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by

J. C. GÓMEZ-LARRAÑAGA, F. GONZÁLEZ-ACUÑA, AND
WOLFGANG HEIL

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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

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ON THE K -CATEGORY OF 3-MANIFOLDS FOR K A WEDGE OF SPHERES OR PROJECTIVE PLANES

J. C. GÓMEZ-LARRAÑAGA, F. GONZÁLEZ-ACUÑA, AND WOLFGANG HEIL

ABSTRACT. For a complex K , a closed 3-manifold M is of K -category $\leq m$, if it can be covered by m open subsets W_1, \dots, W_m such that the inclusions $W_i \rightarrow M$ factor homotopically through maps $W_i \xrightarrow{f_i} K \xrightarrow{\alpha_i} M$. We compute the K -category of closed 3-manifolds M when K is a wedge of 2-spheres and obtain some results for the K -category of M when K is a wedge of two projective planes.

1. INTRODUCTION

The classical Lusternik-Schnirelmann category $cat(X)$ of a space X is the smallest number k such that there is an open cover W_1, \dots, W_k of X with each W_i contractible in X . (If no such finite cover exists, $cat(X) = \infty$.) We are interested in the case when $X = M$, a manifold. Unless stated otherwise, we assume that manifolds are connected, but the W_i 's need not be connected. For M^3 a closed 3-manifold, $2 \leq cat(M^3) \leq 4$, and it has been shown in [5] that $cat(M^3) = 2, 3$, if and only if $\pi_1(M^3) = 1$, free non-trivial, respectively. If the condition that W_i is contractible is replaced by the weaker condition that for every basepoint $* \in W_i$, the inclusion $\iota : W_i \rightarrow X$ induces the trivial homomorphism $\iota_* : \pi_1(W_i, *) \rightarrow \pi_1(X, *)$, then W_i is said to be π_1 -contractible (in X) and $cat_{\pi_1}(X)$ is the smallest number k such that there is an open cover of X by k π_1 -contractible subsets. This invariant is defined by Ralph H. Fox [4] and it is also shown in [5] that $cat_{\pi_1}(M^3) = 1$ (2, respectively; 4, respectively), if and only if $\pi_1(M^3) = 1$ (free non-trivial, respectively; non-free, respectively). By

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Perelman (see e.g., [1]),

$$cat_{\pi_1}(M^3) = \begin{cases} 1 & \text{if and only if } M \text{ is the 3-sphere} \\ 2 & \text{if and only if } M \text{ is a connected sum of } S^2\text{-bundles} \\ & \text{over } S^1 \\ 4 & \text{otherwise.} \end{cases}$$

Mónica Clapp and Dieter Puppe [3] generalized the notion of $cat(X)$ as follows: Let \mathcal{K} be a non-empty class of spaces. A subset W of X is \mathcal{K} -contractible (in X) if the inclusion $\iota : W \rightarrow X$ factors homotopically through some $K \in \mathcal{K}$; i.e., there exist maps $f : W \rightarrow K$, $\alpha : K \rightarrow X$, such that ι is homotopic to $\alpha \cdot f$. The \mathcal{K} -category $cat_{\mathcal{K}}(X)$ of X is the smallest number of open \mathcal{K} -contractible subsets of X that cover X . For closed n -manifolds, $1 \leq cat_{\mathcal{K}}(M) \leq cat(M) \leq n + 1$. When the family \mathcal{K} contains just one space K , one writes $cat_K(M)$ instead of $cat_{\mathcal{K}}(M)$. In particular, if K is a single point, then $cat_K(M) = cat(M)$. The cases for $K = S^1$, $K = S^2$, and $K = P^2$ (the projective plane) have been considered in [6] and [7].

There is also a generalization of the notion of $cat_{\pi_1}(M)$: For a nonempty class of groups \mathcal{G} , we say that a subset W of M is \mathcal{G} -contractible if, for every basepoint $* \in W$, the image $\iota_*(\pi_1(W, *) \subset \pi_1(M, *))$ belongs to \mathcal{G} and we let $cat_{\mathcal{G}}(M)$ be the smallest number of open \mathcal{G} -contractible subsets of M that cover M . It turns out that, if \mathcal{G} is closed under subgroups and quotients, then $cat_{\mathcal{G}}(M) = cat_{\mathcal{A}_{\mathcal{G}}}(M)$, where $\mathcal{A}_{\mathcal{G}} = \{X : \pi_1(X, *) \in \mathcal{G}, \text{ for all } * \in X\}$ [8, Proposition 1]. For closed 3-manifolds, the cases for $\mathcal{G} = ame$ (the class of amenable groups) and $\mathcal{G} = solv$ (the class of solvable groups) have been considered in [9] and [8].

There do not seem to be any known results for $cat_K(M^3)$ when K is a compact 2-complex that is not homotopy equivalent to a closed 2-manifold. In view of the results for $K = \text{point}$, S^2 , and P^2 , one may consider, for example, complexes K that are wedges of these 2-manifolds. As a first result in this direction, we obtain in Theorem 3.5 a complete classification of $cat_K(M^3)$ for $K = \bigvee_n S^2$, the wedge of n copies of S^2 (for $n > 0$).

The case when $K = \bigvee_n P^2$, the wedge of n copies of P^2 , seems to be difficult, especially for $n \geq 3$, since then $\pi_1(K)$ is not a solvable group. The closed 3-manifolds M for which $cat_{P^2}(M) = 2$ are S^3 , P^3 , $P^3 \# P^3$, and $P^2 \times S^1$ [7], and it is easy to see that if M_1 and M_2 are manifolds in this list and $M = M_1 \# M_2$, then $cat_{P^2 \vee P^2}(M) = 2$. We conjecture that these are the only closed 3-manifolds M for which $cat_{P^2 \vee P^2}(M) = 2$.

2. K -CONTRACTIBLE SUBSETS

Let $M = M^n$ be a closed n -manifold and let K be a finite CW-complex.

Recall that a subset W of M is K -contractible (in M) if there are maps $f : W \rightarrow K$ and $\alpha : K \rightarrow M$ such that the inclusion $\iota : W \rightarrow M$ is homotopic to $\alpha \cdot f$. The K -category $cat_K(M)$ is the smallest number m such that M can be covered by m open K -contractible subsets.

W is π_1 -contractible (in M) if, for every basepoint $* \in W$, the inclusion $\iota : W \rightarrow M$ induces the trivial homomorphism $\iota_* : \pi_1(W, *) \rightarrow \pi_1(M, *)$. The π_1 -category $cat_{\pi_1}(M)$ is the smallest number k such that M can be covered by k open π_1 -contractible subsets.

More generally, for a nonempty class of groups \mathcal{G} , a subset W of M is \mathcal{G} -contractible if, for every basepoint $* \in W$, the image $\iota_*(\pi_1(W, *) \subset \pi_1(M, *))$ belongs to \mathcal{G} and $cat_{\mathcal{G}}(M)$ is the smallest number of open \mathcal{G} -contractible subsets of M that cover M .

It is easy to see that cat_K is a homotopy type invariant.

Note that a subset of a K -contractible set is K -contractible.

If \mathcal{G} is closed under subgroups, then a subset of a \mathcal{G} -contractible set is \mathcal{G} -contractible.

Lemma 2.1. (a) *Let M be a closed 3-manifold. If a 2-complex L is a retract of a 2-complex K , then*

$$2 \leq cat_K(M) \leq cat_L(M) \leq cat(M) \leq 4.$$

(b) *If, furthermore, K is simply connected, then*

$$1 \leq cat_{\pi_1}(M) \leq cat_K(M) \leq cat_L(M) \leq cat(M) \leq 4.$$

Proof. (a) Clearly, a set that is contractible in M is L -contractible. For an L -contractible set, there are maps $f : W \rightarrow L$ and $\alpha : L \rightarrow M$ such that $\alpha \cdot f \simeq \iota$. Let $j : L \rightarrow K$ be the inclusion and $r : K \rightarrow L$ be a retraction. Then $(\alpha r) \cdot (j f) \simeq \iota$. Thus, an L -contractible set is K -contractible. Since $id : H_3(M, \mathbb{Z}_2) \rightarrow H_3(M, \mathbb{Z}_2)$ does not factor through $H_3(K, \mathbb{Z}_2) = 0$, $2 \leq cat_K(M)$.

(b) If K is simply connected, it follows that a K -contractible set is π_1 -contractible. \square

For the case when $cat_K(M) = 2$ and $K = S^1$, it is shown in [9] that the two open K -contractible sets can be replaced by compact submanifolds that meet only along their boundaries. The same proof applies for any finite complex K to yield the following.

Proposition 2.2. *Suppose $cat_K M = 2$ ($cat_{\mathcal{G}}(M) = 2$, respectively). Then M can be expressed as a union of two compact, not necessarily connected, K -contractible (\mathcal{G} -contractible, respectively) n -submanifolds W_0 and W_1 such that $W_0 \cap W_1 = \partial W_0 = \partial W_1$.*

In particular, if $\text{cat}_K M = 2$, there are maps f_i and α_i (for $i = 0, 1$) such that the diagram below is homotopy commutative:

$$(*) \quad \begin{array}{ccc} W_i & \xrightarrow{\iota} & M \\ & \searrow f_i & \nearrow \alpha_i \\ & & K \end{array}$$

We now assume that M is a closed 3-manifold with $\text{cat}_K M = 2$ as in Proposition 2.2.

Let $\mathcal{R} = \mathbb{Z}$ or \mathbb{Z}_2 if M is orientable and \mathbb{Z}_2 if M is non-orientable. From the exact homology and cohomology sequences of (M, W_j) , Lefschetz duality, and $(*)$, we obtain a commutative diagram (for $0 \leq j \leq 3$):

$$\begin{array}{ccccc} & & & & H^{3-j}(K; \mathcal{R}) \\ & & & \nearrow \alpha_{1-i}^* & \downarrow f_{1-i}^* \\ H^{3-j}(M, W_{1-i}; \mathcal{R}) & \longrightarrow & H^{3-j}(M; \mathcal{R}) & \xrightarrow{\iota^*} & H^{3-j}(W_{1-i}; \mathcal{R}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_j(W_i; \mathcal{R}) & \xrightarrow{\iota_*} & H_j(M; \mathcal{R}) & \longrightarrow & H_j(M, W_i; \mathcal{R}) \\ & \searrow f_{i*} & \uparrow \alpha_{i*} & & \\ & & H_j(K; \mathcal{R}) & & \end{array}$$

In particular, we have an exact sequence

$$(**) \quad 0 \rightarrow \text{im } \iota_* \rightarrow H_j(M; \mathcal{R}) \rightarrow \text{im } \iota^* \rightarrow 0$$

and obtain the following lemma.

Lemma 2.3. *Suppose $\text{cat}_K(M) = 2$ and K is a 2-dimensional complex such that the \mathbb{Z}_2 -rank $\text{rk}(H_1(K; \mathbb{Z}_2)) \leq m$ and $\text{rk}(H^2(K; \mathbb{Z}_2)) \leq n$ ($m, n \geq 0$). Then $\text{rk}(H_1(M; \mathbb{Z}_2)) \leq m + n$.*

Lemma 2.4. *Suppose M is orientable with $\text{cat}_K(M) = 2$ and K is a 2-dimensional complex such that the rank $\text{rk}(H^1(K)) \leq m$ and $\text{rk}(H_2(K)) \leq n$ ($m, n \geq 0$). Then $\text{rk}(H_2(M)) \leq m + n$.*

3. $cat_K(M^3)$ FOR K A WEDGE OF S^2 'S

We use the following notation:

$\bigvee_m S^2$ denotes the wedge of m copies of S^2 (for $m > 0$).

$S^2 \tilde{\times} S^1$ denotes the trivial or non-trivial S^2 -bundle over S^1 .

$\#_m S^2 \tilde{\times} S^1$ denotes the connected sum of m S^2 -bundles over S^1 (for $m > 0$).

Example 3.1. If $cat_K(M) = 2$, where $K = \bigvee_n S^2$, then $rk(H_1(M; \mathbb{Z}_2)) \leq n$.

This example follows from Lemma 2.3 since $rk(H_1(K; \mathbb{Z}_2)) = 0$ and $rk(H^2(K; \mathbb{Z}_2)) = n$.

Example 3.2. For $K = \bigvee_m S^2$ and $M = \#_m S^2 \tilde{\times} S^1$, we have $cat_K(M) = 2$.

For simplicity, we only describe the case $m = 2$. Then $M = \hat{M}_1 \# \hat{M}_2$, where each \hat{M}_i has a decomposition $S_i^2 \times [-1, 0] \cup S_i^2 \times [0, 1]$. Let M_i be obtained from \hat{M}_i by deleting the interior of a small ball B_i such that ∂B_i is the union of two disks $D_{i,0}$ and $D_{i,1}$ with $D_{i,0} \subset S_i^2 \times [-1, 0]$ and $D_{i,1} \subset S_i^2 \times [0, 1]$. Then $M = M_1 \cup M_2 = W_1 \cup W_2$, where $W_1 = (S_1^2 \times [-1, 0] \cup_{D_{1,0}=D_{2,0}} S_2^2 \times [-1, 0])$ and $W_2 = \cup(S_1^2 \times [0, 1] \cup_{D_{1,1}=D_{2,1}} S_2^2 \times [0, 1])$. Each W_i deformation retracts to $S^2 \vee S^2$ and so W_i is $S^2 \vee S^2$ -contractible.

Lemma 3.3. If $K_n = \bigvee_n S^2$, $M_m = \#_m S^2 \tilde{\times} S^1$, and $1 \leq m \leq n$, then $cat_{K_n}(M_m) = 2$.

Proof. Clearly, $cat_{K_1}(M_1) = cat_{S^2}(S^2 \tilde{\times} S^1) = 2$. Since K_{n-1} is a retract of K_n , $cat_{K_n}(M_m) \leq cat_{K_{n-1}}(M_m)$, so if $m \leq n-1$, then $cat_{K_n}(M_m) \leq 2$ by induction on n . If $m = n$, then $cat_{K_n}(M_m) = 2$ by Example 3.2. \square

Corollary 3.4. $cat_{K_n}(M_m) = \begin{cases} 2 & \text{for } 1 \leq m \leq n \\ 3 & \text{for } m > n. \end{cases}$

Proof. If $cat_{K_n}(M_m) = 2$, then from Example 3.1 it follows that $m = rk(H_1(M_m; \mathbb{Z}_2)) \leq n$. On the other hand, $cat_{K_n}(M_m) \leq cat(M_m) \leq 3$ for all n and m , since M_m can be covered by three balls. Now the corollary follows from Lemma 3.3. \square

Theorem 3.5. Let $K = \bigvee_n S^2$ and let M be a closed 3-manifold. Then

$$cat_K(M) = \begin{cases} 2 & \text{if } M = S^3 \text{ or } M = \#_m S^2 \tilde{\times} S^1 \text{ with } m \leq n \\ 3 & \text{if } M = \#_m S^2 \tilde{\times} S^1 \text{ with } m > n \\ 4 & \text{otherwise.} \end{cases}$$

Proof. If $\pi_1(M)$ is not free, then $4 = cat_{\pi_1}(M) \leq cat_K(M) \leq 4$ by [5] and Lemma 2.1. If $\pi_1(M)$ is free, then (by Perelman) M is S^3 or $M = M_m$. Now Corollary 3.4 applies. \square

In particular, if $n = 1$, we obtain the following corollary as a special case.

$$\text{Corollary 3.6. } \text{cat}_{S^2}(M) = \begin{cases} 2 & \text{if } M = S^3 \text{ or } M = S^2 \tilde{\times} S^1 \\ 3 & \text{if } M = \#_m S^2 \tilde{\times} S^1 \text{ with } m > 1 \\ 4 & \text{otherwise.} \end{cases}$$

Note that the proof shows that in Theorem 3.5 the condition that $K = \bigvee_n S^2$ can be replaced by “ K is a simply connected 2-complex with $\text{rk}(H^2(K; \mathbb{Z}_2)) = n \geq 1$ and K retracts onto K_n .”

$$4. \text{cat}_{P^2 \vee P^2}(M) = 2$$

In this section, let $K = P^2 \vee P^2$.

We denote by $\#_m N^3$ a connected sum of m copies of the 3-manifold N^3 .

If W is a K -contractible subset of a 3-manifold M , then for every basepoint $* \in W$, the image $\iota_* \pi_1(W, *)$ is conjugate to a subgroup of $\alpha_* \pi_1(K, f(*))$, which is a 3-manifold group and a quotient of $\pi_1(K, f(*))$. Furthermore, if W is compact, $\iota_* \pi_1(W, *)$ is finitely generated. Suppose Q is a quotient group of a group G . We say that Q is a *3-manifold quotient* of G if Q is the fundamental group of a 3-manifold. With this notation, a compact K -contractible subset of M is \mathcal{G}_K -contractible, where \mathcal{G}_K denotes the set of finitely generated subgroups of 3-manifold quotients Q of $\pi_1(K, *)$ (for any basepoint $* \in K$), and $1 \leq \text{cat}_{\mathcal{G}_K}(M) \leq \text{cat}_K(M)$.

Lemma 4.1. *The set of subgroups of 3-manifold quotients of $\mathbb{Z}_2 * \mathbb{Z}_2$ is $\mathcal{G}_K = \{1, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_2\}$.*

Proof. Present $\mathbb{Z}_2 * \mathbb{Z}_2$ as the semi-direct product $G = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle r, s : s^2 = 1, srs^{-1} = r^{-1} \rangle$. Then every element of G can be uniquely written as r^m or $r^m s$, for some $m \in \mathbb{Z}$.

Let H be a subgroup of G . If $H \subset \langle r \rangle$, then $H = 1$ or $\cong \mathbb{Z}$; if $H \cap \langle r \rangle = 1$, then $H = 1$ or $\cong \mathbb{Z}_2$; in any other case, $H \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Now let H be a subgroup of a proper quotient $Q \cong G/N$ of G . If $N \not\subset \langle r \rangle$, then $Q \cong \mathbb{Z}_2$; hence, $H = 1$ or \mathbb{Z}_2 . This is the only case that can happen; otherwise, $N = \langle r^m \rangle$ for some integer $m \neq 0$, and then G/N is the finite dihedral group D_{2m} of order $2m$. In this case, if M^3 is a 3-manifold with finite fundamental group D_{2m} , then M^3 can be taken to be compact [10, Theorem 8.1] and ∂M^3 is a union of 2-spheres, so M^3 can be assumed closed. Then by John Milnor [11], every element of order 2 of $\pi_1(M^3)$ is central. However, for $m > 2$, this is not true for D_{2m} . If $m = 2$, then it is well known that D_{2m} , the four group, is not a 3-manifold group (see, for example, [10, Theorem 9.13]). \square

For a 3-manifold C , we denote by \hat{C} the manifold obtained by capping off all 2-sphere boundaries with 3-balls.

The compact 3-manifolds whose fundamental groups belong to \mathcal{G}_K are well known and are listed in the following proposition. Here $P^2 \times I \#_b P^2 \times I$ denotes the disk sum of two copies of $P^2 \times I$.

Proposition 4.2. *Let M be a compact 3-manifold. Then $\text{cat}_{\mathcal{G}_K}(M) = 1$ if and only if*

$$\hat{M} \in \{S^3, S^2 \tilde{\times} S^1, D^2 \tilde{\times} S^1, P^3, P^2 \times I, P^3 \# P^3, P^2 \times I \# P^2 \times I, P^3 \# P^2 \times I, P^2 \times I \#_b P^2 \times I\}.$$

The following lemma is proved in [9] for a more general class \mathcal{G} of groups.

Lemma 4.3. *Let M be a closed 3-manifold with $\text{cat}_{\mathcal{G}_K}(M) \leq 2$. Then there is a closed 2-sided surface F in M , not necessarily connected, such that F and $\overline{M - N(F)}$ are \mathcal{G}_K -contractible and every component of F is a 2-sphere or incompressible.*

Proof. We write $M = W_0 \cup W_1$ as in Proposition 2.2, with $F = \partial W_0 = \partial W_1$. For each component F' of F , $\text{im}(\pi_1(F') \rightarrow \pi_1(M))$ is contained in $\text{im}(\pi_1(W'_i) \rightarrow \pi_1(M))$, where W'_i is a component of W_i , and it follows that F and $\overline{M - N(F)}$ are \mathcal{G}_K -contractible. Now assume that F is a closed 2-sided surface in M of minimal complexity such that F and $\overline{M - N(F)}$ are \mathcal{G}_K -contractible. Here, the complexity $c(F)$ is the sum of all the (orientable or non-orientable) genera of the non-sphere components of F .

If F has compressible non-sphere components, then there is such a component F' with a compressing disk D that does not meet the other components of F . Let $D \times I$ be a regular neighborhood such that $(D \times I) \cap F = \partial D \times I$ and $\partial D \times 0$ is an essential curve in F' . For the component F'_1 of $F_1 = (F - \partial D \times I) \cup (D \times \partial I)$ that contains $D \times \{0\}$ or $D \times \{1\}$, $\text{im}(\pi_1(F'_1) \rightarrow \pi_1(M))$ is a subgroup of $\text{im}(\pi_1(F') \rightarrow \pi_1(M))$. Since F' is \mathcal{G}_K -contractible, so is F_1 .

Furthermore, if M' is the component of $\overline{M - N(F)}$ that contains F' but not D and if M'_1 is the component $M' \cup D \times I$ of $\overline{M - N(F_1)}$, then $\pi_1(M')$ and $\pi_1(M'_1)$ have the same image in $\pi_1(M)$. Since M' is \mathcal{G}_K -contractible, so is M'_1 .

Hence, F_1 and $\overline{M - N(F_1)}$ are \mathcal{G}_K -contractible and $c(F_1) < c(F)$, a contradiction. \square

Theorem 4.4. *Let M be a closed 3-manifold. Then $\text{cat}_{\mathcal{G}_K}(M) \leq 2$ if and only if $M = S^3 \#_m P^3 \#_n (P^2 \times S^1) \#_k (S^2 \tilde{\times} S^1)$, for some $k, m, n \geq 0$.*

Proof. By Lemma 4.3, there is a closed 2-sided surface F of minimal complexity in M such that F and $\overline{M - N(F)}$ are \mathcal{G}_K -contractible and every

component F' of F is a 2-sphere or incompressible. It follows that F' and every component C of $\overline{M - F} \times [0, 1]$ is π_1 -injective (i.e., the inclusions into M induce injections of fundamental groups). Since F' and C are \mathcal{G}_K -contractible, F' and C have fundamental groups belonging to \mathcal{G}_K . In particular, by Lemma 4.1, F' is a 2-sphere or projective plane, and \hat{C} is as in Proposition 4.2, except for $D^2 \tilde{\times} S^1$ and $P^2 \times I \#_b P^2 \times I$, since ∂C is incompressible.

If \hat{C} is one of $P^3 \# P^2 \times I$, $P^2 \times I \# P^2 \times I$, or $P^3 \# P^3$, let S_C be a 2-sphere splitting C into two manifolds homeomorphic to a punctured P^3 or a punctured $P^2 \times I$. Now let Σ be the union of the 2-manifolds of the form S_C and the 2-sphere components of F , then split M along Σ and cap off the 2-sphere boundary components with 3-balls. Every component of the resulting 3-manifold is homeomorphic to $S^2 \tilde{\times} S^1$, to P^3 , or to $P^2 \times S^1$. This shows that M is as in the theorem.

Conversely, if M is of this type, let W_0 be a regular neighborhood of a disjoint collection of 2-spheres and projective planes such that every component of $W_1 := \overline{M - W_0}$ is a punctured $S^2 \tilde{\times} S^1$, P^3 , or $P^2 \times I$. Then W_0 and W_1 are \mathcal{G}_K -contractible (in fact, $\{1, \mathbb{Z}, \mathbb{Z}_2\}$ -contractible), and $M = W_0 \cup W_1$. \square

Corollary 4.5. *Let M be a closed 3-manifold. If $\text{cat}_K(M) = 2$, then $M = S^3 \#_m P^3 \#_n (P^2 \times S^1) \#_k (S^2 \tilde{\times} S^1)$, with $0 \leq k + m + 2n \leq 4$.*

Proof. Since $1 \leq \text{cat}_{\mathcal{G}_K}(M) \leq \text{cat}_K(M)$, this follows from Theorem 4.4 and Lemma 2.3. \square

If M_i is a closed 3-manifold ($i = 1, 2$) with $\text{cat}_{P^2}(M_i) = 2$, then, by [7, Corollary 2], $M_i \in \{S^3, P^3, P^3 \# P^3, P^2 \times S^1\}$.

Thus, Corollary 4.5 implies that if $\text{cat}_K(M) = 2$ and M has no $S^2 \tilde{\times} S^1$ -factors, then $M = M_1 \# M_2$, where $\text{cat}_{P^2}(M_i) = 2$. The converse is given by the following.

Proposition 4.6. *If M_i is a closed 3-manifold with $\text{cat}_{P^2}(M_i) = 2$ for $i = 1, 2$, then $\text{cat}_K(M_1 \# M_2) = 2$.*

Proof. There is a decomposition $M_i = W_{i0} \cup W_{i1}$, with $W_{i0} \cap W_{i1} = \partial W_{i0} = \partial W_{i1}$, and where each $W_{ij} \in \{B^3, P_0^3, P^2 \times I\}$ (here, P_0^3 denotes the once-punctured P^3). For the connected sum $M = M_1 \# M_2$, choose 3-balls $B_i \subset M_i$ so that $\partial B_i \cap W_{ij} = D_{ij}$, a disk. Then $M = W_0 \cup W_1$, with $W_j = W_{1j} \cup W_{2j}$ ($j = 0, 1$), where in the union the disks D_{1j} and D_{2j} are identified. Now $W_j \simeq W_{1j} \vee W_{2j}$ is homotopy equivalent to B^3 , P^2 , or $P^2 \vee P^2$, hence K -contractible. \square

If M is orientable with $\text{cat}_K(M) = 2$, then $\text{rk}(H_2(M)) = 0$ by Lemma 2.4, and therefore M does not contain non-separating 2-spheres. So we obtain the following.

Corollary 4.7. *Let M be a closed orientable 3-manifold. Then*

$$\text{cat}_K(M) = 2 \text{ if and only if } M = S^3 \#_m P^3, \text{ with } 0 \leq m \leq 4.$$

Finally, we show that there can be at most one $S^2 \times S^1$ -factor if M is non-orientable.

Lemma 4.8. *Let $M^3 = N^3 \# (S^2 \times S^1) \# (S^2 \times S^1)$, where N^3 is a closed non-orientable 3-manifold. Then $\text{cat}_K(M^3) > 2$.*

Proof. Suppose $\text{cat}_K(M) \leq 2$ and let W_i be as in Proposition 2.2. Let $p : \tilde{M} \rightarrow M$ be the orientable 2-fold covering and $\tilde{W}_i = p^{-1}(W_i)$. Then, by [7, Lemma 1], \tilde{W}_i is \tilde{K}_i -contractible where \tilde{K}_i is the pullback of $K \xrightarrow{\alpha_i} M$. Now \tilde{K}_i is a (possibly disconnected) 2-fold covering of K and is homotopy equivalent to $S^2 \vee S^2 \vee S^1$, $P^2 \vee S^2 \vee P^2$, or two copies of $P^2 \vee P^2$, so $\text{rk}(H_1(\tilde{K}_i)) \leq 1$ and $\text{rk}(H^2(\tilde{K}_i)) \leq 2$. In the exact sequence (**) (before Lemma 2.3), $\iota_* : H_1(\tilde{W}_0) \rightarrow H_1(\tilde{M})$ factors through $H_1(\tilde{K}_0)$ and $\iota^* : H^2(\tilde{M}) \rightarrow H^2(\tilde{W}_1)$ factors through $H^2(\tilde{K}_1)$; hence, $\text{rk}(H_1(\tilde{M})) \leq 1 + 2 = 3$. However $\tilde{M} = \tilde{N}^3 \#_4(S^1 \times S^2)$, where \tilde{N}^3 is the orientable 2-fold cover of N^3 , so $\text{rk}(H_1(\tilde{M})) \geq 4$, a contradiction. \square

Now let $S^2 \tilde{\times} S^1$ denote the nontrivial S^2 -bundle over S^1 . Since $N^3 \# (S^2 \tilde{\times} S^1) = N^3 \# (S^2 \times S^1)$ for a non-orientable 3-manifold N^3 , we obtain the following corollary.

Corollary 4.9. *Let M be a closed non-orientable 3-manifold. If $\text{cat}_K(M) = 2$, then*

$$M = \begin{cases} (P^2 \times S^1) \# (P^2 \times S^1) & \text{or} \\ \#_m P^3 \# (P^2 \times S^1) \#_k (S^2 \times S^1) & \text{with } 0 \leq m + k \leq 2, 0 \leq k \leq 1, \text{ or} \\ \#_m P^3 \# (S^2 \tilde{\times} S^1) \#_k (S^2 \times S^1) & \text{with } 0 \leq m + k \leq 3, 0 \leq k \leq 1. \end{cases}$$

We conjecture that there are no factors that are S^2 -bundles over S^1 in this corollary; i.e., if M is closed and non-orientable, then $\text{cat}_K(M) = 2$ if and only if $M = (P^2 \times S^1) \# (P^2 \times S^1)$, $P^3 \# (P^2 \times S^1)$, or $P^3 \# P^3 \# (P^2 \times S^1)$.

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(Gómez-Larrañaga) CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS; A.P. 402;
 GUANAJUATO 36000, GTO. MÉXICO
E-mail address: `jcarlos@cimat.mx`

(González-Acuña) INSTITUTO DE MATEMÁTICAS UNAM; 62210 CUERNAVACA,
 MORELOS, MÉXICO AND
 CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS; A.P. 402; GUANAJUATO 36000, GTO.
 MÉXICO
E-mail address: `fico@matem.unam.mx`

(Heil) DEPARTMENT OF MATHEMATICS; FLORIDA STATE UNIVERSITY;
 TALLAHASSEE, FL 32306, USA
E-mail address: `heil@math.fsu.edu`