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A NOTE ON GENERALIZED ORDERED TOPOLOGICAL PRODUCTS

AI-JUN XU AND WEI-XUE SHI

Abstract. In this paper, we present an equivalent definition of a generalized ordered topological product (GOTP) $(X * Y)$ of two generalized ordered spaces X and Y by a mapping p. Moreover, we investigate the mapping p and show p is a continuous mapping from a GOTP$(X * Y)$ to generalized ordered space X under some conditions. Finally, we give other results on the mapping p.

1. Introduction

In [4] and [5], we introduced a new topology on the lexicographic product set $X \times Y$ for two generalized ordered (GO) spaces X and Y and investigated the relationship of properties, such as Lindelöfness, monotone Lindelöfness, paracompactness, and perfectness, of the two GO-spaces and their generalized ordered topological product. However, the definition of the generalized ordered topological product (GOTP) $(X * Y)$ of two GO spaces X and Y (in [4] and [5]) is somewhat complicated and has many parts. In this paper, for GO-spaces X and Y, let $p : X * Y \to X$ be the projection. We refine the definition of the GOTP in Definition 2.3 by the mapping p. We show that if Y has two endpoints, then the mapping p is a closed quotient mapping, and if Y has either a left or a right endpoint, but not both, then X has no neighbor points if and only if p is a continuous mapping. In addition, we know that mapping p is not

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necessarily open by Example 2.12, but we give other interesting results (see Theorem 2.14 and Theorem 2.15).

In this paper, we reserve the symbol $\mathbb{N}$ for the set of positive integers. A *linearly ordered topological space* (*LOTS*) is a triple $(X, \lambda_X, \leq)$, where $(X, \leq)$ is a linearly ordered set and λ_X is the interval topology on $(X, \leq)$. A *generalized ordered space*(*GO-space*) is a triple $(X, \tau_X, \leq)$, where $(X, \leq)$ is a linearly ordered set and τ_X is a topology on $(X, \leq)$ such that $\lambda_X \subseteq \tau_X$ and τ_X has a base consisting of order convex sets, where a set A is called *order convex* if $x \in A$ for every x lying between two points of A. If m and n are points of a GO-space X such that $m < n$ and $(m, n) = \emptyset$, then m and n are said to be *neighbor points* in X; m is called the *left neighbor point* of n and n is the *right neighbor point* of m.

For the undefined terminology and notions, refer to [1] and [3].

2. Main Results

For GO-spaces X and Y, denote by p the mapping of GOTP$(X * Y)$ to X assigning the point $\langle x, y\rangle \in$ GOTP$(X * Y)$ to the point $x \in X$. For a GO-space $X = (X, \tau_X, <)$, let

$$\begin{aligned} L_X &= \{x \in X \mid (\leftarrow, x] \in \tau_X - \lambda_X\}, \\ R_X &= \{x \in X \mid [x, \rightarrow) \in \tau_X - \lambda_X\}, \text{ and} \\ I_X &= \{x \in X \mid x \text{ is an isolated point of } X\}, \end{aligned}$$

where λ_X is the usual order topology on X.

In the following, we present a definition of a GOTP that is equivalent to that of [4] and [5] but is often easier to use.

Definition 2.1 ([2])**.** Let $(X, <_X)$ and $(Y, <_Y)$ be linearly ordered sets. Then the lexicographic product $X * Y$ of $(X, <_X)$ and $(Y, <_Y)$ is defined as the ordered set $(X \times Y, \lessdot)$ where $\lessdot$ is the lexicographic ordering; i.e., if $a = \langle x_1, y_1\rangle$ and $b = \langle x_2, y_2\rangle \in X \times Y$, then

$$a \lessdot b \text{ if and only if } x_1 <_X x_2 \text{ or } x_1 = x_2 \text{ and } y_1 <_Y y_2.$$

Definition 2.2 ([4], [5])**.** Let $(X, \tau_X, <_X)$ and $(Y, \tau_Y, <_Y)$ be GO-spaces, let λ_X and λ_Y be the usual interval topology on X and Y, respectively, and let λ_{X*Y} be the usual interval topology on the linearly ordered set $X * Y$.

We mean a topology on $X * Y$ which has a subbase

$$\begin{aligned} \mathcal{B} =& \lambda_{X*Y} \cup \tau_R \cup \tau_L \\ &\cup \{[\langle x, y\rangle, \rightarrow) \subseteq X * Y \mid x \in X,\ y \in Y \text{ and } [y, \rightarrow) \in \tau_Y - \lambda_Y\} \\ &\cup \{(\leftarrow, \langle x, y\rangle] \subseteq X * Y \mid x \in X,\ y \in Y \text{ and } (\leftarrow, y] \in \tau_Y - \lambda_Y\}, \end{aligned}$$

where either

$$\tau_R = \emptyset \text{ and } \tau_L = \emptyset, \text{ if } Y \text{ does not have endpoints,}$$

or

$$\tau_R = \{[\langle x, y_0\rangle, \rightarrow) \mid x \in X \text{ and } [x, \rightarrow) \in \tau_X - \lambda_X\} \text{ and } \tau_L = \emptyset,$$
$$\text{if } Y \text{ has a left endpoint } y_0, \text{ but no right one,}$$

or

$$\tau_R = \emptyset \text{ and } \tau_L = \{(\leftarrow, \langle x, y_1\rangle] \mid x \in X \text{ and } (\leftarrow, x] \in \tau_X - \lambda_X\},$$
$$\text{if } Y \text{ has a right endpoint } y_1, \text{ but no left one,}$$

or

$$\begin{aligned}\tau_R &= \{[\langle x, y_0\rangle, \rightarrow) \mid x \in X \text{ and } [x, \rightarrow) \in \tau_X - \lambda_X\} \text{ and}\\ \tau_L &= \{(\leftarrow, \langle x, y_1\rangle] \mid x \in X \text{ and } (\leftarrow, x] \in \tau_X - \lambda_X\},\\ &\qquad \text{if } Y \text{ has both a left endpoint } y_0 \text{ and a right endpoint } y_1.\end{aligned}$$

Definition 2.3. Let $(X, \tau_X, <_X)$ and $(Y, \tau_Y, <_Y)$ be GO-spaces and let λ_X, λ_Y, and λ_{X*Y} be the usual interval topology on X, Y, and $X * Y$, respectively. The generalized ordered topology $\mathcal{O}$ on $X * Y$ is generated by the subbase

$$\begin{aligned}\mathcal{P} =& \lambda_{X*Y} \cup \{p^{-1}([x, \rightarrow)) \mid x \in X \text{ and } [x, \rightarrow) \in \tau_X - \lambda_X\}\\ &\cup \{p^{-1}((\leftarrow, x]) \mid x \in X \text{ and } (\leftarrow, x] \in \tau_X - \lambda_X\}\\ &\cup \{[\langle x, y\rangle, \rightarrow) \subseteq X * Y \mid x \in X,\ y \in Y \text{ and } [y, \rightarrow) \in \tau_Y - \lambda_Y\}\\ &\cup \{(\leftarrow, \langle x, y\rangle] \subseteq X * Y \mid x \in X,\ y \in Y \text{ and } (\leftarrow, y] \in \tau_Y - \lambda_Y\}.\end{aligned}$$

As said in [5], for different ordered sets, the orderings may be different. But in almost all cases, we can distinguish them from their contexts. So we will use the symbol $<$ for all the orderings unless another symbol is necessary to avoid confusion.

Now we show that the generalized ordered topology $\mathcal{O}$ in Definition 2.3 and the generalized ordered topology τ_{X*Y} in Definition 2.2 are the same generalized ordered topology on lexicographic product $X * Y$.

Proposition 2.4. *Let $(X, \tau_X, <_X)$ and $(Y, \tau_Y, <_Y)$ be GO-spaces. The generalized ordered topology $\mathcal{O}$ on the lexicographic product $X * Y$ (defined in Definition 2.3) coincides with the generalized ordered topology τ_{X*Y} on the lexicographic product $X * Y$ (defined in Definition 2.2).*

Proof. Let $\mathcal{P}$ and $\mathcal{B}$ be the subbases defined in Definition 2.3 and Definition 2.2, respectively. Now we show that $\mathcal{O} = \tau_{X*Y}$. There are four cases to consider.

Case 1: Y has a left endpoint y_0, but no right one. Then, for each $x \in X$ and $[x, \rightarrow) \in \tau_X - \lambda_X$, $p^{-1}([x, \rightarrow)) = [\langle x, y_0\rangle, \rightarrow)$. By Definition 2.2 and Definition 2.3, it is clear that $\mathcal{B} \subseteq \mathcal{P}$ and $\tau_{X*Y} \subseteq \mathcal{O}$. In addition, for each $x \in X$, if $(\leftarrow, x] \in \tau_X - \lambda_X$, then $p^{-1}((\leftarrow, x]) = \cup\{(\leftarrow, \langle x, y\rangle)|y \in Y\} \subseteq \lambda_{X*Y}$. Hence, each member of $\mathcal{P}$ is a member of τ_{X*Y}. So, $\mathcal{O} \subseteq \tau_{X*Y}$. Consequently, $\tau_{X*Y} = \mathcal{O}$.

Case 2: Y has a right endpoint y_0, but no left one.

Case 3: Y has both a left endpoint y_0 and a right endpoint y_1.

Case 4: Y does not has endpoints.

Cases 2, 3, and 4 can be proved by some modifications. □

In the following, we say that the space $(X * Y, \tau_{X*Y})$ is the GOTP of GO-spaces $(X, \tau_X, <_X)$ and $(Y, \tau_Y, <_Y)$ and denote it by GOTP$(X * Y)$. In GOTP$(X * Y)$, the mapping p is not necessarily a continuous mapping (see Example 2.7). But, if Y has two endpoints, the mapping p is not only a continuous mapping but also a closed quotient mapping (see Theorem 2.6).

Lemma 2.5. *Let X and Y be GO-spaces and y_0 (y_1) be the left (right) endpoint of Y.*

(1) *$[x, \rightarrow)$ is open in X iff $[\langle x, y_0\rangle, \rightarrow)$ is open in* GOTP$(X * Y)$.
(2) *$(\leftarrow, x]$ is open in X iff $(\leftarrow, \langle x, y_1\rangle]$ is open in* GOTP$(X * Y)$.
(3) *x is isolated in X iff $\{x\} * Y$ is open in* GOTP$(X * Y)$.

Proof. (1) Necessity. Assume $[x, \rightarrow)$ is open in X. If x has an immediate predecessor x' in X, then $\langle x', y_1\rangle$ is an immediate predecessor of $\langle x, y_0\rangle$ in $X * Y$. So $[\langle x, y_0\rangle, \rightarrow)$ is open in GOTP$(X * Y)$. If x does not have an immediate predecessor in X, there are two cases to consider:

(a) x is not a left endpoint of X. Then $[x, \rightarrow) \in \tau_X - \lambda_X$. Hence, $[\langle x, y_0\rangle, \rightarrow)$ is open in GOTP$(X * Y)$ by Definition 2.3.

(b) x is a left endpoint of X. Then $\langle x, y_0\rangle$ is a left endpoint of $X * Y$. Hence, $[\langle x, y_0\rangle, \rightarrow)$ is open in GOTP$(X * Y)$.

Sufficiency. Assume $[\langle x, y_0\rangle, \rightarrow)$ is open in GOTP$(X * Y)$. If x is a left endpoint of X, then $[x, \rightarrow)$ is open in X. If x is not a left endpoint of X, there are two possibilities:

(i) x has an immediate predecessor x' in X. Then $[x, \rightarrow)$ is open in X.

(ii) x does not have an immediate predecessor in X. Then $\langle x, y_0\rangle$ does not have an immediate predecessor in GOTP$(X * Y)$ and $[\langle x, y_0\rangle, \rightarrow) \in \tau_{X*Y} - \lambda_{X*Y}$. Hence, $[x, \rightarrow)$ is open in X. Otherwise, $[x, \rightarrow) \notin \tau_X - \lambda_X$. By Definition 2.3, $[\langle x, y_0\rangle, \rightarrow) \notin \tau_{X*Y} - \lambda_{X*Y}$. Contradiction.

The proofs of (2) and (3) are similar. □

Theorem 2.6. *Let X and Y be GO-spaces. If Y has both a left and a right endpoint, then*

(1) *p is a quotient mapping,*
(2) *X is a quotient space,*
(3) *p is a closed mapping.*

Proof. (1) Let y_0 be a left endpoint of Y and y_1 a right endpoint of Y.

Claim 1. O is open in X if $p^{-1}(O)$ is open in $\mathrm{GOTP}(X*Y)$ for every subset O of X. It suffices to prove that, for each $x \in O$, there is an open neighborhood $U(x)$ of x in X such that $U(x) \subseteq O$. There are four cases:

(i) $x \in I_X$. Then let $U(x) = \{x\}$.
(ii) $x \notin I_X$ and $[x, \rightarrow)$ is open in X. Then $(\leftarrow, x]$ is not open in X. By Lemma 2.5(2), $(\leftarrow, \langle x, y_1\rangle]$ is not open in $\mathrm{GOTP}(X*Y)$. Thus, there exists a $v > x$ such that $[\langle x, y_0\rangle, \langle v, y_0\rangle) \subseteq p^{-1}(O)$ since $\langle x, y_1\rangle \in p^{-1}(O)$. Therefore, let $U(x) = [x, v)$.
(iii) $x \notin I_X$ and $(\leftarrow, x]$ is open in X.
(iv) Neither $[x, \rightarrow)$ nor $(\leftarrow, x]$ is open in X.

The proofs of (iii) and (iv) are similar to (ii).

Claim 2. $p^{-1}(O)$ is open in $\mathrm{GOTP}(X*Y)$ for every open convex subset O of X. If O is an open convex subset of X, then there exist $a, b \in X$ such that $O = [a, b)$ or $O = (a, b]$ or $O = (a, b)$ or $O = [a, b]$ or $O = [a, \rightarrow)$ or $O = (\leftarrow, b]$ or $O = (a, \rightarrow)$ or $O = (\leftarrow, b)$. If $O = (a, b)$ or $O = (a, \rightarrow)$ or $O = (\leftarrow, b)$, it is clear that $p^{-1}(O)$ is open in $\mathrm{GOTP}(X*Y)$. Next, we only show the case that $O = [a, b)$; the other cases are proved similarly. If a is a left endpoint of X, then $p^{-1}(O) = [\langle a, y_0\rangle, \langle b, y_0\rangle)$ is open in $\mathrm{GOTP}(X*Y)$. If a is not a left endpoint of X, there are two possibilities:

(i) a has an immediate predecessor a' in X. Then

$$p^{-1}(O) = [\langle a, y_0\rangle, \langle b, y_0\rangle) = (\langle a', y_1\rangle, \langle b, y_0\rangle).$$

Hence, $p^{-1}(O)$ is open in $\mathrm{GOTP}(X*Y)$.

(ii) a does not have an immediate predecessor in X. Then $[a, \rightarrow) \in \tau_X - \lambda_X$. Thus, $[\langle a, y_0\rangle, \rightarrow)$ is open in $\mathrm{GOTP}(X*Y)$ by Definition 2.3. So, $p^{-1}(O)$ is open in $\mathrm{GOTP}(X*Y)$ since $p^{-1}(O) = [\langle a, y_0\rangle, \rightarrow) \cap (\leftarrow, \langle b, y_0\rangle)$.

Therefore, p is a quotient mapping by Claim 1 and Claim 2.

(2) is obvious by (1).

(3) Suppose that A is a closed subset of $\mathrm{GOTP}(X*Y)$. Without loss of generality, we assume that A is a convex subset. Then there exist $x_a, x_b \in X$ and $y_a, y_b \in Y$ such that $A = [\langle x_a, y_a\rangle, \langle x_b, y_b\rangle]$ or $A = [\langle x_a, y_a\rangle, \langle x_b, y_b\rangle)$ or $A = (\langle x_a, y_a\rangle, \langle x_b, y_b\rangle]$ or $A = (\langle x_a, y_a\rangle, \langle x_b, y_b\rangle)$ or $A = [\langle x_a, y_a\rangle, \rightarrow)$ or $A = (\langle x_a, y_a\rangle, \rightarrow)$ or $A = (\leftarrow, \langle x_b, y_b\rangle]$ or $A =$

$(\leftarrow, \langle x_b, y_b\rangle)$. Next, we only show the case that $A = [\langle x_a, y_a\rangle, \langle x_b, y_b\rangle)$. There are two cases to consider:

(i) y_b is not a left endpoint of Y. Then $p(A) = [x_a, x_b]$ is a closed subset of X.

(ii) y_b is a left endpoint of Y. Then $[\langle x_b, y_b\rangle, \rightarrow)$ is open in GOTP$(X * Y)$. So $[x_b, \rightarrow)$ is open in X by Lemma 2.5(1). Hence, $p(A) = [x_a, x_b)$ is a closed subset of X. □

In Lemma 2.5 and Theorem 2.6, the condition that Y has two endpoints cannot be removed (see Example 2.7 and Example 2.8).

Example 2.7. Let $X = Y = \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Then for each $n \in X - \{1\}$, $\{n\}$ is open in X, and neither $[\langle n, 1\rangle, \rightarrow)$ nor $\{n\} * Y$ is open in GOTP$(X * Y)$. It is clear that $p^{-1}(\{n\}) = \{n\} * Y$. Hence, p is not continuous.

Example 2.8. Let S be the Sorgenfrey line and $Y = [0, 1)$ with usual interval topology. Then for each $x \in S$, $[x, \rightarrow) \in \tau_S - \lambda_S$, and $\{x\} * Y$ is open in GOTP$(X * Y)$ by Definition 2.3. In addition, $p^{-1}(\{x\}) = \{x\} * Y$ and $\{x\}$ is not open in S. Hence, p is not a quotient mapping. However, it is easy to prove that p is continuous.

Now we consider the cases in which Y in GOTP$(X * Y)$ does not have the left endpoint or the right endpoint. Obviously, if Y has neither, then GOTP$(X * Y)$ is the topological sum of $|X|$ many copies of Y so that the topology of X does not matter. So it is clear that p is a continuous mapping.

Theorem 2.9. *Let X and Y be GO-spaces. If Y has either the left or the right endpoint, but not both, then X has no neighbor points if and only if p is a continuous mapping.*

Proof. Suppose that Y has the left endpoint but not the right one and y_0 is the left endpoint of Y.

Necessity. Suppose that X has no neighbor points. Let O be any open convex subset of X. We only consider the case that $O = [a, b)$, where $a, b \in X$. The proofs of the other cases are similar. There are two cases to consider:

(1) a is the left endpoint of X. Then $p^{-1}(O)$ is open in GOTP$(X * Y)$ by Definition 2.3.

(2) a is not the left endpoint of X. Then $[a, b) \in \tau_X - \lambda_X$ since a does not have a left neighbor point in X. Thus, $p^{-1}(O)$ is open in GOTP$(X*Y)$ by Definition 2.3. Hence, p is a continuous mapping.

Sufficiency. Suppose that p is a continuous mapping. We assume that X has neighbor points and m is the left neighbor point of n, where $m, n \in$

X. Then $[n,\rightarrow)$ is open in X and $[\langle n, y_0\rangle,\rightarrow) = p^{-1}([n,\rightarrow))$ is open in $\mathrm{GOTP}(X*Y)$. However, $\langle n, y_0\rangle \notin \mathrm{Int}([\langle n, y_0\rangle,\rightarrow))$ by Definition 2.3. Contradiction. Hence, X has no neighbor points.

For the case that Y has the right endpoint but not the left one, the proof is similar. □

Remark 2.10. In Definition 2.3, by Theorem 2.6, if Y has two endpoints, then the subbase $\mathcal{P}$ of $\mathrm{GOTP}(X*Y)$ can be improved to

$$\begin{aligned}\mathcal{P} =&\lambda_{X*Y} \cup \{p^{-1}(U) \mid U \in \tau_X\}\\ &\cup \{[\langle x,y\rangle,\rightarrow) \subseteq X*Y \mid x \in X,\ y \in Y \text{ and } [y,\rightarrow) \in \tau_Y - \lambda_Y\}\\ &\cup \{(\leftarrow,\langle x,y\rangle] \subseteq X*Y \mid x \in X,\ y \in Y \text{ and } (\leftarrow,y] \in \tau_Y - \lambda_Y\}.\end{aligned}$$

In addition, we have other results on the mapping p.

Proposition 2.11. *For every open cover $\mathcal{U}$ of* $\mathrm{GOTP}(X*Y)$*,* $\mathrm{Int}p(\mathcal{U}) = \{\mathrm{Int}p(U) \mid U \in \mathcal{U}\}$ *covers the set $R_X \cup L_X \cup I_X$, and if X has the left (or right) endpoint x_0 (or x_1), then* $\mathrm{Int}p(\mathcal{U})$ *contains x_0 (or x_1).*

Proof. It is clear that $\mathrm{Int}p(\mathcal{U})$ contains I_X for each open cover $\mathcal{U}$ of $\mathrm{GOTP}(X*Y)$. It remains to consider three cases:
(1) $x \in R_X - I_X$;
(2) $x \in L_X - I_X$.
(3) $x \notin I_X$ and x is an endpoint of X;

We only prove that x must be covered by $\mathrm{Int}p(\mathcal{U})$ for (1). For the other cases, we can prove them similarly. Let $x \in R_X - I_X$. There exists a convex open subset V of $\mathrm{GOTP}(X*Y)$ with $\langle x, y_1\rangle \in V \subseteq U \in \mathcal{U}$. In addition, by Lemma 2.5, $(\leftarrow,\langle x, y_1\rangle]$ is not open in $\mathrm{GOTP}(X*Y)$ since $(\leftarrow, x]$ is not open in X. Then there exists a $v \in X$ with $x < v$ and $\langle v, y_0\rangle \in V$. Then $x \in [x, v) \subseteq \mathrm{Int}p(V) \subseteq \mathrm{Int}p(U)$. □

In Theorem 2.6, the mapping p may not necessarily be an open mapping by Example 2.12. Furthermore, we give an interesting result in Theorem 2.15.

Example 2.12. Let $X = Y = [0,1]$ with the same base consisting of all intervals $[x,r)$ and $(y,1]$, where $x < r$ and $x,y,r \in [0,1]$. Then $U = (\langle \frac{1}{3}, \frac{1}{2}\rangle, \langle \frac{2}{3}, \frac{1}{2}\rangle)$ is open in $\mathrm{GOTP}(X*Y)$, but $p(U) = [\frac{1}{3}, \frac{2}{3}]$ is not open in X.

Lemma 2.13. *Let X and Y be GO-spaces and y_0 (y_1) the left (right) endpoint of Y. If U is an open convex subset of* $\mathrm{GOTP}(X*Y)$*, then the set*

$$U^{\Delta} = \{x \in X \,|\, \{x\} * Y \subset U\}$$

is an open convex set in X, and if V is also an open convex subset of $\mathrm{GOTP}(X*Y)$ *with $V \subset U$, then $V^{\Delta} \subset U^{\Delta}$.*

Proof. We only need prove that U^Δ is open and convex in X. Assume that $x', x'' \in U^\Delta$ with $x' < x''$. Let $x \in X$ with $x' < x < x''$. For any $y \in Y$, we have $\langle x', y\rangle \in U$ and $\langle x'', y\rangle \in U$. Also $\langle x', y\rangle < \langle x, y\rangle < \langle x'', y\rangle$ since $x' < x < x''$. Hence, $\langle x, y\rangle \in U$ since U is convex in $\mathrm{GOTP}(X * Y)$. So $\{x\} * Y \subset U$. It follows that $x \in U^\Delta$.

Next we prove that U^Δ is open in X. Let $x \in U^\Delta$. If x is neither the minimum point nor the maximum point of U^Δ, then x is obviously an interior point of U^Δ. Assume that x is the maximum point of U^Δ. Then $\langle x, y_1\rangle \in U$ so that $\langle x, y_1\rangle$ is an interior point of U. If $(\leftarrow, x]$ is not open in X, then x must have no immediate successor in X. It follows that there exists an $x'' \in X$ with $x < x''$ such that $[\langle x, y_1\rangle, \langle x'', y_0\rangle) \subset U$. So for any $x' \in X$ with $x < x' < x''$, we have $\{x'\} * Y \subset U$. Hence, $x' \in U^\Delta$ which is contrary to the maximality of x. So $(\leftarrow, x]$ must be open in X. If, simultaneously, x is not the minimum point of U^Δ, then taking $x' \in U^\Delta$ with $x' < x$, $(x', x]$ is an open neighborhood of x contained in U^Δ. If, simultaneously, x is the minimum point of U^Δ, then we can similarly prove that $[x, \rightarrow)$ is open in X so that $U^\Delta = \{x\}$ is open. For the case that x is the minimum point of U^Δ, we similarly prove that x is an interior point of U^Δ. Thus, U^Δ is open in X. □

Theorem 2.14. *Let X and Y be GO-spaces and y_0 (y_1) the left (right) endpoint of Y. If U is an open convex subset of* $\mathrm{GOTP}(X * Y)$*, then $|p(U) - U^\Delta| \le 2$, and if $p(U) - U^\Delta \neq \emptyset$, then the elements of $p(U) - U^\Delta$ must be the maximum or minimum points of $p(U)$.*

Theorem 2.15. *Let X and Y be GO-spaces and y_0 (y_1) the left (right) endpoint of Y. Suppose that U is an open convex subset of* $\mathrm{GOTP}(X * Y)$ *and $U^\Delta \neq \emptyset$.*

If the maximum point $x_1(U)$ of $p(U) - U^\Delta$ belongs to $L_X \cup I_X$, then $U^\Delta \cup \{x_1(U)\}$ is an open convex subset of X.

If the minimum point $x_0(U)$ of $p(U) - U^\Delta$ belongs to $R_X \cup I_X$, then $U^\Delta \cup \{x_0(U)\}$ is an open convex subset of X.

The proofs of Theorem 2.14 and Theorem 2.15 are easy by Definition 2.3 and Lemma 2.13.

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