if for each countable family $\mathcal{U}$ of open subsets of X, there is a compact $K \subseteq X$ such that, for all $U \in \mathcal{U}$, $K \cap U \neq \emptyset$;

(d) X is p-pseudo-ω-bounded if for each family $\{U_n : n \in \mathbb{N}\}$ of open subsets of X, there is a compact subspace $K \subseteq X$ such that $\{n \in \mathbb{N} : K \cap U_n \neq \emptyset\} \in p$.

We prove:

(1) Every strongly p-compact space is p-compact.

(2) In the class of locally compact spaces, strong p-compactness and p-compactness are equivalent; and p-pseudo-ω-boundedness and p-pseudocompactness are equivalent too.

(3) For two ultrafilters p, $q \in \mathbb{N}^*$, $p \leq_{RK} q$ if and only if every strongly q-pseudocompact space $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$ is strongly p-pseudocompact.

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Notations and basic definitions

Every space in this paper is considered to be Tychonoff and has more than one point. ω is the first infinite cardinal number and ω_1 is the first non-countable cardinal number. The letter $\mathbb{N}$ stands for the space of the natural numbers with its discrete topology. Given a set X, we use the following notation: $[X]^{<\omega} := \{A \subseteq X : |A| < \omega\}$ and $[X]^{\omega} := \{A \subseteq X : |A| = \omega\}$. If X is a topological space and $A \subseteq X$, we use $Cl_X(A)$ (or simply $Cl(A)$ if there is no possibility of confusion) to denote the closure of A in X. For spaces X, Y, $C(X,Y)$ denotes the set of all continuous functions with domain X and range contained in Y. As usual, with βX we denote the Stone-Čech compactification of X, and X^* denote the remainder $\beta X \setminus X$. Given two ultrafilters $p, q \in \beta\mathbb{N}$, we say that $p \leq_{RK} q$ if there exists a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\beta f(q) = p$, where βf is the continuous extension of f to $\beta\mathbb{N}$. This relation is known as the *Rudin-Keisler* preorder on $\beta\mathbb{N}$.

If $\leq$ is a preorder on X, we say that p, $q \in X$ are $\leq$-equivalent if $p \leq q$ and $q \leq p$; p, q are $\leq$-comparable if either $p \leq q$ or $q \leq p$; and p, q are $\leq$-incomparable if they are not $\leq$-comparable.

If X is the cartesian product $\prod_{s\in S} X_s$ of a family $\{X_s : s \in S\}$ of non-empty sets and $s \in S$, then π_s denotes the projection from X to X_s.

Given a space X, $p \in \mathbb{N}^*$, and a sequence $(S_n)_{n\in\mathbb{N}}$ of non-empty subsets of X, we say that $z \in X$ is a *p-limit* of $(S_n)_{n\in\mathbb{N}}$ if for each neighborhood W of z, $\{n \in \mathbb{N} : S_n \cap W \neq \emptyset\} \in p$. A space X is *p-compact* (*p-pseudocompact*) if every sequence of points (of non-empty open subsets) of X has a p-limit point. Of course, if z and y are p-limits of a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X, then $z = y$. If x is the p-limit of $(x_n)_{n\in\mathbb{N}}$, we write $x = p - lim\ x_n$. The set $L(p, (S_n)_{n\in\mathbb{N}})$ of p-limits of a sequence $(S_n)_{n\in\mathbb{N}}$ of non-empty subsets of X is always closed and have more than one point. We say that a space X is *ω-bounded* if every subset $A \in [X]^{\omega}$ is contained in a compact subset of X. The notions used and not defined in this article have the meaning given to them in [5].

Introduction

In 1975, J. Ginsburg and V. Saks introduced the concept of p-pseudocompactness in [8]. This notion, defined in terms of p-convergence of sequences of non-empty open subsets, generalizes pseudocompactness and is related to p-compactness, introduced by Bernstein [3] and analyzed by Ginsburg and Saks [8], in a similar way as pseudocompactness is related to compactness. Furthermore, it is related to the *Rudin-Keisler* preorder: every p-pseudocompact space is q-pseudocompact if and only if $p \leq_{RK} q$. Following the ideas in [3], [6] and [8], we introduce and analyze the concepts of strong p-compactness and strong p-pseudocompactness.

The study of all these concepts is relevant because they determine different kinds of countably compact and pseudocompact spaces with different properties. The property of pseudo-ω-boundedness was inspired in ω-boundedness; in [2] the authors proved that this property characterizes the pseudocompactness of the hyperspace of compact sets.

In Section 1, we introduce the notion of strong p-compactness, and we study its properties and relations with other properties; in particular, we prove: (1) every strongly p-compact space is p-compact, and (2) every locally compact p-compact space is strongly p-compact.

In Section 2, we prove that a Tychonoff product $\prod_{\alpha<\kappa} X_\alpha$ is strongly p-compact if and only if each X_α is strongly p-compact and $|\{\alpha < \kappa : X_\alpha$ is not compact$\}| \leq \omega$. Moreover, if $f : X \to Y$ is an onto continuous and open function and X is strongly p-compact, then Y must be strongly p-compact.

In Section 3, we introduce and study the concepts of strong p-pseudocompactness and pseudo-ω-boundedness. We prove that both are productive properties (a property that pseudocompact spaces don't necessarily have). Finally, we introduce the almost pseudo-ω-bounded and the p-pseudo-ω-bounded spaces, and prove that in the class of locally compact spaces, strong p-compactness, p-compactness, p-pseudo-ω-boundedness and p-pseudocompactness are equivalent, and pseudocompactness is equivalent to almost pseudo-ω-boundedness.

p-pseudocompactness and strong p-pseudocompactness have similar properties including their relation with the Rudin-Keisler preorder; in particular, in Section 4, we show that $p \leq_{RK} q$ if and only if every strongly q-pseudocompact space $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$ is strongly p-pseudocompact; from this fact we derive new results which have a similar flavor to those given in Theorem 1.5 in [6].

Finally, in Section 5, we give an example of a strong p-compact, non-pseudo ω-bounded space and a strong p-compact, non-q-compact space.

1. Strong p-compactness

Definition 1.1. Let $p \in \mathbb{N}^*$. We say that a space X is *strongly p-compact* if X is p-pseudocompact and for each sequence $(x_n)_{n\in\mathbb{N}}$ of points in X, there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X, with $x_n \in U_n$ for each $n \in \mathbb{N}$, such that $L(p, (U_n)_{n\in\mathbb{N}})$ is a non-empty compact subspace of X.

Lemma 1.2. *Let $p \in \mathbb{N}^*$. The following properties are equivalent for a topological space X:*

(1) *X is p-compact;*

(2) *for every sequence* $(S_n)_{n\in\mathbb{N}}$ *of non-empty subsets of* X, $L(p,(S_n)_{n\in\mathbb{N}}) \neq \emptyset$,
(3) *for each sequence* $(D_n)_{n\in\mathbb{N}}$ *of non-empty closed subsets of* X*, it happens that* $L(p,(D_n)_{n\in\mathbb{N}}) \neq \emptyset$*; and*
(4) *for every sequence* $(S_n)_{n\in\mathbb{N}}$ *of non-empty subsets of* X*, we have that the set* $L(p,(S_n)_{n\in\mathbb{N}})$ *is not empty and for each open subset* U *of* X *satisfying*

$$\{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in \mathbb{N}\} \subseteq U,$$

it happens that $\{n \in \mathbb{N} : S_n \subseteq U\} \in p$.

Proof. All the implications are obvious except for the second assertion of $(1 \Rightarrow 4)$. Assume that there is a sequence $(S_n)_{n\in\mathbb{N}}$ of non-empty subsets of X, and assume that U is an open subset of X such that

$$\{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in \mathbb{N}\} \subseteq U$$

and $\{n \in \mathbb{N} : S_n \subseteq U\} \notin p$. In particular, the set $A = \{n \in \mathbb{N} : S_n \nsubseteq U\}$ belongs to p. Take $x_n \in S_n \setminus U$ if $n \in A$, and $x_n \in S_n$ if $n \notin A$. Since X is p-compact, there is $z \in X$ such that $z = p - lim\ x_n$. For each $n \in \mathbb{N}$, $x_n \in S_n$, so $z \in \{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in \mathbb{N}\} \subseteq U$. Moreover, by definition, $\{n \in \mathbb{N} : x_n \notin U\} = A$. Since p is an ultrafilter, $\{n \in \mathbb{N} : x_n \in U\} \notin p$; this is a contradiction. □

Proposition 1.3. *Let* $p \in \mathbb{N}^*$ *and* $A \in p$*. If* X *is a* p*-compact space, then for every sequence* $(S_n)_{n\in\mathbb{N}}$ *of non-empty subsets of* X *we have that*

$$L(p,(S_n)_{n\in\mathbb{N}}) = Cl(\{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in \mathbb{N}\})$$
$$= Cl(\{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in A\}). \quad (i)$$

Proof. Let $A \in p \in \mathbb{N}^*$. Then,

$$\{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in \mathbb{N}\} =$$
$$\{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in A\}.$$

So, the second equality (i) is obvious. Let

$$T = \{x = p - lim\ x_n : x_n \in S_n \text{ for each } n \in \mathbb{N}\}.$$

It is clear that $T \subseteq L(p,(S_n)_{n\in\mathbb{N}})$. Then $Cl(T) \subseteq L(p,(S_n)_{n\in\mathbb{N}})$ because $L(p,(S_n)_{n\in\mathbb{N}})$ is closed. Now we only have to prove that $L(p,(S_n)_{n\in\mathbb{N}}) \subseteq Cl(T)$.

Let $x \notin Cl(T)$ and let U and V be two disjoint open subsets of X such that $Cl(T) \subseteq U$ and $x \in V$. By Lemma 1.2, $\{n \in \mathbb{N} : S_n \subseteq U\} \in p$. Since U and V are disjoint open sets,

$$\{n \in \mathbb{N} : S_n \cap V \neq \emptyset\} \subseteq \{n \in \mathbb{N} : S_n \nsubseteq U\} \notin p.$$

This implies that $x \notin L(p,(S_n)_{n\in\mathbb{N}})$. So, we obtain the required equality. $\square$

We are going to obtain some basic properties about strong p-compactness. First a notation and some preliminary results.

Notation 1.4. Let $p \in \mathbb{N}^*$ and let $\mathcal{B}$ be a family of sequences of nonempty subsets of X. We denote by $\mathcal{L}_{\mathcal{B}}$ the set $\{L(p,B) : B \in \mathcal{B}\}$.

Proposition 1.5. *Let $p \in \mathbb{N}^*$ and let $S = (x_n)_{n\in\mathbb{N}}$ be a sequence in X. For each $x_m \in \{x_n : n \in \mathbb{N}\}$, consider a local base $\mathcal{N}_m$ of x_m in X. Let*

$$\mathcal{B} = \{(U_n)_{n\in\mathbb{N}} : U_n \in \mathcal{N}_n \text{ for each } n \in \mathbb{N}\}.$$

Then, $x = p - lim\ x_n$ if and only if $\bigcap \mathcal{L}_{\mathcal{B}} = \{x\}$.

Proof. It is clear that if $x = p - lim\ x_n$, then $x \in L(p,(U_n)_{n\in\mathbb{N}})$ for each sequence of open subsets $(U_n)_{n\in\mathbb{N}} \in \mathcal{B}$. Thus, $x \in \cap\mathcal{L}_{\mathcal{B}}$.

Let x be an element of $\bigcap \mathcal{L}_{\mathcal{B}}$ and assume that x is not a p-limit of sequence S. Then, there is a neighborhood W of x such that $\{n \in \mathbb{N} : x_n \in W\} \notin p$. Let V be an open neighborhood of x such that $Cl(V) \subseteq W$. For each $n \in \mathbb{N}$ such that $x_n \notin cl(V)$, let $B_n \in \mathcal{N}_n$ such that $B_n \cap Cl(V) = \emptyset$, and for each $n \in \mathbb{N}$ such that $x_n \in cl(V)$, choose whatever $B_n \in \mathcal{N}_n$.

Then,

$$\{n \in \mathbb{N} : B_n \cap V \neq \emptyset\} \subseteq \{n \in \mathbb{N} : x_n \in Cl(V)\} \subseteq \{n \in \mathbb{N} : x_n \in W\};$$

so $\{n \in \mathbb{N} : B_n \cap V \neq \emptyset\}$ does not belong to p, but this is not possible because our hypothesis says that

$$x \in \bigcap \mathcal{L}_{\mathcal{B}} \subseteq L(p,(B_n)_{n\in\mathbb{N}}).$$

Hence, $x = p - lim\ x_n$. Since the p-limits of sequences are unique, we obtain $\bigcap \mathcal{L}_{\mathcal{B}} = \{x\}$. $\square$

Lemma 1.6. *Let $p \in \mathbb{N}^*$ and let X be a p-pseudocompact space. Then, for every sequence S and every family $\mathcal{B}$ defined as in Proposition 1.5, the collection $\mathcal{L}_{\mathcal{B}}$ has the finite intersection property.*

Proof. Take $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$. Since for each $n \in \mathbb{N}$, $\mathcal{N}_n$ is a local base of x_n, there is $B_n \in \mathcal{N}_n$ such that $x_n \in B_n \subseteq V_n \cap U_n$. Since X is p-pseudocompact,

$$\emptyset \neq L(p,(B_n)_{n\in\mathbb{N}}) \in \mathcal{L}_{\mathcal{B}} \quad \text{and}$$

$$L(p,(B_n)_{n\in\mathbb{N}}) \subseteq L(p,(U_n)_{n\in\mathbb{N}}) \cap L(p,(V_n)_{n\in\mathbb{N}}).$$

This concludes our proof. $\square$

Now, we establish a basic fact about the relationship between p-compactness and strong p-compactness.

Theorem 1.7. *For every ultrafilter $p \in \mathbb{N}^*$, every strongly p-compact space is p-compact.*

Proof. Let X be a strongly p-compact space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of X such that $L(p,(U_n)_{n\in\mathbb{N}})$ is compact and $x_n \in U_n$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $\mathcal{N}_n$ be the family of all open subsets of X containing the point x_n and contained in U_n. It is clear that for each $n \in \mathbb{N}$, $\mathcal{N}_n$ is a local base of x_n.

Now we define the families

$$\mathcal{B} = \{(V_n)_{n\in\mathbb{N}} : V_n \in \mathcal{N}_n\} \quad \text{and} \quad \mathcal{L}_\mathcal{B} = \{L(p,(V_n)_{n\in\mathbb{N}}) : (V_n)_{n\in\mathbb{N}} \in \mathcal{B}\}.$$

Note that for each sequence $(V_n)_{n\in\mathbb{N}} \in \mathcal{B}$, the set $L(p,(V_n)_{n\in\mathbb{N}})$ is contained in $L(p,(U_n)_{n\in\mathbb{N}})$ because for each $V_n \in \mathcal{N}_n$, $V_n \subseteq U_n$. Since X is p-pseudocompact, $\mathcal{L}_\mathcal{B}$ is a family of compact sets with the finite intersection property (Lemma 1.6); so, $\bigcap \mathcal{L}_\mathcal{B} \neq \emptyset$. Because of Theorem 1.5, the sequence $(x_n)_{n\in\mathbb{N}}$ has a p-limit point. Therefore, X is p-compact. □

Finally we are going to give an example of a p-compact space which is not strongly p-compact. Recall that, given an ultrafilter $p \in \beta\mathbb{N}$, $P_{RK}(p)$ denotes the set $\{q \in \beta\mathbb{N} : q \leq_{RK} p\}$.

Remark 1.8. If $p \in \mathbb{N}^*$ and $P = \{z \in \beta\mathbb{N} : \text{there is a sequence } (x_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $z = p\text{-}\lim x_n\}$, then $P \cup \mathbb{N} = P_{RK}(p)$.

It is well-known that every closed subset of a p-compact space is p-compact, and the product of a collection of p-compact spaces is still p-compact. So, for every Tychonoff space X and every free ultrafilter p, there is a unique space, up to homeomorphism, $\beta_p(X)$ satisfying:

(1) X is dense in $\beta_p(X)$,
(2) $\beta_p(X)$ is p-compact, and
(3) for every p-compact space Y and every function $f \in C(X,Y)$, there is a function $F \in C(\beta_p(X),Y)$ such that $F|_X = f$.

It is also known that $\beta_p(X)$ is the intersection of all the p-compact subspaces of βX containing X (see [8] and Chapter 5 of [9]).

Example 1.9. For every free ultrafilter p on $\mathbb{N}$, the space $X = \beta_p(\mathbb{N})$ is not strongly p-compact.

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ in infinite sets and pick $x_n \in U_n^* \cap \beta_p(\mathbb{N})$ for each $n \in \mathbb{N}$. Then $L(p,(V_n)_{n\in\mathbb{N}})$ is infinite for every sequence of open sets $(V_n)_{n\in\mathbb{N}}$, where $x_n \in V_n \subseteq cl_X(U_n)$. Since $|\beta_p(\mathbb{N})| = 2^\omega$ and every infinite closed subset of $\beta\mathbb{N}$ has cardinality 2^{2^ω}, $L_X(p,(V_n)_{n\in\mathbb{N}})$ can not be compact. So we must have that $\beta_p(\mathbb{N})$ is not strongly p-compact. □

It is known that every p-compact space is countably compact; so, countable compactness does not imply strong p-compactness. Below, in Example 5.3, we show a strongly p-compact subspace of $\beta\mathbb{N}$ which contains $\mathbb{N}$.

Theorem 1.10. *The property of being strongly p-compact is inherited by closed subsets.*

Proof. Assume that X is a strongly p-compact space. Let $B \subset X$ be a non-empty closed subset of X, $(x_n)_{n\in\mathbb{N}}$ be a sequence in B, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of X such that $L(p,(U_n)_{n\in\mathbb{N}})$ is compact and $x_n \in U_n$ for each $n \in \mathbb{N}$. It is clear that $(B \cap U_n)_{n\in\mathbb{N}}$ is a sequence of open subsets of B with $x_n \in B \cap U_n$. By Theorem 1.7, X is p-compact, so B is p-compact (Theorem 2.4 in [8]). If $x = p - lim\ x_n$, then

$$x \in L_B(p,(B \cap U_n)_{n\in\mathbb{N}}) \subseteq B \cap L(p,(U_n)_{n\in\mathbb{N}}).$$

So, $L_B(p,(B\cap U_n)_{n\in\mathbb{N}})$ is not empty. Since $L_B(p,(B\cap U_n)_{n\in\mathbb{N}})$ is closed in B, it is closed in the compact subspace $B \cap L(p,(U_n)_{n\in\mathbb{N}})$. Hence, we conclude that $L_B(p,(B \cap U_n)_{n\in\mathbb{N}})$ is compact. Moreover, B is p-pseudocompact because it is p-compact. □

Theorem 1.11. *Every locally compact p-compact space is strongly p-compact.*

Proof. Let X be a locally compact p-compact space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Since X is p-compact, there is $x \in X$ such that $x = p - lim\ x_n$. Now, take an open neighborhood U of x such that $Cl(U)$ is compact. Let V be a neighborhood of x satisfying $Cl(V) \subseteq U$. For each $n \in \mathbb{N}$ such that $x_n \notin Cl(V)$, take a neighborhood U_n of x_n such that $U_n \subseteq X \setminus Cl(V)$, and for each $n \in \mathbb{N}$ with $x_n \in Cl(V)$, take a neighborhood U_n of x_n such that $U_n \subseteq U$.

We claim that $X \setminus Cl(U)$ is a subset of $X \setminus L(p,(U_n)_{n\in\mathbb{N}})$. Indeed, assume that $z \notin Cl(U)$. Then $X\setminus Cl(U)$ is an open subset of X containing z. Furthermore,

$$\{n : U_n \cap (X \setminus Cl(U)) \neq \emptyset\} \cap \{n : U_n \cap V \neq \emptyset\} = \emptyset.$$

Since $\{n \in \mathbb{N} : U_n \cap V \neq \emptyset\} \in p$, we have that $\{n : U_n \cap (X \setminus Cl(U)) \neq \emptyset\} \notin p$. This means that $z \notin L(p,(U_n)_{n\in\mathbb{N}})$. Therefore, $L(p,(U_n)_{n\in\mathbb{N}}) \subseteq Cl(U)$. Since $Cl(U)$ is compact and $L(p,(U_n)_{n\in\mathbb{N}})$ is closed, $L(p,(U_n)_{n\in\mathbb{N}})$ is compact. □

Corollary 1.12. *Every compact space is strongly p-compact.*

Observe that ω_1, with its order topology, is ω-bounded, locally compact, collectionwise normal, first countable, strongly p-compact for every $p \in \mathbb{N}^*$, but it is not compact.

Corollary 1.13. *Let X be a p-compact space and let Y be a locally compact space. If there is a continuous and onto function $f : X \to Y$, then Y is strongly p-compact.*

Proof. By Lemma 2.3 in [8], Y is p-compact, and by Theorem 1.11, Y is strongly p-compact. □

Remark 1.14. It is natural to ask what happens if we replace the condition $\mathbf{L(p, (U_n)_{n\in\mathbb{N}})}$ **is compact** by the condition $\mathbf{L(p, (U_n)_{n\in\mathbb{N}})}$ **is p-compact** in Definition 1.1. Of course, every p-compact space satisfies this new property. We consider that the most interesting question which arises from this new concept is if every space with the new property is p-compact (or at least countable compact). Our conjecture is that this property does not imply p-compactness. In the proof of Theorem 1.7 the hypothesis **the closed subsets $\mathbf{L(p, (U_n)_{n\in\mathbb{N}})} \in \mathcal{L}_\mathcal{B}$ are compact** can not be weakened by the hypothesis **the closed subsets $\mathbf{L(p, (U_n)_{n\in\mathbb{N}})} \in \mathcal{L}_\mathcal{B}$ are p-compact** because this last assertion only guarantees that $\bigcap \mathcal{L}_\mathcal{B} \neq \emptyset$ when $\mathcal{B}$ is a countable family. Although at this moment we do not have a counterexample, we think that it is possible to construct such an example and this is an interesting non-trivial problem.

2. Products, images and preimages of strongly p-compact spaces

It is known that the product of p-compact spaces and the continuous image of a p-compact space is a p-compact space (Lemma 2.3 in [8] and Theorem 4.2 in [3]). With respect to the images and productivity of strongly p-compactness we have:

Theorem 2.1. *For every $p \in \beta\mathbb{N}$, strong p-compactness is invariant under continuous open functions.*

Proof. Let X be a strongly p-compact space, $f \in C(X, Y)$ open and onto, and let $p \in \beta\mathbb{N}$. Also, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y. For each $n \in \mathbb{N}$, pick $x_n \in f^{-1}(y_n)$. Since X is strongly p-compact, there exist open subsets U_n of X such that $x_n \in U_n$ for each $n \in \mathbb{N}$, and $L(p, (U_n)_{n\in\mathbb{N}})$ is compact.

For each $n \in \mathbb{N}$, denote by V_n the open set $f[U_n]$. We have that $y_n \in V_n$, so it will be enough to show the equality

$$L(p, (V_n)_{n\in\mathbb{N}}) = f[L(p, (U_n)_{n\in\mathbb{N}})].$$

By Theorem 1.3.(3) of [6], $L(p, (V_n)_{n\in\mathbb{N}}) \supseteq f[L(p, (U_n)_{n\in\mathbb{N}})]$. Now, take

$$z \in L(p, (V_n)_{n\in\mathbb{N}}) \setminus f[L(p, (U_n)_{n\in\mathbb{N}})].$$

Since $L(p, (U_n)_{n\in\mathbb{N}})$ is compact, $f[L(p, (U_n)_{n\in\mathbb{N}})]$ is compact too, so we can find disjoint open subsets B_1, B_2 in Y such that $z \in B_1$

and $f[L(p,(U_n)_{n\in\mathbb{N}})] \subseteq B_2$. By Proposition 1.3, there exist $z' \in B_1$ and $z_n \in V_n$ such that $z' = p - lim\ z_n$; then, $A = \{n \in \mathbb{N} : z_n \in B_1\} \in p$. Therefore, $f^{-1}(z_n) \subseteq f^{-1}[B_1]$ for all $n \in A$. Moreover, it is clear that $f^{-1}[B_1]$ and $f^{-1}[B_2]$ are disjoint open subsets of X and

$$L(p,(U_n)_{n\in\mathbb{N}}) \subseteq f^{-1}[B_2].$$

Let $B = \{n \in \mathbb{N} : U_n \subseteq f^{-1}[B_2]\}$. It is clear that $A \cap B = \emptyset$, but Lemma 1.2 guarantees that $B \in p$. Since p is an ultrafilter and $A \in p$, it happens that $\emptyset = A \cap B \in p$ which is a contradiction. Then, we must have $z \in f[L(p,(U_n)_{n\in\mathbb{N}})]$ and we conclude that $L(p,(V_n)_{n\in\mathbb{N}}) = f[L(p,(U_n)_{n\in\mathbb{N}})]$. □

Theorem 2.2. *Let $p \in \mathbb{N}^*$, let $\mathcal{X} = \{X_s : s \in S\}$ be a family of topological spaces and let X be the topological product of such a family. Then, X is strongly p-compact if and only if X_s is strongly p-compact for every $s \in S$, and $|\{s \in S : X_s$ is not compact $\}| \leq \omega$.*

Proof. ($\Rightarrow$) Assume that X is strongly p-compact. By Theorem 1.10, each X_t is strongly p-compact because it is homeomorphic to a closed subset of X.

Now, assume that $T \subseteq S$ is such that X_t is not compact for each $t \in T$. Suppose that $|T| > \omega$. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence of points and $(U_n)_{n\in\mathbb{N}}$ a sequence of open subsets of X such that $x_n \in U_n$ for each $n \in \mathbb{N}$. By Theorem 1.7, X_t is p-compact for each $t \in S$. So, X is p-compact (Theorem 4.2 of [3]). Let x_0 be the p-limit point of the sequence $(x_n)_{n\in\mathbb{N}}$.

For each $n \in \mathbb{N}$, there is $A_n \in [S]^{<\omega}$, and there is, for each $s \in A_n$, an open set W_n^s of X_s such that $x_n \in \bigcap_{s\in A_n} \pi_s^{-1}[W_n^s] \subseteq U_n$. The set $A = \bigcup_{n\in\mathbb{N}} A_n$ is countable. Now define the set

$$B = \{x \in X : \pi_s(x) = \pi_s(x_0) \text{ for all } s \in A\}.$$

Claim: $B \subseteq L(p,(U_n)_{n\in\mathbb{N}})$.

Indeed, take $x \in B$, $F \in [S]^{<\omega}$ and $V = \cap_{s\in F}\pi_s^{-1}[V_s]$ where V_s is an open subset of X_s such that $x \in V$. By definition of A, we have $\pi_s[U_n] = X_s$ for each $s \notin A$; so, if $F \cap A = \emptyset$, then $V \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$.

Assume now that $F \cap A \neq \emptyset$. Then,

$$\{n \in \mathbb{N} : V \cap U_n \neq \emptyset\} \supseteq \{n \in \mathbb{N} : (\bigcap_{s\in F\cap A} \pi_s^{-1}[V_s]) \cap U_n \neq \emptyset\} \supseteq$$

$$\bigcap_{s\in F\cap A} \{n \in \mathbb{N} : V_s \cap W_n^s \neq \emptyset\} \supseteq \bigcap_{s\in F\cap A} \{n \in \mathbb{N} : \pi_s(x_n) \in V_s\}.$$

By Lemma 2.3 in [8], $\pi_s(x_0) = p - lim\ \pi_s(x_n)$ for each $s \in S$; so, for each $s \in A$, $\pi_s(x) = p - lim\ \pi_s(x_n)$. Then, for each $s \in A$,

$$\{n \in \mathbb{N} : \pi_s(x_n) \in V_s\} \in p.$$

Since $F \cap A$ is finite, we have that

$$\bigcap_{s \in F \cap A} \{n \in \mathbb{N} : \pi_s(x_n) \in V_s\} \in p;$$

this implies that $x \in L(p, (U_n)_{n\in\mathbb{N}})$, and the proof of the Claim has concluded.

Since $|A| < |T|$, we can take $t \in T \setminus A$. The space X_t is homeomorphic to a closed subset of X contained in $B \subseteq L(p, (U_n)_{n\in\mathbb{N}})$. But X_t is not compact, so $L(p, (U_n)_{n\in\mathbb{N}})$ cannot be compact; this is a contradiction because we have supposed that X is strongly p-compact. So, we must have $|T| \leq \omega$.

($\Leftarrow$) Now suppose that X_s is strongly p-compact for every $s \in S$. Let $T = \{s \in S : X_s$ is not compact$\}$ and assume that $T \neq \emptyset$. We have to consider two cases:

I. $|T| = \omega$. Define $X_T = \prod_{t\in T} X_t$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X_T, and for each $n < \omega$, $t \in T$ let U_n^t be an open subset of X_t such that $\pi_t(x_n) \in U_n^t$ and $L(p, (U_n^t)_{n\in\mathbb{N}})$ is compact. Enumerate the set T as $\{t_n : n \in \mathbb{N}\}$ and, for each $n < \omega$, consider the set $V_n = \bigcap_{m\leq n} \pi_{t_m}^{-1}[U_n^{t_m}]$. Each V_n is a canonical open subset of X_T containing x_n.

We are going to show that $L(p, (V_n)_{n\in\mathbb{N}}) \subseteq \prod_{t\in T} L_{X_t}(p, (U_n^t)_{n\in\mathbb{N}}) = L$. In fact, take $x \notin L$, so there is $r \in T$ such that $\pi_r(x) \notin L_{X_r}(p, (U_n^r)_{n\in\mathbb{N}})$. Let W be an open neighborhood of $\pi_r(x)$ such that $\{n : W \cap U_n^r = \emptyset\} \in p$. Observe that $V_n \subseteq \pi_r^{-1}[U_n^r]$ for every $n > m$, where $r = t_m$. Hence

$$\{n \in \mathbb{N} : \pi_r^{-1}[W] \cap V_n = \emptyset\} \supseteq \{n \geq t_m : W \cap U_n^r = \emptyset\} \in p.$$

Since $x \in \pi_r^{-1}[W]$, we obtain $x \notin L(p, (V_n)_{n\in\mathbb{N}})$, and so $L(p, (V_n)_{n\in\mathbb{N}})$ is compact.

II. $|T| < \omega$. This case is a consequence of Case I and Theorem 2.1.

If $T = S$ the proof is finished. Suppose $T \neq S$. Since X is the product of a strongly p-compact and a compact space, by Case II, X is strongly p-compact. □

Remark 2.3. From Theorems 1.11 and 2.2, we can deduce that if X is a strongly p-compact non-compact space, then X^ω is strongly p-compact and it is not locally compact. On the other hand, every infinite discrete space is locally compact and it is not strongly p-compact.

The following lemma is Theorem 3.7.26 in [5].

Lemma 2.4. *Let P be a property which is inherited by closed subsets and every product of a compact space with a space satisfying P has P. If Y has P and $f : X \to Y$ is a continuous and perfect function, then X has property P.*

Corollary 2.5. *Let $p \in \mathbb{N}^*$. Then every continuous and perfect preimage of a strongly p-compact space is strongly p-compact.*

Proof. Let $f : X \longrightarrow Y$ be a continuous perfect and onto function. Assume that Y is a strongly p-compact space. By Theorem 1.10, the case II in Theorem 2.2 and Lemma 2.4, we conclude that X is a strongly p-compact space. □

Question 2.6. *Is it possible to find a strongly p-compact space X, a space Y which is not strongly p-compact and a (closed or perfect) continuous function $f : X \to Y$?*

3. STRONG p-PSEUDOCOMPACTNESS AND PSEUDO-ω-BOUNDEDNESS

Definition 3.1. Let X be a topological space and $p \in \mathbb{N}^*$. We say that X is:

(1) *strongly p-pseudocompact* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X and there is $x \in X$ such that $x = p - lim\ x_n$ and $x_n \in U_n$ for all $n \in \mathbb{N}$,

(2) *pseudo-ω-bounded* if for each countable family $\mathcal{U}$ of open subsets of X, there is a compact $K \subseteq X$ such that $K \cap U \neq \emptyset$ for all $U \in \mathcal{U}$.

Observe that every p-compact space is strongly p-pseudocompact.

Theorem 3.2. *Let $p \in \mathbb{N}^*$.*

(1) *Every pseudo-ω-bounded space is strongly p-pseudocompact and every strongly p-pseudocompact space is p-pseudocompact.*

(2) *If X contains a dense strongly p-pseudocompact subspace, then X is strongly p-pseudocompact.*

(3) *Regular closed subsets inherit the property of being strongly p-pseudocompact.*

(4) *X is pseudo-ω-bounded if and only if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X, there exist points $x_n \in U_n$ such that, for every $p \in \mathbb{N}^*$, the sequence $(x_n)_{n\in\mathbb{N}}$ has a p-limit.*

Proof. (1) We are going to prove the first assertion because the second one can be proved easily. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X. Let K be a compact subset of X which has a non-empty intersection with U_n for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, there is a point $x_n \in K \cap U_n$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ is contained in K, there is $x \in K$ such that $x = p - lim\ x_n$.

(2) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of X. For each $n \in \mathbb{N}$, let V_n be the set $U_n \cap Y$. Then, $(V_n)_{n\in\mathbb{N}}$ is a sequence of open sets of Y, so there are points $x_n \in V_n \subseteq U_n$ and $x \in Y \subseteq X$ such that x is the p-limit of the sequence $(x_n)_{n\in\mathbb{N}}$ in Y. Now it is easy to show that x is the p-limit of the sequence $(x_n)_{n\in\mathbb{N}}$ in X. Thus, we conclude that X is strongly p-compact.

(3) Let D be a regular closed subset of X and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of D. Since $D = Cl(Int(D))$, for each $n \in \mathbb{N}$, the set $V_n = U_n \cap Int(D)$ is open in X. Since X is strongly p-pseudocompact, there are points $x_n \in V_n \subseteq U_n$ and $x \in X$ such that $x = p - lim\ x_n$. Since $(x_n)_{n\in\mathbb{N}}$ is a sequence in D, D is closed and $x \in Cl\{x_n : n \in \mathbb{N}\}$, then $x \in D$.

(4) ($\Rightarrow$) Suppose that X is pseudo-ω-bounded. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of X and let K be a compact subspace of X such that $K \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, take an element $x_n \in K \cap U_n$. Since $\{x_n : n \in \mathbb{N}\} \subseteq K$ and K is compact, $Cl_X(\{x_n : n \in \mathbb{N}\})$ is compact; so, the sequence $(x_n)_{n\in\mathbb{N}}$ has a p-limit point in X for each $p \in \mathbb{N}^*$.

($\Leftarrow$) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points such that $x_n \in U_n$ for all $n \in \mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ has a p-limit point in X for each $p \in \mathbb{N}^*$. Now, Theorem 3.4 in [3] guarantees that $K = Cl_X(\{x_n : n \in \mathbb{N}\})$ is compact. Moreover, for each $n \in \mathbb{N}$, $x_n \in K \cap U_n$. □

Remark 3.3. Every ω-bounded space is pseudo-ω-bounded and every p-compact space is strongly p-pseudocompact. On the other hand, it is possible to find pseudo-ω-bounded spaces which are not strongly p-compact (for example, Σ-products of compact spaces). In Example 5.3, below, we show an strongly p-compact space which is not pseudo-ω-bounded.

Theorem 3.4. *The property of being strongly p-pseudocompact is invariant under continuous images.*

Proof. Let X be a strongly p-pseudocompact space. Let $f : X \to Y$ be a continuous and onto function. Finally, let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of Y. It is clear that $(V_n)_{n\in\mathbb{N}}$, where $V_n = f^{-1}[U_n]$ for each $n \in \mathbb{N}$, is a sequence of open subsets of X; so, there exist points $x_n \in V_n$ and $x \in X$ such that $x = p - lim\ x_n$. Then, $f(x_n) \in U_n$, $f(x) \in Y$ and $f(x) = p - lim\ f(x_n)$ (Lemma 2.3 in [8]). □

Theorem 3.5. *Let $\{X_s : s \in S\}$ be a family of topological spaces and X the Tychonoff product of such a family. Then, X is strongly p-pseudocompact if and only if for each $s \in S$, X_s is strongly p-pseudocompact.*

Proof. ($\Rightarrow$) This implication is a consequence of Theorem 3.4.

($\Leftarrow$) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of X. For each $n \in \mathbb{N}$ and each $s \in S$, let V_n^s be a non-empty open subset of X_s such that $V_n = \cap_{s\in S}\pi_s^{-1}[V_n^s] \subseteq U_n$. Since X_s is strongly p-pseudocompact for each $s \in S$, there is a sequence $(x_n^s)_{n\in\mathbb{N}}$ of points in X_s and there is a point $x^s \in X_s$ such that $x_n^s \in V_n^s$ for each $n \in \mathbb{N}$ and $x^s = p - lim\ x_n^s$. Take the point $x \in X$ such that $\pi_s(x) = x^s$ for each $s \in S$. Finally, for each $n \in \mathbb{N}$, take $x_n \in X$ such that $\pi_s(x_n) = x_n^s$ for each $s \in S$. It is clear that $x_n \in U_n$ for each $n \in \mathbb{N}$ and $x = p - lim\ x_n$. We conclude that X is strongly p-pseudocompact. □

Corollary 3.6. *Pseudo-ω-boundedness is a productive property and invariant under continuous functions. Also, regular closed subsets inherit this property. Furthermore, if a space X contains a dense pseudo-ω-bounded subspace, then X is pseudo-ω-bounded.*

Theorem 3.7. *Let $A \in p \in \mathbb{N}^*$ and let X be a strongly p-pseudocompact space. Then, for each sequence $(U_n)_{n\in\mathbb{N}}$ of non-empty open subsets of X, it happens that*

$$L(p, (U_n)_{n\in\mathbb{N}}) = Cl(Q)$$

where $Q = \{x \in X :$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in U_n$ for each $n \in A$ and $x = p - lim\ x_n\}$.

Proof. Using arguments similar to those given in the proof of Theorem 1.3, we have $Cl(Q) \subseteq L(p, (U_n)_{n\in\mathbb{N}})$. We are only going to prove the relation $L(p, (U_n)_{n\in\mathbb{N}}) \subseteq Cl(Q)$.

Let $z \notin Cl(Q)$ and let V, W be disjoint open subsets of X such that $z \in V$ and $Cl(Q) \subseteq W$. We will show that $\{n \in \mathbb{N} : V \cap U_n \neq \emptyset\} \notin p$. Assume the contrary: $\{n : V \cap U_n \neq \emptyset\} \in p$ and take, for each $n \in \mathbb{N}$, $V_n = U_n \cap V$ if $U_n \cap V \neq \emptyset$ and $V_n = U_n$ otherwise.

Since X is strongly p-pseudocompact, there are points $x_n \in V_n$ and $x \in X$ such that $x = p - lim\ x_n$. For each $n \in \mathbb{N}$, $V_n \subseteq U_n$, so $x \in Q$ and W is an open neighborhood of x. This means that $\{n \in \mathbb{N} : x_n \in W\} \in p$, but this is not possible because V and W have an empty intersection; so, we must have

$$\{n \in \mathbb{N} : x_n \in W\} \subseteq \{n : V \cap U_n = \emptyset\} \notin p.$$

This concludes our proof. □

Definition 3.8. Let X be a space and $p \in \mathbb{N}^*$.

(1) We say that X is almost pseudo-ω-bounded if for each infinite countable family $\mathcal{U}$ of open subsets of X, there is a compact subspace $K \subseteq X$ such that $|\{U \in \mathcal{U} : K \cap U \neq \emptyset\}| = \omega$.
(2) We say that X is p-pseudo-ω-bounded if for each family $\{U_n : n \in \mathbb{N}\}$ of open subsets of X, there is a compact subspace $K \subseteq X$ such that $\{n \in \mathbb{N} : K \cap U_n \neq \emptyset\} \in p$.

Theorem 3.9. *If X is locally compact, then the following statements are equivalent for every ultrafilter $p \in \mathbb{N}^*$:*

(1) *X is p-pseudo-ω-bounded,*
(2) *X is strongly p-pseudocompact, and*
(3) *X is p-pseudocompact.*

Proof. (1) $\Rightarrow$ (2). Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X and let K be a compact set such that $A = \{n \in \mathbb{N} : K \cap U_n \neq \emptyset\} \in p$. Pick $x_n \in K \cap U_n$ if $n \in A$ and choose an arbitrary $x_n \in U_n$ when $n \notin A$. Since $(x_n)_{n\in\mathbb{N}}$ is a sequence in K which is compact it has p-limit.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X and let $x \in L(p, (U_n)_{n\in\mathbb{N}})$. Let W be a compact neighborhood of x. Consider the set

$$A = \{n \in \mathbb{N} : U_n \cap int(W) \neq \emptyset\}.$$

For each $n \in \mathbb{N}$, we take $x_n \in U_n \cap int(W)$ if $n \in A$ and let x_n be equal to x if $n \notin A$. Since $(x_n)_{n\in\mathbb{N}}$ is a sequence in $Cl(W)$ which is compact, then $Cl(\{x_n : n \in \mathbb{N}\})$ is compact. □

Note that, in the last result, the locally compactness is necessary just for the implication (3) $\Rightarrow$ (1).

Corollary 3.10. *If X is locally compact, then X is pseudocompact if and only if it is almost pseudo-ω-bounded.*

The following questions are inspired in a question posed by M. Sanchiz and Á. Tamariz-Mascarúa [10], which remain without an answer.

Question 3.11. *Is it true that for every free ultrafilter p on $\mathbb{N}$ every (normal, first countable) topological space (topological group) X is strongly p-pseudocompact if and only if it is p-pseudocompact?*

Question 3.12. *Is it true that for every free ultrafilter p on $\mathbb{N}$ every normal or first countable) topological space (topological group) X is p-compact if and only if it is strongly p-pseudocompact?*

Question 3.13. *Is there some countable compact (strongly pseudocompact) space non-strongly p-pseudocompact for all p on $\mathbb{N}$?*

Where a space X is strongly pseudocompact if, for each sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X there is $p \in \mathbb{N}^*$ and there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in U_n$ for all $n \in \mathbb{N}$ and the sequence $(x_n)_{n\in\mathbb{N}}$ has p-limit.

4. Strong p-pseudocompactness and the Rudin-Keisler pre-orden on $\beta\omega$

Theorem 4.1. *Let $p \in \mathbb{N}^*$ and let X be a space having a dense subset of isolated points S. Then, X is strongly p-pseudocompact if and only if X is p-pseudocompact.*

Proof. Assume that X is p-pseudocompact. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X. Since S is dense in X, for each $n \in \mathbb{N}$, we can take a point $x_n \in U_n \cap S$. Since the points in S are isolated and X is p-pseudocompact, $(\{x_n\})_{n\in\mathbb{N}}$ is a sequence of non-empty open subsets of X and $L(p, (\{x_n\})_{n\in\mathbb{N}}) \neq \emptyset$. If $x \in L(p, (\{x_n\})_{n\in\mathbb{N}})$, then $x = p - lim\ x_n$. □

Corollary 4.2. *Let p, $q \in \mathbb{N}^*$. Then, the following assertions are equivalent:*

(1) *$p \leq_{RK} q$,*
(2) *every q-pseudocompact space is p-pseudocompact,*
(3) *$P_{RK}(q)$ is strongly p-pseudocompact,*
(4) *every strongly q-pseudocompact space $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$ is strongly p-pseudocompact.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) and the implication (3) $\Rightarrow$ (1) are consequences of Theorem 1.5 in [6]. Finally, the implication (2) $\Rightarrow$ (3) and the equivalence (3) $\Leftrightarrow$ (4) follow from Theorem 4.1 and Lemma 1.9 in [6] which says that a space X with $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$ is p-pseudocompact if and only if $P_{RK}(p) \subseteq X$. □

Question 4.3. *Is it true that for every free ultrafilter p on ω every (normal, first countable) space X is p-compact if and only if it is strongly p-pseudocompact?*

Definition 4.4. Let X be a topological space and let $\mathcal{D}$ be a non-empty subset of $\mathbb{N}^*$. We say that X is pseudo-$\mathcal{D}$-bounded if for each sequence $(U_n)_{n\in\mathbb{N}}$ of non-empty open subsets of X, there are both a sequence of points $(x_n)_{n\in\mathbb{N}}$ in X and a set $\{x_p : p \in \mathcal{D}\} \subseteq X$ such that $x_n \in U_n$ and $x_p = p\text{-lim}\ x_n$.

Theorem 4.5. *Let $\mathcal{D} \subseteq \mathbb{N}^*$ and $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$. Then, the following assertions are equivalent:*

(1) *X is pseudo-$\mathcal{D}$-bounded,*
(2) *X is strongly p-pseudocompact for all $p \in \mathcal{D}$, and*
(3) *X is p-pseudocompact for every $p \in \mathcal{D}$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (1). Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X. For each $n \in \mathbb{N}$, take $x_n \in U_n \cap \mathbb{N}$. By Lemma 1.9 in [6] and Remark 1.8 above, for each $p \in \mathcal{D}$, $p - lim\ x_n \in P_{RK}(p) \subseteq X$. □

Notation 4.6. *Let $q \in \beta\mathbb{N}$. We will denote by $S_{RK}(q)$ the set of Rudin-Keisler successors of q: $S_{RK}(q) = \{p \in \beta(\mathbb{N}) : p \geq_{RK} q\}$.*

Theorem 4.7. *Let $\mathcal{D} \subseteq \mathbb{N}^*$ and $\mathbb{N} \subseteq X \subseteq \beta(\mathbb{N})$. Then, the following assertions are equivalent:*

(1) *$X = \beta\mathbb{N}$,*
(2) *X is pseudo-ω-bounded,*
(3) *X is pseudo-$\mathbb{N}^*$-bounded,*
(4) *X is strongly p-pseudocompact for every $p \in \mathbb{N}^*$,*
(5) *X is p-pseudocompact for every $p \in \mathbb{N}^*$,*
(6) *X is pseudo-$\mathcal{D}$-bounded and for each $q \in \mathbb{N}^*$, $\mathcal{D} \cap S_{RK}(q) \neq \emptyset$,*
(7) *for every $q \in \mathbb{N}^*$ there is $p \in S_{RK}(q)$ such that X is strongly p-pseudocompact, and*
(8) *for all $q \in \mathbb{N}^*$, there is $p \in S_{RK}(q)$ such that X is p-pseudocompact.*

Proof. The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (2) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) are evident. The equivalence (2) $\Leftrightarrow$ (3) is (4) from Theorem 3.2. The implication (5) $\Rightarrow$ (1) follows from Lemma 1.9 in [6]. Finally, (8) $\Rightarrow$ (5) is a consequence of Theorem 1.5 in [6]. □

Corollary 4.8. *Let $\mathcal{D} \subseteq \mathbb{N}^*$ and $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$. If X is pseudo-$\mathcal{D}$-bounded and it is not pseudo-ω-bounded, then there is $q \in \mathbb{N}^*$ such that $\mathcal{D} \subseteq \mathbb{N}^* \setminus S_{RK}(q)$.*

5. Strong p-compactness and strong p-pseudocompactness

Recall that a point $x \in X$ is a *weak P-point* in X if x is not an accumulation point of any countable subset of X. The following result is known.

Lemma 5.1. *([11]) There are 2^{2^ω} weak P-points in $\mathbb{N}^*$ which are pairwise $\leq_{RK}$-incomparable.*

Definition 5.2. Let X be a topological space and $\mathcal{D} \subseteq \mathbb{N}^*$. We say that X is strongly $\mathcal{D}$-compact if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X, there is a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets such that, for each $n \in \mathbb{N}$, $x_n \in U_n$ and for each $p \in \mathcal{D}$, X is p-pseudocompact and $L(p, (U_n)_{n\in\mathbb{N}})$ is compact.

We finish this paper with one example of a space X with properties closer to pseudo-ω-boundedness which do not imply that X must be pseudo-ω-bounded. The spirit of this last example is to reinforce the relevance of the pseudo-ω-boundedness.

Example 5.3. Let $q \in \mathbb{N}^*$ be a weak P-point. Let $\mathcal{D}$ be the set of all ultrafilters on $\mathbb{N}$ which are RK-incomparable with q. Denote by $\mathcal{Q}$ the set $\mathbb{N}^* \setminus S_{RK}(q)$. Then, $X = \beta\mathbb{N} \setminus \{q\}$ and $\mathcal{Q}$ satisfy the following properties:

1. X is locally compact,
2. X is strongly $\mathcal{D}$-compact,
3. X is pseudo-$\mathcal{Q}$-bounded and $\mathcal{Q}$ is dense in $\mathbb{N}^*$,
4. X is not q-compact, and
5. X is not pseudo-ω-bounded.

Besides, we can choose q in such a way that $|\mathcal{Q}| = |\mathcal{D}| = 2^{2^\omega}$.

Proof. It is evident that X is not q-compact because $q \notin X$. Since X is open in $\beta\mathbb{N}$, it is locally compact. It is also clear that $\mathcal{D} \subseteq \mathcal{Q}$. By Lemma 5.1, there are 2^{2^ω} weak P-points in $\mathbb{N}^*$ which are pairwise RK-incomparable; so, we can assume that $|\mathcal{D}| = |\mathcal{Q}| = 2^{2^\omega}$. Moreover, for each $p \in \mathcal{Q}$, $P_{RK}(p) \subseteq X$; thus, $\mathcal{Q}$ is dense in $\mathbb{N}^*$. By Corollary 4.8, X is pseudo-$\mathcal{Q}$-bounded and it is not pseudo-ω-bounded.

Finally, we are going to show that X is strongly $\mathcal{D}$-compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Consider the set $A = \{x_n : n \in \mathbb{N}\} \cap \mathbb{N}^*$. Let U, V be two disjoint clopen subsets of $\beta\mathbb{N}$ such that $Cl_X(A) \subseteq U$ and $q \in V$. For each $n \in \mathbb{N}$, take $U_n = \{x_n\}$ if $x_n \in \mathbb{N}$ and let $U_n \subseteq U$ be a canonical clopen neighborhood of x_n if $x_n \in \mathbb{N}^*$. Let $p \in \mathcal{D}$. If $B = \{n \in \mathbb{N} : x_n \in \mathbb{N}^*\} \in p$, then, by Proposition 1.3,

$$L_X(p, (U_n)_{n\in\mathbb{N}}) = L_X(p, (U_n)_{n\in B}) = L_U(p, (U_n)_{n\in B}).$$

Since U is a non-empty compact space, $L_U(p, (U_n)_{n\in B})$ is compact too. On the other hand, if $C = \{n \in \mathbb{N} : x_n \in \mathbb{N}\} \in p$, then, by Proposition 1.3 and Remark 1.8,

$$L_{\beta(\mathbb{N})}(p, (U_n)_{n\in\mathbb{N}}) = \{p - lim_C\ x_n\} \subseteq P_{RK}(p) \subseteq X.$$

Therefore, $L_X(p, (U_n)_{n\in\mathbb{N}})$ is a non-empty compact subspace of X. $\square$

In particular, if $q \in \mathbb{N}^*$ is a weak P-point and $p \in \mathbb{N}^*$ is $\leq_{RK}$-incomparable with q, then the space $X = \beta\mathbb{N} \setminus \{q\}$ is strongly p-compact, locally compact, not q-compact and not pseudo-ω-bounded.

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