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A CONSTRUCTION OF HEWITT REALCOMPACTIFICATION IN TERMS OF NETS

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Abstract. We give a construction of Hewitt realcompactification of a completely regular Hausdorff space by using nets.

1. Introduction

Throughout the paper, topological spaces are assumed to be Hausdorff. The set of real-valued continuous functions on a topological space X is denoted by $C(X)$. We consider $C(X)$ as an algebra and a Riesz space with the pointwise algebraic operations and the pointwise order. That is, for each f and g in $C(X)$ and every real number α, we define

$$(\alpha f)(x) := \alpha f(x), \quad (f+g)(x) := f(x) + g(x), \quad (fg)(x) := f(x)g(x),$$

and the corresponding order is defined by

$$f \leqslant g \quad \text{if and only if} \quad f(x) \leqslant g(x) \quad \text{in } \mathbb{R} \text{ for all } x \in X.$$

The symbol $C_b(X)$ denotes the set of real-valued continuous and bounded functions on X. Clearly, $C_b(X)$ is a subalgebra and a Riesz subspace of $C(X)$. A great deal of information about the algebras $C(X)$ and $C_b(X)$ can be found in [6].

Recall that a topological space X is called *completely regular* if for each closed set $F \subseteq X$ and $x \in X \setminus F$ there exists a bounded continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(F) = 0$. It is well-known that for each topological space X, there exists a completely regular space Y such that $C(X)$ and $C(Y)$ are algebraically isomorphic spaces (see, [6, Theorem 3.9]).

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Recall that a linear map T from a Riesz space E into another Riesz space F is called a *Riesz homomorphism* if $T(x \vee y) = T(x) \vee T(y)$ for all x and y in E, where $x \vee y$ denotes the supremum of x and y. If T is a one-t-one, onto Riesz homomorphism and T^{-1} is a Riesz homomorphism, then T is said to be a *Riesz isomorphism*, and in this case E and F are called *Riesz isomorphic*. For some basic facts about Riesz space theory, we refer to [9]. Let now X and Y be topological spaces. If $\pi : C(X) \to C(Y)$ is an algebra homomorphism, then it is a Riesz homomorphism (see [6, Theorem 1.6]), or [10, Theorem 14.1] for a more general setting). The converse of this also true if $T(1) = 1$: this follows from Lemma 2.4 below (see [11, Theorem 18.8] for the general case). So, without loss of generality, we can suppose that there is no ambiguity in distinguishing between a Riesz homomorphism and an algebra homomorphism between $C(X)$ and $C(Y)$.

It is well-known that a completely regular Hausdorff space is called *real-compact* if it is homeomorphic to a closed subset of a product space of the reals. The notion of a real-compact space was introduced by Hewitt [8]. More details on real-compact spaces can be found in [6] and [3].

The following characterization of real-compact spaces in terms of Riesz homomorphisms is well-known. In ZF, a proof of this can be found in [5]. It should also be mentioned that W.W. Comfort, in [2], points out that some constructions and properties reminiscent of the Stone-Čech compactification and the Hewitt realcompactification can be achieved.

Theorem 1.1. *Let X be a completely regular Hausdorff space. The following are equivalent:*

(i) *X is realcompact.*
(ii) *For each Riesz homomorphism $\pi : C(X) \to \mathbb{R}$ with $\pi(1) = 1$, there exists a unique $x \in X$ such that $\pi(f) = f(x)$ for all $x \in X$.*

Theorem 1.1 implies the following Banach-Stone Theorem.

Theorem 1.2. *Let X and Y be real-compact spaces. The following are equivalent:*

(i) *The spaces X and Y are homeomorphic.*
(ii) *The spaces $C(X)$ and $C(Y)$ are Riesz isomorphic.*
(iii) *The spaces $C(X)$ and $C(Y)$ are algebraically isomorphic.*

A topological space Y is called a *compactification* of a topological space X if X is homeomorphic to a dense subspace of Y. Each completely regular space X has a unique compactification βX such that each bounded continuous function can be extended to a unique continuous function $f^\beta : \beta X \to \mathbb{R}$; moreover, $C_b(X)$ and $C(\beta X)$ are Riesz and algebraically isomorphic. Fore more about the construction of the Stone-Čech compactification, we refer to [3] and [6].

A real-compact space Y is called a *realcompactification* of a topological space X if X is homeomorphic to a dense subspace of Y. Hewitt [8] showed that each completely regular space X has a realcompactification υX such that each continuous function $f \in C(X)$ has a unique extension to υX such that $C(X)$ and $C(\upsilon X)$ are Riesz and algebraically isomorphic. By Theorem 1.2, υX is unique (up to homeomorphism). The space υX is called the *Hewitt realcompactification* of X. For different constructions of υX, one can consult on [3] and [6]. See also [1] and [7].

In this note, we will give another construction of υX in terms of nets.

2. The construction of υX

Let X be a completely regular space. A net (x_i) in X is called *C-net* if $\lim_i f(x_i)$ exists for all $f \in C(X)$. Let Y_0 be the set of C-nets in X. We define a relation $\equiv$ on Y_0 by

$$(x_i) \equiv (y_j) \quad \text{if and only if} \quad \lim f(x_i) = \lim f(y_j) \quad \text{for all } f \in C(X).$$

Then $\equiv$ defines an equivalence relation on Y_0. For each $(x_i) \in Y_0$, $[(x_i)]$ denotes the equivalence class of (x_i), and Y denotes the set of equivalence class of C-nets. We define a topology on Y, which is the weak topology determined by $\{\pi(f) : f \in C(X)\}$, where $\pi(f) : Y \to \mathbb{R}$ defined by

$$\pi(f)([(x_i)]) := \lim f(x_i).$$

That is, the family

$$\{\pi(f)^{-1}(U) : U \subseteq \mathbb{R} \text{ is open and } f \in C(X)\}$$

is a subbase of the topological space Y. We have, then, the following result.

Theorem 2.1. *The space Y is homeomorphic to υX.*

To prove Theorem 2.1, we need the following lemmata.

Lemma 2.2. *The space Y is completely regular.*

Proof. Let $[(x_i)] \neq [(y_j)]$ in Y. By definition, $\pi(f)([(x_i)]) \neq \pi(f)([(x_i)])$ for some $f \in C(X)$. So, Y is Hausdorff. Since the topology on Y is the weak topology determined by $\{\pi(f) : f \in C(X)\}$, it follows that Y is completely regular ([6, theorem 3.7]). □

Lemma 2.3. *Let $([(x_i^\alpha)_{i\in I_\alpha}])_\alpha$ be a net in Y and $[(y_j)_{j\in J}] \in Y$. Then*

$$[(x_i^\alpha)_{i\in I_\alpha}] \to [(y_j)] \overset{\alpha}{\iff} \lim_{i\in I_\alpha} f(x_i^\alpha) \to \lim f(y_j) \quad \textit{for all} \quad f \in C(X).$$

Proof. The proof immediately follows from the definition of the topology of Y. □

The proof of the above lemma is given under the assumption that the Hewitt realcompactification of any completely regular space exists (see [4]). We can reprove this in terms of βX as follows.

Lemma 2.4. *Let $\pi : C(X) \to \mathbb{R}$ be a Riesz homomorphism with $\pi(1) = 1$. Then there exists a C-net (x_i) such that*

$$\pi(f) = \lim_i f(x_i) \quad \textit{for all} \quad f \in C(X).$$

Proof. Since $C_b(X)$ and $C(\beta X)$ are Riesz isomorphic, there exists a unique $y \in \beta X$ such that

$$\pi(f) = f^\beta(y) = \lim_i f(x_i)$$

for all $f \in C_b(X)$, where f^β denotes the extension of f in $C(\beta X)$ and $x_i \to y$ in βX. Let $0 \leqslant f$ be given. Choose $n \in \mathbb{N}$ such that $\pi(f) < n$. Then

$$\pi(f) = \pi(f) \wedge n = \pi(f \wedge n) = \lim_i (f \wedge n)(x_i) = \lim_i (f(x_i) \wedge n) < n.$$

This implies that there exists an i_0 such that $\lim_i (f(x_i) < n$ for all $i \geqslant i_0$, whence

$$\lim f(x_i) = \lim_i (f(x_i) \wedge n).$$

Therefore, for all $0 \leqslant f \in C(X)$, we have

$$\pi(f) = \pi(f \wedge n) = \lim_i (f(x_i) \wedge n) = \lim f(x_i).$$

For any $f \in C(X)$, since $f = f^+ - f^-$, we also have

$$\pi(f) = \lim_i (f(x_i)),$$

which completes the proof. □

Lemma 2.5. *The space Y is real-compact.*

Proof. Define

$$\sigma : Y \to \prod_{f \in C(X)} \mathbb{R}, \quad \sigma([(x_i)]) = (\lim_i f(x_i))_{f \in C(X)}.$$

By Lemma 2.3, X and $\sigma(X)$ are homeomorphic. For each α, let $y = (a_f)_{f \in \sigma(X)} \in \overline{\sigma(Y)}$ be given. Choose a net (y_α) in Y with $\sigma(y_\alpha) \to y$. Put $y_\alpha := [(x_i^\alpha)_{i \in I_\alpha}]$. Then, for each $f \in C(X)$, we have

$$\lim_\alpha \lim_{i \in I_\alpha} f(x_i^\alpha) = a_f.$$

Define

$$\pi : C(X) \to \mathbb{R}, \quad \pi(f) = a_f.$$

It is clear that π is a Riesz homomorphism with $\pi(1) = 1$. Then there exists a C-net (x_j) in X such that

$$a_f = \pi(f) = \lim_i f(x_i)$$

for all $f \in C(X)$. Hence

$$\sigma(y_\alpha) \to (a_f)_{f\in C(X)} = (\lim_i f(x_i)) = \sigma([(x_j)]).$$

It follows that $\sigma(Y)$ is a closed subspace of $\prod_{f\in C(X)} \mathbb{R}$, whence Y is real-compact. □

Lemma 2.6. *The space Y is a realcompactification of X.*

Proof. Let I and J be directed sets. For each $x \in X$, one has $[(x_i)_{i\in I}] = [(x_j)_{j\in J}]$ in Y, where $x_i = x_j = x$. In this case we write $[(x)]$ instead of $[(x_i)_{i\in I}$. Define $\tau : X \to Y$ by $\tau(x) = [(x)]$. It is obvious that τ is a homeomorphism (into). Let $y = [(x_i)_{i\in I}] \in Y \setminus \overline{\tau(X)}$. Choose $g \in C(Y)$ with $g(y) = 1$ and $g(\overline{\sigma(X)}) = 0$. It is now obvious that $\tau(x_j) \to y$ in Y. This implies, moreover, that

$$0 = g(\tau(x_i)) \to g(y) = 1,$$

which is a contradiction. Therefore, $\overline{\tau(X)} = Y$, so that Y is a realcompactification of X. □

Lemma 2.7. *The spaces $C(X)$ and $C(Y)$ are Riesz and algebraically isomorphic.*

Proof. Let τ be a homeomorphism from X into Y as in the proof of Lemma 2.6. Define

$$\pi : C(Y) \to C(X), \quad \pi(f) = f \circ \tau.$$

Then π is the required algebraic and Riesz isomorphism. □

Proof of Theorem 2.1. By the previous Lemma, $C(X)$ and $C(Y)$ are algebraically and Riesz isomorphic, and by Hewitt's Theorem, $C(X)$ and $C(\upsilon X)$ are algebraically and Riesz isomorphic. It follows that $C(Y)$ and $C(\upsilon X)$ are algebraically and Riesz isomorphic. Moreover, Y and υX are real-compact; Theorem 1.2 now applies and Y and υX are homeomorphic. □

Using the new construction of the Hewitt realcompactification obtained above, we can reprove the following well-known theorems.

Theorem 2.8. *The space υX is an (embeddable) subspace of βX.*

Proof. Let $\pi : C(\beta X) \to C(\upsilon X)$ be defined by

$$\pi(f^\beta)([(x_i)]_{i\in I}) := \lim_i f(x_i).$$

It is obvious that π is a Riesz homomorphism. Let $y = [(x_i)_{i\in I}] \in Y$ be given. Then $\pi_y \circ \pi : C(\beta X) \to \mathbb{R}$ is a Riesz homomorphism, where $\pi_y : C(\upsilon X) \to \mathbb{R}$ is defined by $\pi_y(f) := \lim_i f(x_i)$. From Theorem 2.1, there exists a unique $\sigma(y) \in \beta X$ such that

$$\pi(f^\beta) = f^\beta(\sigma(y)).$$

Hence, a map $\sigma : \upsilon X \to \beta X$ is defined, and it is straightforward to check that σ is a homeomorphism into βX. □

Theorem 2.9. *The space Y is the smallest realcompactification amongst the real-compact spaces Z satisfying $X \subseteq Z \subseteq \beta X$.*

Proof. Let Z be a real-compact space such that $X \subseteq Z \subseteq \beta X$. Let $y = [(x_i)_{i\in I}] \in Y$ be given. Define $\pi : C(Z) \to \mathbb{R}$ by

$$\pi(f) = \lim(f|_X)(x_i).$$

Then π is a Riesz homomorphism. By Theorem 1.1, there exists a unique $\sigma(y) \in Z$ such that, for all $f \in C(Z)$, one has

$$\pi(f) = f(\sigma(y)).$$

Hence we have defined a map $\sigma : Y \to Z$, which is a homeomorphism (into). This completes the proof. □

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