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PRESERVATION OF CONVERGENCE OF A SEQUENCE TO A SET

AKIRA IWASA, MASARU KADA, AND SHIZUO KAMO

Abstract. We say that a sequence of points converges to a set if every open set containing the set contains all but finitely many terms of the sequence. We investigate preservation of convergence of a sequence to a set in forcing extensions.

1. Introduction

Let $\langle X, \tau\rangle$ be a topological space and $\mathbb{P}$ a notion of forcing. Let $\mathbf{V}^{\mathbb{P}}$ denote the forcing extension of $\mathbf{V}$ by $\mathbb{P}$. In $\mathbf{V}^{\mathbb{P}}$, we define a topology $\tau^{\mathbb{P}} = \{\bigcup \mathcal{S} : \mathcal{S} \subseteq \tau\}$ on X, that is, $\tau^{\mathbb{P}}$ is the topology generated by τ in $\mathbf{V}^{\mathbb{P}}$. We say that a topological property φ is *preserved* by $\mathbb{P}$ if whenever $\langle X, \tau\rangle$ satisfies φ, $\langle X, \tau^{\mathbb{P}}\rangle$ also satisfies φ. First, let us observe the following.

Theorem 1.1. *Convergence of a sequence (to a point) is preserved by any forcing.*

Proof. If a sequence $\{x_n : n \in \omega\}$ in $\langle X, \tau\rangle$ converges to y, then in $\langle X, \tau^{\mathbb{P}}\rangle$, it still converges to y because in $\mathbf{V}^{\mathbb{P}}$, τ serves as a base for $\tau^{\mathbb{P}}$. □

By the above theorem, there is nothing to investigate about preservation of convergence of a sequence. So let us generalize the concept of convergence.

Definition 1.2. We say that a sequence of points $\{x_n : n \in \omega\}$ *converges to a set* A if $x_n \notin A$ for all $n \in \omega$ and if for every open set U containing A, there exists $k \in \omega$ such that for every $n \geq k$, $x_n \in U$.

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Let us illustrate the fact that convergence of a sequence to a set is not necessarily preserved by forcing.

Example 1.3. There exists a space X and a sequence $\{x_n : n \in \mathbb{N}\}$ in X such that

(1) in $\mathbf{V}$, the sequence $\{x_n : n \in \mathbb{N}\}$ converges to a set A, and
(2) in $\mathbf{V}^{\mathbb{P}}$ for some forcing $\mathbb{P}$, $\{x_n : n \in \mathbb{N}\}$ does not converge to A.

Proof. Let $[0,1]^{\mathbf{V}}$ be the unit interval in $\mathbf{V}$ equipped with the usual topology and let $\{q_n : n \in \mathbb{N}\}$ enumerate the rationals in $[0,1]^{\mathbf{V}}$. In the example,

- $X = [0,1]^{\mathbf{V}} \times [0,1]^{\mathbf{V}}$ equipped with the usual topology,
- $x_n = (q_n, \frac{1}{n})$ for each $n \in \mathbb{N}$,
- $A = [0,1]^{\mathbf{V}} \times \{0\}$, and
- $\mathbb{P}$ is any forcing notion which adjoins a new real.

The sequence $\{(q_n, \frac{1}{n}) : n \in \mathbb{N}\}$ converges to the set $[0,1]^{\mathbf{V}} \times \{0\}$. We shall show that in $\mathbf{V}^{\mathbb{P}}$, the sequence $\{(q_n, \frac{1}{n}) : n \in \mathbb{N}\}$ does not converge to the set $[0,1]^{\mathbf{V}} \times \{0\}$. Let r be a new real adjoined by $\mathbb{P}$ such that $0 < r < 1$. Take subsequences $\{q_{n_i} : i \in \mathbb{N}\}$ and $\{q_{m_i} : i \in \mathbb{N}\}$ of rationals converging to r such that

$$q_{n_1} < q_{n_2} < q_{n_3} < \cdots \ r \ \cdots < q_{m_3} < q_{m_2} < q_{m_1}.$$

Consider the following points:

$$\begin{gathered}(0,1),\ (q_{n_1}, \tfrac{1}{2n_1}),\ (q_{n_2}, \tfrac{1}{2n_2}),\ \cdots \\ (r,0)\ \cdots,\ (q_{m_2}, \tfrac{1}{2m_2}),\ (q_{m_1}, \tfrac{1}{2m_1}),\ (1,1).\end{gathered}$$

These points form a V-shape with $(r,0)$ being the tip of the letter V, and the tip $(r,0)$ is not in $[0,1]^{\mathbf{V}} \times \{0\}$. Using the V-shape, we can define an open set containing the set $[0,1]^{\mathbf{V}} \times \{0\}$ and missing points $(q_{n_i}, \frac{1}{n_i})$ and $(q_{m_i}, \frac{1}{m_i})$ for all $i \in \mathbb{N}$. This implies that in $\mathbf{V}^{\mathbb{P}}$, the sequence $\{(q_n, \frac{1}{n}) : n \in \mathbb{N}\}$ does not converge to the set $[0,1]^{\mathbf{V}} \times \{0\}$. □

In this note, we investigate under what circumstances convergence of a sequence to a set is preserved by forcing. Let us give definitions.

Definition 1.4. We say that a sequence $\{x_n : n \in \omega\}$ is of *discrete points* if, for each $k \in \omega$, $x_k \notin \overline{\{x_n : n \in \omega \setminus \{k\}\}}$. (In this paper, we only consider sequences of discrete points.)

We write $\{x_n\}_n$ for $\{x_n : n \in \omega\}$ and $\{x_{n_i}\}_i$ for $\{x_{n_i} : i \in \omega\}$.

A Tychonoff space X is said to be *pseudocompact* if every continuous real-valued function on X is bounded.

A point p is an *accumulation point* of a set A if for every neighborhood U of p, $U \cap A$ is infinite.

A point p is a *cluster point* of a family $\{A_n : n \in \omega\}$ if for every neighborhood U of p, $U \cap A_n \neq \emptyset$ for infinitely many $n \in \omega$.

A space X is said to be *perfect* if for every $x \in X$, $x \in \overline{X \setminus \{x\}}$.

$Fn(\kappa, 2)$ is the set of all finite partial functions from a cardinal κ to 2. (Forcing with $Fn(\kappa, 2)$ adjoins κ-many Cohen reals [7].)

A space X is said to be *scattered* if for every subspace $S \subseteq X$, there exists an $x \in S$ such that $x \notin \overline{S \setminus \{x\}}$.

For a space X, X can be uniquely represented as $X = P \cup S$, where P is a perfect set, S is a scattered set, and $P \cap S = \emptyset$; we say P is the *perfect kernel* of X and S is the *scattered kernel* of X ([3, Problem 1.7.10]).

In this paper, we mainly deal with Tychonoff spaces. The property that a space is Tychonoff is preserved by any forcing ([2, Lemma 22]).

2. Convergence of a Sequence to a Set

In this section, we study the concept of convergence of a sequence to a set. A proof of the following proposition is routine.

Proposition 2.1. *Let $\{x_n\}_n$ be a sequence of discrete points in a space X and let A be a subset of X such that $\{x_n\}_n \cap A = \emptyset$. The following are equivalent:*

(1) *$\{x_n\}_n$ converges to A;*
(2) *for every subsequence $\{x_{n_i}\}_i$ of $\{x_n\}_n$, $\overline{\{x_{n_i}\}_i} \cap A \neq \emptyset$.*

We use the following characterization of pseudocompact spaces.

Proposition 2.2 ([9, pp. 177–178]; [3, Theorem 3.10.23])**.** *For a Tychonoff space X, the following are equivalent:*

(1) *X is pseudocompact.*
(2) *If $\mathcal{U} = \{U_n : n \in \omega\}$ is a sequence of nonempty open subsets of X such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$, then $\mathcal{U}$ has a cluster point in X.*
(3) *For every decreasing sequence $U_1 \supseteq U_2 \supseteq \cdots$ of non-empty open subsets of X, $\bigcap_{n \in \omega} \overline{U_n} \neq \emptyset$.*

The following are equivalent conditions for a sequence to converge to a set.

Proposition 2.3. *Suppose that X is a Tychonoff space and that $\{x_n\}_n$ is a sequence of discrete points in X. The following are equivalent:*

(1) *$\{x_n\}_n$ converges to a set.*
(2) *$\{x_n\}_n$ converges to the set $\overline{\{x_n\}_n} \setminus \{x_n\}_n$.*

(3) *For every subsequence* $\{x_{n_i}\}_i$ *of* $\{x_n\}_n$*,* $\{x_{n_i}\}_i$ *has an accumulation point, that is,* $\overline{\{x_{n_i}\}_i} \setminus \{x_{n_i}\}_i \neq \emptyset$*.*
(4) *The closure* $\overline{\{x_n\}_n}$ *of* $\{x_n\}_n$ *is pseudocompact.*

Proof. (1) $\Longrightarrow$ (2). Suppose that a sequence $\{x_n\}_n$ does not converge to $\overline{\{x_n\}_n} \setminus \{x_n\}_n$. Take an arbitrary set $A \subseteq X$ such that $A \cap \{x_n\}_n = \emptyset$. We shall show that the sequence $\{x_n\}_n$ does not converge to A. By Proposition 2.1, there exists a subsequence $\{x_{n_i}\}_i$ such that $\overline{\{x_{n_i}\}_i} \cap [\overline{\{x_n\}_n} \setminus \{x_n\}_n] = \emptyset$. Since $\overline{\{x_{n_i}\}_i} \subseteq \overline{\{x_n\}_n}$, it must be the case that $\overline{\{x_{n_i}\}_i} \subseteq \{x_n\}_n$. Since $\{x_n\}_n \cap A = \emptyset$, we have $\overline{\{x_{n_i}\}_i} \cap A = \emptyset$. By Proposition 2.1, the sequence $\{x_n\}_n$ does not converges to A.

(2) $\Longrightarrow$ (3). Assume on the contrary that $\overline{\{x_{n_i}\}_i} = \{x_{n_i}\}_i$. Then $\overline{\{x_{n_i}\}_i} \cap [\overline{\{x_n\}_n} \setminus \{x_n\}_n] = \emptyset$, which implies that $\{x_n\}_n$ does not converge to $\overline{\{x_n\}_n} \setminus \{x_n\}_n$ by Proposition 2.1.

(3) $\Longrightarrow$ (4). Suppose $Y := \overline{\{x_n\}_n}$ is not pseudocompact. According to Proposition 2.2, there exists in Y a family of nonempty open sets $\mathcal{U} = \{U_i : i \in \omega\}$ such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$ and $\mathcal{U}$ has no cluster point in Y. For each $i \in \omega$, pick $x_{n_i} \in U_i$. $\{x_{n_i}\}_i$ does not have an accumulation point in Y. Since Y is closed, $\{x_{n_i}\}_i$ does not have an accumulation point in X either.

(4) $\Longrightarrow$ (1). It is clear that (2) implies (1), so it suffices to prove (4) implies (2). We assume on the contrary that $\{x_n\}_n$ does not converge to $\overline{\{x_n\}_n} \setminus \{x_n\}_n$. By Proposition 2.1, there exists a subsequence $\{x_{n_i}\}_i$ such that $\overline{\{x_{n_i}\}_i} \cap [\overline{\{x_n\}_n} \setminus \{x_n\}_n] = \emptyset$. This implies that $\overline{\{x_{n_i}\}_i} \subseteq \{x_n\}_n$. Since $\{x_n\}_n$ is a sequence of discrete points, it follows that $\overline{\{x_{n_i}\}_i} = \{x_{n_i}\}_i$. Thus, $\{\{x_{n_i}\} : i \in \omega\}$ is a family of open subsets of $\overline{\{x_n\}_n}$ with no cluster point. By Proposition 2.2, $\overline{\{x_n\}_n}$ is not pseudocompact. □

3. Preservation of Convergence to a Compact Set

In this section, we investigate preservation of convergence of a sequence to a *compact* set. Let us look at a proposition.

Proposition 3.1. *Let* $\langle X, \tau \rangle$ *be a compact space and* $\mathbb{P} = Fn(\kappa, 2)$ *for some cardinal* κ*. The following are equivalent:*

(1) $\langle X, \tau^{\mathbb{P}} \rangle$ *is compact.*
(2) $\langle X, \tau^{\mathbb{P}} \rangle$ *is countably compact.*
(3) $\langle X, \tau^{\mathbb{P}} \rangle$ *is pseudocompact.*

Proof. Clearly, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). In order to show (3) $\Longrightarrow$ (1), we note that any forcing preserves regularity and that adjoining Cohen reals preserves Lindelöfness (see [4]). Therefore, $\langle X, \tau^{\mathbb{P}} \rangle$ is a regular Lindelöf space and, in particular, it is normal ([3, Theorem 3.8.2]). Every normal pseudocompact space is countably compact ([3, Theorem 3.10.21]) and every countably compact Lindelöf space is compact. □

Here is a useful fact.

Proposition 3.2 ([5, Lemma 7]; [1, Proposition 5.5])**.** *For a compact Hausdorff space X, the following are equivalent:*

(1) *The compactness of X is preserved by any forcing.*
(2) *The compactness of X is preserved by adjoining a Cohen real.*
(3) *X is scattered.*

Using Proposition 3.2, we obtain the following theorem.

Theorem 3.3. *Let X be a Tychonoff space. Suppose that a sequence $\{x_n\}_n$ of discrete points in X converges to a compact set K. The following are equivalent:*

(1) *In $\mathbf{V}^{\mathbb{P}}$ with any forcing $\mathbb{P}$, the sequence $\{x_n\}_n$ still converges to K.*
(2) *In $\mathbf{V}^{Fn(\omega,2)}$, the sequence $\{x_n\}_n$ still converges to K.*
(3) *The closure $\overline{\{x_n\}_n}$ of $\{x_n\}_n$ is scattered.*

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3). Since $\{x_n\}_n$ converges to a compact set, the closure $\overline{\{x_n\}_n}$ is compact as well. Assume that $\overline{\{x_n\}_n}$ is not scattered. Then $\overline{\{x_n\}_n}$ is not compact in $\mathbf{V}^{Fn(\omega,2)}$ by Proposition 3.2. By Proposition 3.1, $\overline{\{x_n\}_n}$ is not pseudocompact in $\mathbf{V}^{Fn(\omega,2)}$. By Proposition 2.3, the sequence $\{x_n\}_n$ does not converge to any set in $\mathbf{V}^{Fn(\omega,2)}$.

(3) $\Longrightarrow$ (1). By Proposition 3.2, $\overline{\{x_n\}_n}$ remains compact in $\mathbf{V}^{\mathbb{P}}$. Therefore, in $\mathbf{V}^{\mathbb{P}}$, $\{x_n\}_n$ converges to $\overline{\{x_n\}_n} \setminus \{x_n\}_n$ by Proposition 2.3. It is not difficult to see that $\overline{\{x_n\}_n} \setminus \{x_n\}_n \subseteq K$. Thus, in $\mathbf{V}^{\mathbb{P}}$, the sequence $\{x_n\}_n$ converges to K. □

The following example shows that the assumption of the compactness of K is necessary in both of the implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) in Theorem 3.3. In section 4, we remove the assumption of the compactness of K in the implication (1) $\Longrightarrow$ (3).

Example 3.4 ([8])**.** For an infinite almost disjoint family $\mathcal{A}$ on ω, we define a topological space $\Psi(\mathcal{A})$ as follows. Let $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ as a set, each point from ω is isolated, and a neighborhood base of $A \in \mathcal{A}$ is the collection of sets of the form $\{A\} \cup (A \smallsetminus F)$ where F is a finite subset of A. Then $\Psi(\mathcal{A})$ is a scattered space and we have $\overline{\omega} = \Psi(\mathcal{A})$. $\Psi(\mathcal{A})$ is called the Mrówka space when $\mathcal{A}$ is a maximal almost disjoint family. We consider ω as a sequence of points in $\Psi(\mathcal{A})$. Note that $\mathcal{A}$ is an infinite closed discrete subspace of $\Psi(\mathcal{A})$ and so it is not compact.

Claim 3.5. *The sequence ω converges to $\mathcal{A}$ in $\Psi(\mathcal{A})$ if and only if $\mathcal{A}$ is a maximal almost disjoint family.*

Proof. Suppose that $\mathcal{A}$ is not maximal and let X be an infinite subset of ω which is almost disjoint from every $A \in \mathcal{A}$. Let $U = (\omega \setminus X) \cup \mathcal{A}$. Then U is an open set which contains $\mathcal{A}$ and misses infinitely many points from ω. On the other hand, suppose that ω does not converge to $\mathcal{A}$. Then there is an infinite subset X of ω which has no accumulation point in $\mathcal{A}$. By the definition of a neighborhood of $A \in \mathcal{A}$, this means that $X \cap A$ is finite for every $A \in \mathcal{A}$. □

Now we can easily observe the following fact. In $\mathbf{V}$, take a maximal almost disjoint family $\mathcal{A}$ on ω. Then ω converges to $\mathcal{A}$ in $\Psi(\mathcal{A})$. For a forcing notion $\mathbb{P}$, if $\mathcal{A}$ remains maximal in $\mathbf{V}^{\mathbb{P}}$, then ω still converges to $\mathcal{A}$ in $\mathbf{V}^{\mathbb{P}}$, and otherwise ω does not converge to $\mathcal{A}$ in $\mathbf{V}^{\mathbb{P}}$.

In order to show that the assumption of the compactness of K is necessary in the implication (3) $\Longrightarrow$ (2) in Theorem 3.3, take a maximal almost disjoint family $\mathcal{A}$ in $\mathbf{V}$ obtained by extending the set of all branches through $2^{<\omega}$ (identified with ω through a bijection). According to [7, VIII, Exercise A14], $\mathcal{A}$ is no longer maximal in $\mathbf{V}^{\mathbb{P}}$, where $\mathbb{P}$ is any forcing notion which adjoins a new real. Therefore, the convergence of ω to $\mathcal{A}$ in $\Psi(\mathcal{A})$ is destroyed by adjoining any new real.

To see that (2) $\Longrightarrow$ (1) in Theorem 3.3 does not hold without assuming K is compact, we note that if $\mathbf{V}$ satisfies the continuum hypothesis, then in $\mathbf{V}$ there is a maximal almost disjoint family $\mathcal{A}$ on ω which is still maximal in $\mathbf{V}^{\mathrm{Fn}(\omega,2)}$ [7, VIII, Theorem 2.3]. For such an $\mathcal{A}$, the convergence of ω to $\mathcal{A}$ in $\Psi(\mathcal{A})$ is preserved by $\mathrm{Fn}(\omega, 2)$ but destroyed by some forcing. (Maximality of any maximal almost disjoint family on ω can be destroyed by forcing: Use the ccc poset defined in [7, II, Definition 2.7], or just collapse the cardinality of the family to ω.)

4. Destroying Convergence to a Set

In this section, we show that the implication (1) $\Longrightarrow$ (3) in Theorem 3.3 holds without assuming that K is compact. In other words, we define a forcing notion which can destroy convergence of a sequence $\{x_n\}_n$ to a set, just assuming that $\overline{\{x_n\}_n}$ is not scattered. First, let us look at a lemma that says adjoining a real makes a perfect space non-pseudocompact.

Lemma 4.1. *Suppose that $\langle X, \tau\rangle$ is a Tychonoff space and that forcing with $\mathbb{P}$ adjoins a real. If $\langle X, \tau\rangle$ is perfect, then $\langle X, \tau^{\mathbb{P}}\rangle$ is not pseudocompact.*

Proof. If $\langle X, \tau\rangle$ is not pseudocompact, then there exists a continuous unbounded real-valued function f on X. f remains continuous and unbounded in $\mathbf{V}^{\mathbb{P}}$, so $\langle X, \tau^{\mathbb{P}}\rangle$ is not pseudocompact.

Now we assume that $\langle X, \tau\rangle$ is pseudocompact. We construct a Cantor scheme [6, Definition 6.1, Theorem 6.2], which is a family of non-empty open sets $\{U_s : s \in 2^{<\omega}\}$ such that

(1) $U_{s\frown 0} \cap U_{s\frown 1} = \emptyset$ for $s \in 2^{<\omega}$;
(2) $\overline{U_{s\frown i}} \subseteq U_s$ for $s \in 2^{<\omega}$ and $i \in \{0, 1\}$.

Let $U_\emptyset = X$. Given U_s for $s \in 2^{<\omega}$, pick $x \in U_s$ and $y \in U_s$ with $x \neq y$. (This is possible because there is no isolated point.) Using regularity, choose open sets $U_{s\frown 0}$ and $U_{s\frown 1}$ such that $U_{s\frown 0} \cap U_{s\frown 1} = \emptyset$, $x \in U_{s\frown 0} \subseteq \overline{U_{s\frown 0}} \subseteq U_s$, and $y \in U_{s\frown 1} \subseteq \overline{U_{s\frown 1}} \subseteq U_s$.

Since X is pseudocompact, $\bigcap\{\overline{U_s} : s \subseteq r\} \neq \emptyset$ for each $r \in 2^\omega$ by Proposition 2.2. Working in $\mathbf{V}^{\mathbb{P}}$, take a generic real $r^* \in 2^\omega \setminus \mathbf{V}$.

Claim. $\bigcap\{\overline{U_s} : s \subseteq r^*\} = \emptyset$.

This claim implies that $\langle X, \tau^{\mathbb{P}}\rangle$ is not pseudocompact by Proposition 2.2 and completes the proof of the lemma.

Proof of Claim: Assume on the contrary that $\bigcap\{\overline{U_s} : s \subseteq r^*\} \neq \emptyset$ and pick $x \in \bigcap\{\overline{U_s} : s \subseteq r^*\}$. It is not difficult to see that for every $r \in 2^\omega \cap \mathbf{V}$, $x \notin \bigcap\{\overline{U_s} : s \subseteq r\}$. Observe that

$$\bigcup_{r \in 2^\omega \cap \mathbf{V}} \left[\bigcap\{\overline{U_s} : s \subseteq r\}\right] = \bigcap_{n \in \omega} \left[\bigcup\{\overline{U_s} : s \in 2^n\}\right].$$

Since x does not belong to the set on the left-hand side, $x \notin \bigcup\{\overline{U_s} : s \in 2^n\}$ for some $n \in \omega$. This implies that $x \notin \overline{U_{r^*\restriction n}}$, which contradicts the fact that $x \in \overline{U_s}$ for all $s \subseteq r^*$. □

The following lemma is crucial.

Lemma 4.2. *Let X be a Tychonoff space. Suppose that*

(1) *a sequence $\{x_i\}_i$ of discrete points of X converges to a set,*
(2) *the closure $\overline{\{x_i\}_i}$ of $\{x_i\}_i$ is not scattered, and*
(3) *the perfect kernel of $\overline{\{x_i\}_i}$ is not pseudocompact.*

Then there is a forcing notion $\mathbb{Q}$, satisfying the ccc such that in $\mathbf{V}^{\mathbb{Q}}$, the sequence $\{x_i\}_i$ no longer converges to the set.

Proof. Let P be the perfect kernel of $\overline{\{x_i\}_i}$. Since P is non-pseudocompact, by Proposition 2.2 we find a family $\{V_n : n < \omega\}$ of pairwise disjoint nonempty open subsets of P without cluster point in P. For each n, pick any point d_n from V_n and set $D = \{d_n : n < \omega\}$. Then D does not accumulate anywhere in P, and neither does it in X since P is closed in X. We can find, for each n, a neighborhood U_n of d_n in X so that U_n's are pairwise disjoint. For each n, let $\mathcal{U}_n$ be a neighborhood base of d_n inside U_n (e.g., $\mathcal{U}_n = \{V : V \text{ open in } X, d_n \in V \subseteq U_n\}$). Also, for each n, find a subset B_n of $U_n \cap \{x_i\}_i$ such that $d_n \in \overline{B_n}$.

We define a forcing notion $\mathbb{Q}$ by the following. A condition p of $\mathbb{Q}$ is of the form $p = (s^p, W^p)$, where

(1) $s^p \in \bigcup_{l<\omega}(\prod_{n<l} B_n)$, and
(2) $W^p \in \prod_{n<\omega} \mathcal{U}_n$.

For $p = (s^p, W^p)$, $q = (s^q, W^q)$ in $\mathbb{Q}$, $p \leq_{\mathbb{Q}} q$ if

(1) $s^p \supseteq s^q$
(2) for all $n < \omega$, $W^p(n) \subseteq W^q(n)$, and
(3) for all $n \in dom(s^p) \setminus dom(s^q)$, $s^p(n) \in W^q(n)$.

We observe that for each $s \in \bigcup_{l<\omega}(\prod_{n<l} B_n)$, $\{p \in \mathbb{Q} : s^p = s\}$ is centered (every finite subset has a lower bound), and therefore the set $\mathbb{Q}$ ordered by $\leq_{\mathbb{Q}}$ is a σ-centered (and hence ccc) forcing poset.

Let G be any $\mathbb{Q}$-generic filter over $\mathbf{V}$, and in $\mathbf{V}[G]$, let $S = \bigcup\{s^p : p \in G\}$. For each $n < \omega$, $\{p \in \mathbb{Q} : |s^p| \geq n\}$ is dense in $\mathbb{Q}$ and so S is an infinite subsequence of $\{x_i\}_i$.

Claim. S does not accumulate anywhere in X.

By Proposition 2.1, this claim implies that the sequence $\{x_i\}_i$ does not converge to any set and hence finishes the proof of the theorem.

Proof of Claim: Fix $x \in X$. We shall show that x is not an accumulation point of S. Working in $\mathbf{V}$, since D has no accumulation point in X, we can find a neighborhood V of x such that $\overline{V}$ meets D in at most one point. For each $n < \omega$ with $d_n \notin \overline{V}$, choose $H_n \in \mathcal{U}_n$ so that $H_n \cap V = \emptyset$. If $d_n \in \overline{V}$, then pick any $H_n \in \mathcal{U}_n$. The set $\{p \in \mathbb{Q} : (\forall n < \omega)(W^p(n) \subseteq H_n)\}$ is dense in $\mathbb{Q}$, so we can find $p \in G$ such that $W^p(n) \subseteq H_n$ for all $n < \omega$. This implies that $S(n) \in W^p(n)$ for all $n < \omega$, and hence S does not accumulate at x. □

Combining Lemma 4.1 and Lemma 4.2, we prove the main theorem in this section.

Theorem 4.3. *Let X be a Tychonoff space. Suppose that*

(1) *a sequence $\{x_i\}_i$ of discrete points in X converges to a set, and*
(2) *the closure $\overline{\{x_n\}_i}$ of $\{x_i\}_i$ is not scattered.*

Then there is a forcing notion $\mathbb{P}$, satisfying the ccc such that in $\mathbf{V}^{\mathbb{P}}$, the sequence $\{x_i\}_i$ no longer converges to the set.

Proof. Let P be the perfect kernel of $\overline{\{x_i\}_i}$. By assumption, $P \neq \emptyset$. Let $\mathbb{P}$ be a forcing notion which satisfies the ccc and adjoins a real. By Lemma 4.1, in $\mathbf{V}^{\mathbb{P}}$, P is not pseudocompact. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for the poset defined in Lemma 4.2. Then a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies the ccc and in $\mathbf{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ the sequence $\{x_i\}_i$ does not converge to the set by Lemma 4.2. □

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