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by

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ABSTRACT. Let $FP_G(X)$ and $FP(X)$ be the free paratopological groups on a topological space X in the senses of Graev and Markov, respectively. In this paper, we prove that the groups $FP_G(X)$ and $FP(X)$ are discrete if X is discrete and the group $FP_G(X)$ is indiscrete if X is indiscrete while the group $FP(X)$ is the union of infinite indiscrete subspaces if X is indiscrete. Then we give a class of spaces X for which the groups $FP_G(X)$ and $FP(X)$ are locally invariant and another class of spaces X where they are not. Finally, we provide another proof for the existence of the free paratopological group $FP(X)$ by embedding the space X in an infinite direct product of paratopological groups.

1. Introduction

The existence of the Graev free paratopological group $FP_G(X)$ on a pointed topological space X were proved by Romaguera, Sanchis and Tkačenko [11] in 2003. An analogous proof of [11] was used by Elfard [3] to prove the existence of the Markov free paratopological group $FP(X)$ on a topological space X.

In this paper, we prove that both groups $FP_G(X)$ and $FP(X)$ are discrete if X is discrete and the group $FP_G(X)$ is indiscrete if X is indiscrete while the group $FP(X)$ is the union of infinite indiscrete subspaces if X is indiscrete. Then we introduce the Graev paratopological group topology on the underlying set of the group $FP(X)$ on a topological space X (see [1]) and then we give a class of spaces X for which the groups $FP_G(X)$ and $FP(X)$ are locally invariant and another class of spaces X where they are not.

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S. Kakutani [5] proved the existence of the Hausdorff free topological group $F(X)$ on a Tychonoff space X in the sense of Markov. His method of proof is based on the use of embedding a Tychonoff space as a closed subspace of an infinite direct product of topological groups. Hewitt and Ross [8], in Theorem 8.8, used a similar method of [5] to prove the existence of the Hausdorff free topological group $F(X)$ on any Tychonoff space X. Finally, we use a similar argument as in Theorem 8.8 of [8] to prove the existence of the free paratopological group $FP(X)$ on any topological space X.

2. Graev and Markov free paratopological groups

The Graev free paratopological group on a pointed topological space X with a basepoint e is a paratopological group $FP_G(X)$ that, algebraically, is a free group with $X \setminus \{e\}$ as a free base and such that any continuous mapping, $f\colon X \to G$, from X to a paratopological group G sending e to the identity of G, $f(e) = e_G$, can be extended uniquely to a continuous homomorphism $\hat{f}\colon FP_G(X) \to G$ on the group $FP_G(X)$.

For different choices of a basepoint $e \in X$, the resulting Graev free paratopological groups are isomorphic [11].

The Markov free paratopological group on a topological space X is a paratopological group $FP(X)$ that, algebraically, is a free group with X as a free base and such that any continuous mapping, $f\colon X \to G$, from X to a paratopological group G can be extended uniquely to a continuous homomorphism $\hat{f}\colon FP(X) \to G$ on the group $FP(X)$.

We denote the free topology of the free paratopological groups $FP_G(X)$ and $FP(X)$ by $\mathcal{T}_{FP}$. We remark that, in both senses Graev and Markov, the free topology $\mathcal{T}_{FP}$ is the strongest paratopological group topology on the abstract free group $F_a(X)$ that induce the original topology on X.

The relation between Graev and Markov free paratopological groups on a topological space X is determined by the fact that the Markov free paratopological $FP(X)$ is the Graev free paratopological group $FP_G(X \oplus \{e\})$ on the space $X \oplus \{e\}$, where '$\oplus$' denotes the topological sum. Graev's concept is more general than Markov's concept, in the sense that every Markov free paratopological group is a Graev free paratopological group.

Let X be a set. Then for all $k \in \mathbb{Z}$ we define the subset $Z_k(X) = \{x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n} \in F_a(X) : \sum_{i=1}^{n} \epsilon_i = k\}$ of the free group $F_a(X)$. For every $k_1, k_2 \in \mathbb{Z}$, the sets $Z_{k_1}(X)$ and $Z_{k_2}(X)$ are disjoint whenever $k_1 \neq k_2$. The set $Z_0(X)$ is the smallest normal subgroup containing the set $Z_F = \bigcup_{x \in X} x^{-1}X$.

Let $f : X \to \mathbb{Z}$ be the mapping defined by $f(x) = 1$ for all $x \in X$. Then f is continuous and hence extends to a continuous homomorphism $\hat{f} : FP(X) \to \mathbb{Z}$. It follows that the collection of sets $Z_k(X) = \hat{f}^{-1}(\{k\})$ for $k \in \mathbb{Z}$ forms a partition of $FP(X)$ into homeomorphic clopen subspaces.

Remark 2.1. If G is a paratopological group and g is a fixed element in G, then the translations $r_g \colon x \mapsto xg$ and $l_g \colon x \mapsto gx$ of G onto G are homeomorphisms of G.

Proposition 2.2. *Let X be a topological space. Then the subgroup $Z_0(X)$ of the Markov free paratopological group $FP(X)$ can be identified with the Graev free paratopological group $FP_G(X)$.*

Proof. Fix $x_0 \in X$. Let $f \colon x_0^{-1}X \to G$ be a continuous mapping of the subspace $x_0^{-1}X$ of $Z_0(X)$ to a paratopological group G such that $f(e) = e_G$. Let $\theta \colon X \to G$ defined by $\theta(x) = f(x_0^{-1}x)$ for all $x \in X$. Then θ is a continuous mapping, because it is the composition of two continuous mappings, the translation mapping and f. So we have $\theta(x_0) = e_G$. We extend θ to a continuous homomorphism $\hat{\theta} \colon FP(X) \to G$.

Now if $x \in X$, then $\hat{\theta}(x_0^{-1}x) = (\hat{\theta}(x_0))^{-1}\hat{\theta}(x) = e_G\theta(x) = f(x_0^{-1}x)$, which implies that $\hat{\theta}|_{(x_0^{-1}X)} = f$. Thus we have $\hat{\theta}|_{Z_0(X)} = \hat{f} \colon Z_0(X) \to G$ is the extension of f. Therefore, by Remark 2.1, $Z_0(X)$ is the Graev free group $FP_G(X)$ on X. □

We note that if X is a topological space, then by Theorem 4.2 of [1], the topology of the subspace X^{-1} of the free paratopological group $FP(X)$ has as an open base the collection $\{C^{-1} : C \text{ is closed in } X\}$.

Proposition 2.3. *Let X be an indiscrete space. Then the subspace X^{-1} of $FP(X)$ and the subspace X^{-1} of $FP_G(X)$ are indiscrete.*

Proof. Since the subspace X^{-1} of $FP(X)$ has the collection $\{\emptyset, X^{-1}\}$ as a base, X^{-1} is an indiscrete subspace of $FP(X)$.

Now for a fixed point $x_0 \in X$, by Remark 2.1, we have x_0X^{-1} is an indiscrete subspace of $FP(X)$ and also x_0X^{-1} is an indiscrete subspace of $Z_0(X)$, which is the Graev free group $FP_G(X)$ on X by Proposition 2.2. Therefore, again by Remark 2.1, X^{-1} is an indiscrete subspace of $FP_G(X)$. □

Theorem 2.4. *Let X be a topological space. Then the free paratopological group $FP_G(X)$ is indiscrete if and only if the space X is indiscrete.*

Proof. $\Longrightarrow$: It is clear.

$\Longleftarrow$: Assume that X is indiscrete. Let U be a neighborhood of e in $FP_G(X)$ and let $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \in F_a(X)$. Then there exists a neighborhood V of e in $FP_G(X)$ such that $V^n \subseteq U$. Since X and X^{-1} are indiscrete and contain e, $V \cap X = X$ and $V \cap X^{-1} = X^{-1}$. Thus $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \in \underbrace{V \cdots V}_{n-times} = V^n \subseteq U$, which means that $U = F_a(X)$. Therefore, $FP_G(X)$ is indiscrete. □

If X is an indiscrete space, then by Theorem 2.4 and Proposition 2.2, the subspace $Z_0(X)$ of the Markov free paratopological group $FP(X)$ is indiscrete. Therefore, the Markov free paratopological group on an indiscrete space X is the union of disjoint indiscrete subspaces $Z_k(X)$, $k \in \mathbb{Z}$.

Theorem 2.5. *Let X be a topological space. Then X is discrete if and only if $FP(X)$ is discrete.*

Proof. $\Longrightarrow$: Assume that X is discrete. Since the free topology of the free topological group $F(X)$ is discrete and it is a paratopological group topology induces the original topology on X so it is the free topology $\mathcal{T}_{FP}$ of $FP(X)$.

$\Longleftarrow$: Assume that $FP(X)$ is discrete. Since X is a subspace of $FP(X)$, it is clear that X is discrete. □

Since the Graev free group $FP_G(X)$ on a space X is a subgroup of the Markov free group $FP(X)$, then as a corollary of Theorem 2.5 we have the following result.

Corollary 2.6. *A space X is discrete if and only if the Graev free paratopological group $FP_G(X)$ is discrete.*

3. Local invariance of free paratopological groups

If d is a quasi-pseudometric on a space X, then for each $x \in X$ we define $d_x(y) = d(x, y)$ for all $y \in X$.

Our main reference for the following is Elfard and Nickolas [1].

Let X be a topological space ($e \notin X$) and let $\mathcal{D}_1$ be the family of all quasi-pseudometrics d on X, which are bounded by 1 and are such that d_x is upper semi-continuous for all $x \in X$. For every $d \in \mathcal{D}_1$ let $\hat{d}$ be the Graev extension of d to the abstract free group $F_a(X)$ and let $\mathcal{T}_d$ be the paratopological group topology generated by $\hat{d}$ on $F_a(X)$. Let $\mathcal{T}_{GFP}$ be the supremum of all the topologies $\mathcal{T}_d$ on $F_a(X)$ for all $d \in \mathcal{D}_1$. Therefore, $\mathcal{T}_{GFP}$ is a paratopological group topology on $F_a(X)$.

We refer to the topology $\mathcal{T}_{GFP}$ as the *Graev paratopological group topology* on the group $F_a(X)$.

Proposition 3.1. *Let X be a topological space and let $d \in \mathcal{D}_1$. Then for all $w \in FP(X)$, $(\hat{d})_w$ is upper semi-continuous on $FP(X)$.*

Proof. By Theorem 3.8 of [1] and the fact that the free topology $\mathcal{T}_{FP}$ of $FP(X)$ is finer than the Graev topology $\mathcal{T}_{GFP}$, the balls $B_{\hat{d}}(e, \epsilon)$ for $\varepsilon > 0$ are in the free topology $\mathcal{T}_{FP}$. Therefore, by Theorem 2.2 of [2], $(\hat{d})_w$ is upper semi-continuous for all $w \in FP(X)$. □

A paratopological group G is *locally invariant* if it has a base at e of neighborhoods U with the property that $xUx^{-1} = U$ for all $x \in G$.

The Graev topology $\mathcal{T}_{GFP}$ as defined above is the finest locally invariant paratopological group topology on $F_a(X)$ inducing the original topology on X [1]. Therefore, the Graev topology on the underlying set of a free paratopological group equals their free topology if and only if the free topology is locally invariant.

Proposition 3.2. *Let X be a discrete space. Then the free paratopological groups $FP(X)$ and $FP_G(X)$ are locally invariant.*

Proof. By Theorem 2.5 and Corollary 2.6, the free paratopological groups $FP(X)$ and $FP_G(X)$ are discrete and then locally invariant. □

Now let X be an Alexandroff space. Let N_F be the smallest normal submonoid of $F_a(X)$ containing the set $U_F = \bigcup_{x\in X} x^{-1}U(x)$, where $U(x) = \bigcap\{U : U \text{ is open in } X \text{ and } x \in U\}$ and then define $\mathcal{N}_F = \{N_F\}$. By Theorem 4.4 of [4] (see also Theorem 3.2.5 of [3]), $\mathcal{N}_F$ is a neighborhood base at the identity of the free paratopological group $FP(X)$.

Proposition 3.3. *Let X be an Alexandroff space. Then the free paratopological group $FP(X)$ is locally invariant.*

Proof. Since the free paratopological group $FP(X)$ has the collection $\mathcal{N}_F$ as a neighborhood base at the identity e, it is clear that the free paratopological group $FP(X)$ is locally invariant. □

We note that the same result of Theorem 4.4 of [4] is true for the Graev free paratopological group $FP_G(X)$ on an Alexandroff space X. Therefore, the same result of Proposition 3.3 is true for the free paratopological group $FP_G(X)$ on Alexandroff space X.

Next we give a class of spaces X where the free paratopological groups $FP(X)$ and $FP_G(X)$ are not locally invariant. This result is similar to a result in Proposition 1.2.1 of [9] for free topological groups.

Proposition 3.4. *Let X be a Tychonoff space. Assume that X contains a (non-trivial) sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in X$. Then the free paratopological group $FP(X)$ ($FP_G(X)$) is not locally invariant.*

Proof. Let $d \in \mathcal{D}_1$. Then we have $d(x_n, x) \to 0$. Let $\hat{d}$ be the Graev extension of d. Thus we have

$$\begin{aligned}\hat{d}(x^n x_n x^{-(n+1)}, e) &= \hat{d}(x^n x_n x^{-1} x^{-n}, e)\\ &= \hat{d}(x_n x^{-1}, e)\\ &= \hat{d}(x_n, x) \to 0.\end{aligned}$$

So the sequence $\{x^n x_n x^{-(n+1)}\}_{n\in\mathbb{N}}$ converges to e in the Graev topology $\mathcal{T}_{GFP}$ on $F_a(X)$. Now assume that $\mathcal{T}_{FP} = \mathcal{T}_{GFP}$. Let $\hat{\phi}\colon FP(X) \to F(X)$ be the extension of the identity mapping $\phi\colon X \to F(X)$, where $F(X)$ is the free topological group on X. Then $\hat{\phi}$ is a continuous homomorphism mapping. Hence the sequence

$$\{[\hat{\phi}(x)]^n \hat{\phi}(x_n)[\hat{\phi}(x)]^{-(n+1)}\}_{n\in\mathbb{N}}$$

converges to e in the free topology of $F(X)$. This contradicts a result of Abels [7] that the free topology of $F(X)$ cannot contain a convergent sequence of increasing length. Therefore, $\mathcal{T}_{GFP} \neq \mathcal{T}_{FP}$ and then the free paratopological group $FP(X)$ is not locally invariant.

The same proof can be applied for $FP_G(X)$ when we use the Graev topology $\mathcal{T}_{GFP}$ as defined in [11]. □

4. A Kakutani-type embedding

We mentioned earlier that Kakutani [5] proved the existence of the Hausdorff free topological group $F(X)$ on a Tychonoff space X by embedding a Tychonoff space as a closed subspace of an infinite direct product topological group. Hewitt and Ross [8], in Theorem 8.8 used a similar method of [5] to prove the existence of the Hausdorff free topological group $F(X)$ on any Tychonoff space X. In Theorem 4.1, we use a similar argument as in Theorem 8.8 of [8] to prove the existence of the free paratopological group $FP(X)$ on any space X.

Let X be a topological space and let $\{Y_i\}_{i\in I}$ be a family of topological spaces. Let $\{f_i\}_{i\in I}$ be a family of continuous mappings, where $f_i : X \to Y_i$. We say that the family $\{f_i\}_{i\in I}$ *separates points* of X if for every pair of distinct points $x, y \in X$, there exists $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$. If for every $x \in X$ and every closed set $A \subseteq X$ and $x \notin A$, there exists $i_0 \in I$ such that $f_{i_0}(x) \notin \mathrm{cl}_{Y_{i_0}}(f_{i_0}(A))$, then we say that the family $\{f_i\}_{i\in I}$ *separates points and closed sets.*

Let $\{\{k, k+1, \ldots\}\colon k \in \mathbb{Z}\}$ be a collection of subsets of the additive group $\mathbb{Z}$. Then it is a base for a paratopological group topology on $\mathbb{Z}$ and we denote the corresponding paratopological group by $\mathbb{Z}^*$.

Theorem 4.1. *Let X be a topological space. Then there exists a paratopological group $FP'(X)$ such that*

(i) *X is a subspace of $FP'(X)$;*
(ii) *algebraically, $FP'(X)$ is the free group generated by X; and*
(iii) *for every continuous mapping ϕ of X into any paratopological group G, there exists a continuous homomorphism $\hat{\phi}$: $FP'(X) \to G$ extending ϕ.*

Proof. Let $\mathbb{Z}_2$ be the additive group with the indiscrete topology. Then $\mathbb{Z}_2$ is a topological group. For every $x, y \in X$, $x \neq y$, we define a continuous mapping $f_{x,y} : X \to \mathbb{Z}_2$ by

$$f_{x,y}(z) = \begin{cases} 0 & \text{if } z = x, \\ 1 & \text{otherwise} \end{cases}$$

for all $z \in X$. We denote the family of all such mappings $\{f_{x,y}\}$ by $\{f_j\}_{j \in J}$. Then the family $\{f_j\}_{j \in J}$ separates points of X.

For every closed set A in X and every point $y \in X$ such that $y \notin A$, we define a continuous mapping $g_{y,A} : X \to \mathbb{Z}^*$ by

$$g_{y,A}(x) = \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X$. Fix $y \in X$ and a closed set $A \subseteq X$ such that $y \notin A$. Since $g_{y,A}(y) = 0 \notin \{\dots, -2, -1\} = \mathrm{cl}_{\mathbb{Z}^*}(g_{y,A}(A))$, the mapping $g_{y,A}$ separates the point y from A. Denote the family of all such mappings $g_{y,A}$ by $\{g_k\}_{k \in K}$.

Let Y be a set, where $|Y| = \aleph_0$ and let $F_a(Y)$ be the abstract free group on the set Y with the indiscrete topology. Then $F_a(Y)$ is a topological group. Let $\{\psi_l\}_{l \in L}$ be the collection of all mappings (of course they are continuous) of X into the topological group $F_a(Y)$.

Let $\mathscr{G}$ be a family of paratopological groups which contains the groups $\mathbb{Z}_2, \mathbb{Z}^*$ and $F_a(Y)$ and satisfying the following conditions;

(1) $|G| \leq \max(|X|, \aleph_0)$ for all $G \in \mathscr{G}$; and
(2) distinct members of $\mathscr{G}$ are not topologically isomorphic; and
(3) if H is a paratopological group with $|H| \leq \max(|X|, \aleph_0)$, then H is topologically isomorphic with some $G \in \mathscr{G}$.

Let $\{(G_i, \eta_i)\}_{i \in I}$ consist of all pairs (G_i, η_i), where $G_i \in \mathscr{G}$, $\eta_i : X \to G_i$ is a continuous mapping for all $i \in I$, where $J, K, L \subseteq I$ and $\eta_i = f_i$ for all $i \in J$, $\eta_i = g_i$ for all $i \in K$ and $\eta_i = \psi_i$ for all $i \in L$, where J, K and L as defined above.

If H is a paratopological group, $|H| \leq \max(|X|, \aleph_0)$ and β is a continuous function of X into H, then there is $i_0 \in I$ and a topological isomorphism ξ of G_{i_0} onto H such that $\xi \circ \eta_{i_0} = \beta$.

Now the collection $\mathscr{G}$ has enough functions to separate points of X and has enough functions to separate points from closed sets in X.

Let $\hat{G} = \prod \mathscr{G}$. Then $\hat{G}$ is a paratopological group (Ravsky [10]). Let e denote the identity of $\hat{G}$. Let $\nu : X \to \hat{G}$ be the mapping defined by $\nu(x)_i = \eta_i(x)$ for each $x \in X$. The mapping ν is a homeomorphism of X onto $\nu(X)$ (Kelley [6], p.116). So we identify X with $\nu(X)$; that is, X is regarded as a subspace of $\hat{G}$. Let $FP'(X)$ be the subgroup generated by X.

We show (ii). Let $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ be any reduced word formed from the elements of X such that $w \neq e$. Then there is a continuous function $\psi_{l_0}, l_0 \in L$ of the space X into $F_a(Y)$ such that $\psi : F_a(X) \to F_a(Y)$ is the extension homomorphism of ψ_{l_0}. Then we have

$$\begin{aligned}\psi(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) &= [\psi(x_1)]^{\varepsilon_1} [\psi(x_2)]^{\varepsilon_2} \cdots [\psi(x_n)]^{\varepsilon_n} \\ &\neq e.\end{aligned}$$

Therefore, algebraically $FP'(X)$ is the free group generated by$X$.

We show (iii). Let $\phi\colon X \to G$ be a continuous mapping of the space X to a paratopological group G. Let D be a subgroup containing $\phi(X)$ in G with the property that $|D| \leq \max(|X|, \aleph_0)$. Thus D is a paratopological group and then (D, ϕ) topologically isomorphic to a pair (G_{i_0}, η_{i_0}). So we can consider $\phi\colon X \to D$ as the continuous map $\pi_{i_0}|_{\nu(X)} \to G_{i_0}$, where $\pi_{i_0}\colon \prod_{i \in I} G_i \to G_{i_0}$ is the projection map. So we extend ϕ to a homomorphism $\hat{\phi}\colon FP'(X) \to D \subseteq G$. Since $\hat{\phi} = \pi_{i_0}|_{FP'(X)}$, $\hat{\phi}$ is continuous. ☐

Remark 4.2. The paratopological group $FP'(X)$ on a topological space X, which described in Theorem 4.1 satisfies the conditions of the definition of the free paratopological group $FP(X)$ on the space X (see [1]) and since $FP(X)$ is unique, we identify $FP'(X)$ with $FP(X)$.

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