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METRIC STRUCTURES FOR CW COMPLEXES

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Abstract. We develop a theory of distance between CW complexes. This is accomplished by using the numerical invariants of cone-length and Lusternik–Schnirelmann category as a framework to induce several (discrete and nondiscrete) metric structures. We investigate some rudimentary point-set properties of the nondiscrete metric.

1. Introduction

The concept of distance between certain kinds of spaces has led to beautiful mathematics and many interesting applications. For example, the Gromov–Hausdorff distance [7] between compact metric spaces, which is a fruitful and rewarding subject to study in its own right, has proved useful in studying a wide range of problems from those in differential geometry [6] to those in cosmology [13]. Our goal in this paper is to take a first step at developing a metric structure on CW complexes by considering several metrics as well as several ways to measure how "different" two spaces are. This point of view is interesting because it is important to know how close two spaces are from the perspective of homotopy type. As we illustrate throughout the paper, there are many ways to measure this. The metrics we define are based on the classical notion of the cone length of a space, a Lusternik–Schnirelmann (LS) type invariant. Our reference for the relationship between LS theory and cone length is [4, Chapter 3]. In §2, we introduce five metrics and determine some relationships with the cone length and killing length. Many of the results in this section are related to the work of Martin Arkowitz, Donald Stanley, and

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Jeffrey Strom [2] who studied the cone length of maps with respect to a collection of spaces. In §2.4, we demonstrate a relationship between the metrics. We are able to show inequalities relating all five of them so that, in particular, they induce the same topology. We give examples to show that the inequalities and estimates throughout §2.4 can be strict. Section 3 is devoted to studying a nondiscrete metric based on the connectivity of the spaces. We will show that this metric is indeed not discrete and that the cone length and killing length are continuous functions from this metric space into the nonnegative reals. We note that our metric space is disconnected with infinitely many components, and we conclude with some open questions.

We work with finite CW complexes, sometimes with abelian fundamental group. We use $*$ to denote a space with the homotopy type of a point and $\equiv$ to denote the same homotopy type of spaces.

By a *collection* $\mathcal{F}$, we mean a collection of spaces such that $* \in \mathcal{F}$ and if $A \in \mathcal{F}$ with $A \equiv B$, then $B \in \mathcal{F}$. The collection $\mathcal{F}$ is *closed* under suspension if, whenever $A \in \mathcal{F}$, we have $\Sigma A \in \mathcal{F}$.

Definition 1.1. We recall the definitions introduced in [11]. Let $\mathcal{F}$ be any collection of spaces. A *weight function* $\omega\colon \mathcal{F} \to \mathbb{R}^{\geq 0}$ is any function such that

(a) $\omega(*) = 0$;
(b) $\omega(A_1 \vee A_2) \leq \omega(A_1) + \omega(A_2)$ for all spaces A_1 and A_2;
(c) $\omega(A_1) = \omega(A_2)$ whenever $A_1 \equiv A_2$;
(d) for all $A \in \mathcal{F}$, $\omega(A) \leq C$ for some constant C.

Let $f\colon X \to Y$, let $\mathcal{F}$ be a collection of spaces, and let ω be a weight function. If f is a homotopy equivalence, set $\ell^{\omega}_{\mathcal{F}}(f) = 0$. Otherwise, let D be a homotopy commutative diagram

$$\begin{array}{ccccccccc} A_0 & & A_1 & & & & A_{m-1} & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ X_0 & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X_{m-1} & \longrightarrow & X_m \\ \| & & & & & & & & \| \\ X & & & & \xrightarrow{\quad f \quad} & & & & Y \end{array}$$

where each $A_i \longrightarrow X_i \longrightarrow X_{i+1}$ is a mapping cone sequence with $A_i \in \mathcal{F}$. Set $\widetilde{\ell}^{\omega}_{D}(f) = \sum_{i=0}^{m-1} \omega(A_i)$. The *$\omega$-length of f* is the number $\ell^{\omega}_{\mathcal{F}}(f) = \inf_D\{\widetilde{\ell}^{\omega}_{D}(f)\}$ where the infimum is taken over all such decompositions D. If no such diagram D exists, we say that $\ell^{\omega}_{\mathcal{F}}(f) = \infty$.

The weighted length is then defined as follows.

Definition 1.2. Let X and Y be spaces and let ω be a weight function. Define $\ell^{\omega}_{\mathcal{F}}(X,Y) = \inf_f\{\ell^{\omega}_{\mathcal{F}}(f)\}$. We define the ω *killing length* by $\mathrm{kl}^{\omega}_{\mathcal{F}}(X) = \ell^{\omega}_{\mathcal{F}}(X,*)$ and the ω *cone length* by $\mathrm{cl}^{\omega}_{\mathcal{F}}(X) = \ell^{\omega}_{\mathcal{F}}(*,X)$.

Because ω may take on noninteger values, there may be an infinite sequence of ω-decompositions of f which have values that converge to some number not realized by any ω-decomposition (see [11, Example 12]). Hence, for $\ell^{\omega}_{\mathcal{F}}(f) < \infty$, let $\epsilon > 0$ be given. Then there is a decomposition D_ϵ such that $\widetilde{\ell}^{\omega}_{D_\epsilon}(f) \leq \ell^{\omega}_{\mathcal{F}}(f) + \epsilon$. The diagram D_ϵ is called an *ϵ-approximation of f.*

The following properties of $\ell^{\omega}_{\mathcal{F}}$ were shown in [11, Proposition 7]. We list them here for completeness.

Proposition 1.3. *Let $f\colon X \to Y$ be any map, let ω be a weight function, and let $\mathcal{F}$ be a collection. Then*

(a) *if* $A \longrightarrow X \xrightarrow{f} Y$ *is a mapping cone sequence, then* $\ell^{\omega}_{\mathcal{F}}(f) \leq \omega(A)$*;*
(b) $\ell^{\omega}_{\mathcal{F}}(fg) \leq \ell^{\omega}_{\mathcal{F}}(f) + \ell^{\omega}_{\mathcal{F}}(g)$*;*
(c) *if* $f \equiv g$*, then* $\ell^{\omega}_{\mathcal{F}}(f) = \ell^{\omega}_{\mathcal{F}}(g)$*.*

In §3 we will define a weight function based on the connectivity of the space in question (see Definition 3.2) and fix $\mathcal{F}$ to be all spaces while only considering distance between spaces with abelian fundamental group. Throughout sections of this paper, we will usually write $\ell_{\mathcal{F}}$, ℓ^{ω}, or ℓ when the collection $\mathcal{F}$ and weight function ω are clear from the context or if there is no need to make reference to $\mathcal{F}$ or ω.

Definition 1.4. The Lusternik–Schnirelmann (LS) category of a map $f\colon X \to Y$ is the least integer k for which X has a cover by open sets

$$X = X_0 \cup X_1 \cup \ldots \cup X_k$$

such that $f|_{X_i} \simeq *$ for each i. When $f = \mathrm{id}_X$, we write $\mathrm{cat}(X) = \mathrm{cat}(\mathrm{id}_X)$, and when $i\colon A \hookrightarrow X$ is the inclusion, we write $\mathrm{cat}_X(A) = \mathrm{cat}(i)$. When A has the homotopy type of the n-skeleton $X^n \subseteq X$, we write $\mathrm{cat}_X(X^n) = \mathrm{cat}(X^n)$ since X is clear from the context.

See [4] for an excellent discussion of the LS category. For some other current work being done on LS category, see [3], [10], and [8].

The following theorem is a corollary of Theorem 3.3.

Theorem 1.5. *Let X and Z be spaces and let m be the smallest integer such that $\mathrm{cat}(X^m) = k$ and that $\mathrm{cat}(Z^m) = i < k$. Then any cone decomposition of Z into X must have at least $k - i$ attachments so that if $\omega(X) = 1$, $k - i \leq \ell(Z,X)$.*

Example 1.6. We give an example of the use of this theorem. Let $\mathcal{F} = \{\text{all spaces}\}$, let $\omega(X) = 1$, and consider $Z = S^5 \times S^5$ and $X = \mathbb{C}\mathrm{P}^5$. We apply Theorem 1.5 to obtain a lower bound for $\ell(Z, X)$. This amounts to locating the first i such that $\mathrm{cat}(X^i) - \mathrm{cat}(Z^i)$ is maximum, which happens when $i = 8$ ([9], Corollary 17) since $\mathrm{cat}(X^8) - \mathrm{cat}(Z^8) = 4 - 1 = 3$. This is easily seen from the following table:

$i = dim$	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{cat}((\mathbb{C}\mathrm{P}^5)^i)$	0	0	1	1	2	2	3	3	4	4	5
$\mathrm{cat}((S^5 \times S^5)^i)$	0	0	0	0	0	1	1	1	1	1	2.

Hence, $3 \leq \ell(S^5 \times S^5, \mathbb{C}\mathrm{P}^5)$ by Theorem 1.5. On the other hand, all spaces have killing length 1 and $\mathrm{cl}(\mathbb{C}\mathrm{P}^5) = 5$ which yields an upper bound of $\ell(S^5 \times S^5, \mathbb{C}\mathrm{P}^5) \leq 1 + 5 = 6$ so that

$$3 \leq \ell(S^5 \times S^5, \mathbb{C}\mathrm{P}^5) \leq 6.$$

2. Metrics

We recall the definition of an extended metric.

Definition 2.0.1. An *extended metric* on a space X is a function $d\colon X \times X \to \mathbb{R} \cup \{\infty\}$ such that

(a) $d(x, y) \geq 0$ for all $x, y \in X$;
(b) $d(x, y) = 0 \Leftrightarrow x = y$;
(c) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let $\mathcal{F}$ be a collection of spaces and let ω be some weight function. By Proposition 1.3, it is easy to see that ℓ is a *pre-metric*; i.e., it satisfies all the properties of an extended metric except symmetry. We first convert the pre-metric ℓ into a metric by considering simple operations.

2.1. Arithmetic metrics.

Definition 2.1.1. Define a metric $d^{\omega}_M = d_M$ by

$$d_M(X, Y) = \frac{\ell(X, Y) + \ell(Y, X)}{2}.$$

Proposition 2.1.2. *The function d_M is an extended metric.*

Proof. We show that d_M satisfies Definition 2.0.1. Conditions (a) and (b) are obvious. By definition, d_M is symmetric so condition (c) is satisfied. The triangle inequality follows from Proposition 1.3. □

Rather than take the average of $\ell(X, Y)$ and $\ell(Y, X)$, the maximum or the minimum of $\{\ell(X, Y), \ell(Y, X)\}$ provides a reasonable metric to work with.

Definition 2.1.3. Let X and Y be spaces. Define $d_B^\omega = d_B$ by $d_B(X,Y) = \max\{\ell(X,Y), \ell(Y,X)\}$.

Proposition 2.1.4. *The function d_B is an extended metric. Furthermore, if $\mathcal{F}$ is closed under suspension, then $d_B(X,*) = \mathrm{cl}_{\mathcal{F}}(X)$.*

Proof. It is obvious that d_B is an extended metric. Corollary 4.8 of [2] asserts that if $\mathcal{F}$ is closed under suspension, then $\mathrm{kl}_{\mathcal{F}}(X) \leq \mathrm{cl}_{\mathcal{F}}(X)$ so that $d_B(*,X) = \max\{\mathrm{kl}_{\mathcal{F}}(X), \mathrm{cl}_{\mathcal{F}}(X)\} = \mathrm{cl}_{\mathcal{F}}(X)$. □

Definition 2.1.5. Let X and Y be spaces. Define $d_H^\omega = d_H$ by $d_H(X,Y) = \min\{\ell(X,Y), \ell(Y,X)\}$.

Proposition 2.1.6. *The function d_H is an extended metric. Furthermore, if $\mathcal{F}$ is closed under suspension, then $d_H(X,*) = \mathrm{kl}_{\mathcal{F}}(X)$.*

Example 2.1.7. Let $X = \mathbb{C}\mathrm{P}^5$, let $Z = S^5 \times S^5$, and choose $\omega(X) = 1$. By Example 1.6, $3 \leq \ell(S^5 \times S^5, \mathbb{C}\mathrm{P}^5) \leq 6$. To estimate $\ell(\mathbb{C}\mathrm{P}^5, S^5 \times S^5)$, observe that $\mathrm{cl}(S^5 \times S^5) = 2$ so that $\ell(\mathbb{C}\mathrm{P}^5, S^5 \times S^5) \leq 3$. We compare the values of the arithmetic metrics. Since $\max\{\ell(X,Z), \ell(Z,X)\} = \ell(X,Z)$ and $\min\{\ell(X,Z), \ell(Z,X)\} = \ell(Z,X)$, we have

$$\begin{aligned} 3 \leq d_M(X,Z) &\leq \tfrac{9}{2} \\ 3 \leq d_B(X,Z) &\leq 6 \\ 1 \leq d_H(X,Z) &\leq 3. \end{aligned}$$

2.2. Definition of d_S.

Definition 2.2.1. Consider all sequences of the form

$$X \equiv X_0 \xrightarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xrightarrow{f_2} \dots \xleftarrow{f_{n-1}} X_n \equiv Y.$$

Define $d_S^\omega = d_S$ by $d_S(X,Y) = \min\{\sum \ell(f_i) : \text{all such sequences}\}$.

Since d_S is clearly symmetric, we have the following proposition.

Proposition 2.2.2. *The function $d_S(X,Y)$ is an extended metric.*

We now show that if our collection of spaces is closed under suspension, then we only need to consider the diagram $X \to M \leftarrow Y$ over all spaces M to compute $\ell(X,Y)$. We first state a lemma that we will need for the proof.

Lemma 2.2.3 ([2, Corollary 4.1.1(a)])**.** *Let $\mathcal{F}$ be any collection of spaces and let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C & \longrightarrow & D \end{array}$$

be a homotopy pushout square. Then $\ell(g) \leq \ell(f)$.

Proposition 2.2.4. *If $\mathcal{F}$ is any collection, then* $d_S(X,Y) = \min\{\ell(X,M) + \ell(Y,M) : M$ *any space*$\}$.

Proof. Consider any sequence as above and form the pushout of f_1 and f_2

$$\begin{array}{ccccccccc} X_0 \xrightarrow{f_0} & X_1 & \xleftarrow{f_1} X_2 \xrightarrow{f_2} & X_3 & \xleftarrow{f_3} \dots \xleftarrow{f_{n-1}} X_n \\ & {\scriptstyle f_1'} \searrow & & \swarrow {\scriptstyle f_2'} & \\ & & P_1 & & \end{array}$$

with maps f_1' and f_2'. By Lemma 2.2.3, $\ell(f_2') \leq \ell(f_1)$ and $\ell(f_1') \leq \ell(f_2)$ so that

$$\begin{aligned} \ell(f_1'f_0) + \ell(f_2'f_3) &\leq \ell(f_1') + \ell(f_0) + \ell(f_2') + \ell(f_3) \\ &\leq \ell(f_0) + \ell(f_1) + \ell(f_2) + \ell(f_3). \end{aligned}$$

This gives

$$X_0 \xrightarrow{f_1'f_0} P_1 \xleftarrow{f_2'f_3} X_4 \xrightarrow{f_4} \dots \xleftarrow{f_{n-1}} X_n$$

with $\ell(f_1'f_0) + \ell(f_2'f_3) + \sum_{i=4}^{n} \ell(f_i) \leq \sum_{i=1}^{n} \ell(f_i)$. Continuing in this manner, we form the diagram

$$X_0 \xrightarrow{f} P_{n-3} \xleftarrow{g} X_n$$

where P_{n-3} is the pushout of a certain diagram and $\ell(f)+\ell(g) \leq \sum\limits_{i=1}^{n} \ell(f_i)$. □

Corollary 2.2.5. *If $\mathcal{F}$ is any collection, then* $\mathrm{kl}_{\mathcal{F}}(X) = d_S(X,*)$.

Proof. By Proposition 2.2.4, there exists a space M such that $d_S(X,*) = \ell(X,M) + \ell(*,M)$. We have

$$\begin{aligned} \mathrm{kl}_{\mathcal{F}}(X) &= \ell(X,*) \\ &\leq \ell(X,M) + \ell(M,*) \\ &\leq \ell(X,M) + \ell(*,M) \\ &= d_S(X,*). \end{aligned}$$

Since $d_S(X,*) \leq \ell(X,*) + \ell(*,*) = \mathrm{kl}_{\mathcal{F}}(X)$, the result follows. □

Example 2.2.6. Let $X = \mathbb{C}\mathrm{P}^n$ and let $\mathcal{F}$ be all spaces. For $\omega(X) = 1$, it is obvious that $d_S(X,*) = \mathrm{kl}(X) = 1$. Now let $\mathcal{C}_n = \{X : \mathrm{cat}(X) \leq n\}$ and let $\omega(X) = \mathrm{cat}(X)$. M. Cuvilliez and Y. Félix [5] have shown that any path-connected space X can be killed using at most $\lceil \log_2(\mathrm{cl}(X)+1) \rceil$ spheres, where $\mathrm{cl}(X)$ is the classical cone length of X. Hence, with $\omega(X) = \mathrm{cat}(X)$, we have $d_S^{\omega}(X,*) = \mathrm{kl}^{\omega}(X) \leq \lceil \log_2(n+1) \rceil$. The same estimate is obtained if, letting $\mathcal{F}$ be the collection of spheres, we let $\omega(X) = 1$ or $\omega(X) = \mathrm{cat}(X)$.

For the following proposition, the hypothesis that $\mathcal{F}$ is closed under suspension is needed in order to apply [2, Corollary 4.9.1(a)].

Proposition 2.2.7. *If $\mathcal{F}$ is closed under suspensions, then $|\mathrm{kl}_{\mathcal{F}}(X) - \mathrm{kl}_{\mathcal{F}}(Y)| \le d_S(X,Y)$.*

Proof. Let $f\colon X \to M$ and $g\colon Y \to M$ with $d_S(X,Y) = \ell(f) + \ell(g)$. By [2, Corollary 4.9.1(a)], $\ell(f) \ge |\mathrm{kl}_{\mathcal{F}}(X) - \mathrm{kl}_{\mathcal{F}}(M)|$ and $\ell(g) \ge |\mathrm{kl}_{\mathcal{F}}(Y) - \mathrm{kl}_{\mathcal{F}}(M)|$. Hence,

$$\begin{aligned} d_S(X,Y) &= \ell(f) + \ell(g) \\ &\ge |\mathrm{kl}_{\mathcal{F}}(X) - \mathrm{kl}_{\mathcal{F}}(M)| + |\mathrm{kl}_{\mathcal{F}}(Y) - \mathrm{kl}_{\mathcal{F}}(M)| \\ &\ge |\mathrm{kl}_{\mathcal{F}}(X) - \mathrm{kl}_{\mathcal{F}}(Y)|, \end{aligned}$$

which proves the result. □

2.3. **Definition of d_A.**

Definition 2.3.1. Define $d_A^{\omega} = d_A$ by

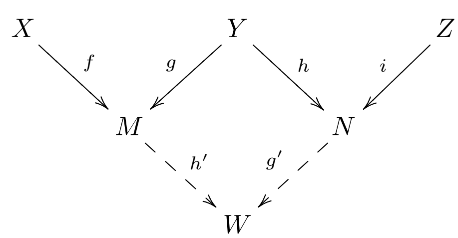
$$d_A(X,Y) = \min_M \max\{\ell(X,M), \ell(Y,M)\}.$$

Proposition 2.3.2. *The function d_A is an extended metric.*

Proof. We check the triangle inequality. Let $d_A(X,Y) = m$ and let $d_A(Y,Z) = n$. We have the diagram

$$X \xrightarrow{f} M \xleftarrow{g} Y \xrightarrow{h} N \xleftarrow{i} Z$$

with $\max\{\ell(f), \ell(g)\} = m$ and $\max\{\ell(h), \ell(i)\} = n$. Consider the homotopy pushout W of the diagram



By Lemma 2.2.3, $\ell(h') \le \ell(h)$ and $\ell(g') \le \ell(g)$. Hence,

$$\begin{aligned} d_A(X,Z) &\le \max\{\ell(h'f), \ell(g'i)\} \\ &\le \max\{\ell(h') + \ell(f), \ell(g') + \ell(i)\} \\ &\le \max\{\ell(h) + \ell(f), \ell(g) + \ell(i)\} \\ &\le m + n = d_A(X,Y) + d_A(Y,Z). \end{aligned}$$

Thus, d_A satisfies the triangle inequality so that d_A is an extended metric. □

2.4. Comparison of metrics.

We show the relationships between our five metrics. In §2.5, we show that each of these inequalities is strict.

Proposition 2.4.1. *Let X and Y be spaces. Then* $\frac{1}{2}d_S(X,Y) \le d_A(X,Y) \le d_S(X,Y) \le d_H(X,Y) \le d_M(X,Y) \le d_B(X,Y)$.

Proof. The fact that $d_H(X,Y) \le d_M(X,Y) \le d_B(X,Y)$ is obvious from their definitions. Assume without loss of generality that $d_H(X,Y) = \ell(X,Y)$. The diagram

$$X \longrightarrow Y \xleftarrow{\text{id}} Y$$

shows that $d_S(X,Y) \le \ell(X,Y)$. Thus, $d_S(X,Y) \le \ell(X,Y) = d_H(X,Y)$.

Since $d_A(X,Y) = \min_M \max\{\ell(X,M), \ell(Y,M)\}$, there is a space M such that $d_A(X,Y) = \max\{\ell(X,M), 4\ell(Y,M)\}$.

Observe that $X \longrightarrow M \longleftarrow Y$ estimates $d_S(X,Y)$ so that

$$\begin{aligned} d_S(X,Y) &\le \ell(X,M) + \ell(Y,M) \\ &\le 2\max\{\ell(X,M), \ell(Y,M)\} \\ &= 2d_A(X,Y). \end{aligned}$$

Finally, Proposition 2.2.4 gives us $d_S(X,Y) = \ell(X,M) + \ell(Y,M)$ for some space M. We have

$$\begin{aligned} d_A(X,Y) &\le \max\{\ell(X,M), \ell(Y,M)\} \\ &\le \ell(X,M) + \ell(Y,M) \\ &= d_S(X,Y). \end{aligned}$$ □

Corollary 2.4.2. *For any collection of spaces,* $\mathrm{kl}_{\mathcal{F}}(X) \le 2d_A(X,*)$.

Proof. This follows immediately from Corollary 2.2.5 and Proposition 2.4.1 □

2.5. Strict inequality.

Example 2.5.1. We first show that $d_H < d_M < d_B$. Let $\omega(X) = 1$, $\mathcal{F} = \{\text{all spaces}\}$, and $X \ne \Sigma A$ for any space A. Then $\mathrm{cl}(X) > 1$ and $\mathrm{kl}(X) = 1$. Hence, $d_H(X,*) = \mathrm{kl}(X) < d_M(X,*) = \frac{\mathrm{cl}(X)+\mathrm{kl}(X)}{2} < \mathrm{cl}(X) = d_B(X,*)$ by Proposition 2.1.6 and Proposition 2.1.4.

Example 2.5.2. Let $\omega(X) = 1$. We show not only that $d_S < d_H$, but also that the difference $d_H - d_S$ can be made arbitrarily large when $\mathcal{F}$ is closed under suspensions and contains the spaces in question (e.g., when $\mathcal{F}$ is all spaces). We apply Theorem 1.5. Let $n \geq 2$ be any integer and let $X = \mathbb{C}\mathrm{P}^n$ and $Z = \prod_{i=1}^{2n} S^{2n+1}$. If both X and Z are in $\mathcal{F}$, then $d_S(X,Z) \leq \ell(X,*) + \ell(Z,*) = 2$. We now show that $n \leq d_H(X,Z) = \min\{\ell(X,Z), \ell(Z,X)\}$. We first observe that the smallest integer i, such that $\mathrm{cat}(X^i) = n$, is $i = 2n$ and $\mathrm{cat}(Z^{2n}) = 0$, so by Theorem 1.5, we have $n - 0 \leq \ell(Z,X)$. On the other hand, the smallest integer i, such that $\mathrm{cat}(Z^i) = 2n$, is $i = (2n+1)2n$. Since $\mathrm{cat}(X^{(2n+1)2n}) = n$, we have $2n - n \leq \ell(X,Z)$.

Example 2.5.3. To see that $d_A < d_S$, let $\mathcal{F}$ be all spaces, let $\omega(X) = 1$, and observe that $d_A(X,Y) \leq \max\{\ell(X,*), \ell(Y,*)\} = 1$. Hence, we find spaces X and Y such that $2 \leq d_S(X,Y)$. Let $X = \mathbb{C}\mathrm{P}^3$ and $Y = \mathbb{C}\mathrm{P}^5$. Now the only way that $d_S(X,Y) < 2$ is if one of $\ell(X,M)$ and $\ell(Y,M)$ is 0; i.e., $M = X$ or $M = Y$. Clearly, $M = \mathbb{C}\mathrm{P}^5$ does not work since $\ell(\mathbb{C}\mathrm{P}^3, \mathbb{C}\mathrm{P}^5) = 2$. Hence, we show that there is no space A for which $A \to \mathbb{C}\mathrm{P}^5 \to \mathbb{C}\mathrm{P}^3$ is a mapping cone sequence. If such a space and mapping cone sequence existed, consider the induced map on the cohomology rings $H^*(\mathbb{C}\mathrm{P}^3) \to H^*(\mathbb{C}\mathrm{P}^5)$. This map is zero which implies that $H^2(\Sigma A) \to H^2(\mathbb{C}\mathrm{P}^3)$ is nonzero. Hence, $H^*(\Sigma A)$ would have nontrivial cup products, which is impossible. Thus, $2 \leq \ell(\mathbb{C}\mathrm{P}^5, \mathbb{C}\mathrm{P}^3)$ and $2 \leq d_S(\mathbb{C}\mathrm{P}^3, \mathbb{C}\mathrm{P}^5)$.

Example 2.5.4. We give an example to show that the inequality in Proposition 2.2.7 can be strict. Let $n \geq 2$ be an integer and $\mathcal{C}_n = \{X : \mathrm{cat}(X) \leq n\}$. Let $\omega(X) = \mathrm{cat}(X)$ be a weight function, and let $X = \mathbb{C}\mathrm{P}^3$ and $Y = \mathbb{C}\mathrm{P}^5$. Let $\mathcal{F}$ be the collection of all spaces. Since $\mathcal{C}_n \subseteq \mathcal{F}$, Example 2.5.3 implies that $2 \leq d_S(\mathbb{C}\mathrm{P}^3, \mathbb{C}\mathrm{P}^5)$. It remains to show that $|\mathrm{kl}_{\mathcal{C}}(\mathbb{C}\mathrm{P}^3) - \mathrm{kl}_{\mathcal{C}}(\mathbb{C}\mathrm{P}^5)| \neq 2$. Using the result stated in Example 2.2.6, combined with the fact that $\mathrm{cat}(S^n) = 1$, we have the estimates $\mathrm{kl}_{\mathcal{C}}(\mathbb{C}\mathrm{P}^3) \leq \lceil \log_2(4) \rceil = 2$ and $\mathrm{kl}_{\mathcal{C}}(\mathbb{C}\mathrm{P}^5) \leq \lceil \log_2(6) \rceil = 3$. Hence, it suffices to show that $\mathrm{kl}_{\mathcal{F}}(\mathbb{C}\mathrm{P}^3) \neq 1$. Suppose there were a mapping cone sequence $A \to \mathbb{C}\mathrm{P}^3 \to *$ with $\mathrm{cat}(A) = 1$. By the long exact cohomology sequence, $H^n(\mathbb{C}\mathrm{P}^3) \cong H^n(A)$ for all n. Now $1 = \mathrm{cat}(A) \geq \cup(A)$ [4, Proposition 1.5] where $\cup(X)$ denotes the cup-length of X. But $H^*(\mathbb{C}\mathrm{P}^3)$ contains non-trivial cup products, which is a contradiction. Hence, $|\mathrm{kl}_{\mathcal{C}}(\mathbb{C}\mathrm{P}^3) - \mathrm{kl}_{\mathcal{C}}(\mathbb{C}\mathrm{P}^5)| < 2 \leq d_S(\mathbb{C}\mathrm{P}^3, \mathbb{C}\mathrm{P}^5)$.

Example 2.5.5. We give an example of a weight function for which the inequality in Corollary 2.4.2 is strict for all path-connected spaces X. Let $\omega(X) = \frac{\mathrm{Hdim}(X)}{\mathrm{Hdim}(X)+1} = 1 - \frac{1}{\mathrm{Hdim}(X)}$. This weight function was shown to have the property that for any space A, $\mathrm{cl}^\omega(\Sigma A) < \mathrm{kl}^\omega(A)$. Since X

is path-connected, $\mathrm{Hdim}(X) \geq 1$ so that $\frac{1}{2} \leq \ell(X,Y)$ for $X \not\equiv Y$, and hence $\frac{1}{2} \leq d_A(X,Y)$. By [11, Corollary 30], $\mathrm{kl}^{\omega}(X) = \omega(X)$. Thus, for $\omega(X) = \frac{\mathrm{Hdim}(X)}{\mathrm{Hdim}(X)+1}$, we have $\mathrm{kl}^{\omega}(X) < 1 \leq 2d_A(X,*)$ for all spaces X.

We now estimate the kinds of values that all five of our metrics give in the case of $X = \mathbb{R}\mathrm{P}^n$ and $Y = \mathbb{R}\mathrm{P}^{n+k}$. We will use the following estimate.

Proposition 2.5.6 ([1])**.** *Let $\mathcal{F}$ be any collection. Then $\cup(Y) - \cup(X) \leq \ell(X,Y)$.*

Example 2.5.7. Let $\mathcal{F}$ be all spaces and let $\omega(X) = 1$. Consider $d(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{n+k})$, $n,k > 0$. By Proposition 2.5.6, $\cup(\mathbb{R}\mathrm{P}^{n+k}) - \cup(\mathbb{R}\mathrm{P}^n) = k \leq \ell(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{n+k}) \leq k$ by the usual CW construction of $\mathbb{R}\mathrm{P}^{n+k}$. The best way to compute $\ell(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n)$ is not obvious. Of course, one can always kill the space and build up from a point. If we take this approach, then $\ell(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n) \leq n+1$.

This yields the following estimates:

$$\begin{aligned}
d_A(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n) &= 1;\\
d_S(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n) &\leq 2;\\
d_H(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n) &\leq \min\{k, n+1\};\\
d_M(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n) &\leq \frac{1}{2}(n+1+k);\\
d_B(\mathbb{R}\mathrm{P}^{n+k}, \mathbb{R}\mathrm{P}^n) &\leq \max\{k, n+1\}.
\end{aligned}$$

We conclude this section by showing how a metric d behaves with respect to the functor Σ.

Proposition 2.5.8. *Let $\mathcal{F}$ be closed under suspensions and let $\omega(X)$ be a weight function with the property that $\omega(\Sigma X) \leq \omega(X)$ for all X. Then $\ell(\Sigma X, \Sigma Y) \leq \ell(X,Y)$.*

Proof. Let

$$\begin{array}{ccccccccc}
A_0 & & A_1 & & & & A_{n-1} & & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & X_n \\
\| & & & & & & & & \| \\
X & & & & \xrightarrow{\quad f \quad} & & & & Y
\end{array}$$

be any decomposition X into Y. Since attaching ΣA_i provides a decomposition of ΣX into ΣY, $\ell(\Sigma X, \Sigma Y) \leq \ell(X,Y)$. In other words, we suspend the above decomposition diagram to obtain $\ell(\Sigma X, \Sigma Y) \leq \ell(X,Y)$. ☐

Corollary 2.5.9. *If $\mathcal{F}$ is closed under suspension and $\omega(X)$ satisfies the property in Proposition 2.5.8, then $d(\Sigma X, \Sigma Y) \leq d(X,Y)$ for $d = d_A, d_S, d_H, d_M, d_B$.*

3. Nondiscrete Metric

We now define the connectivity weight ω_C.

Definition 3.1. For a CW complex A, we define the *connectivity* of A, denoted $\mathrm{conn}(A)$, to be the largest integer n (or ∞) such that $\pi_i(A) = 0$ for $i < n+1$. If A is not path connected, we say that $\mathrm{conn}(A) = -1$.

Definition 3.2. Let X and Y be path-connected CW complexes and let $\mathcal{F}$ be the collection of all CW complexes with abelian fundamental group. Define

$$\omega_C(A) = \begin{cases} 0 \text{ if } A \equiv * \\ 2 \text{ if } A \text{ is not path-connected} \\ \frac{1}{\mathrm{conn}(A)+1} \text{ otherwise.} \end{cases}$$

We say that ω_C is the *connectivity weight* and that $\ell^{\omega_C}(X,Y)$ is the *connectivity length between* X and Y.

We will apply the following estimate below.

Theorem 3.3 ([12, Theorem 11])**.** *Let X and Z be spaces with $m_1 \leq m_2 \leq \ldots \leq m_N < \infty$ where each m_i is the smallest integer such that $\mathrm{cat}(X^{m_i}) - \mathrm{cat}(Z^{m_i}) = i > 0$ for $1 \leq i \leq N$. Then $2 + \sum_{i=2}^{N} \frac{1}{m_i - 1} \leq \ell^{\omega}(Z,X)$ if $\mathrm{cat}(X^1) - \mathrm{cat}(Z^1) = 1$. Otherwise, $\sum_{i=1}^{N} \frac{1}{m_i-1} \leq \ell^{\omega}(Z,X)$.*

In this section, we work with the collection of finite CW complexes with the homotopy type of path-connected spaces with abelian fundamental group. Note that while elements of the metric space need to be path connected, the spaces we attach need not be. Denote this collection by $\mathcal{CA}$. Since many of the metrics in §2 induce the discrete topology, we seek a metric that induces a nondiscrete topology. This is achieved if $\omega = \omega_C$ as seen in Proposition 3.4. We write $B(X;\delta)$ to denote the ball centered at X with radius δ.

Set $\omega_C(X) = \frac{1}{1+\mathrm{conn}(X)}$. We will work with d_B^{ω} in this section. Recall that a set $\mathcal{U} \in \mathcal{CA}$ is open in the topology induced by the metric d if and only if for every $X \in \mathcal{U}$ there exists $\delta > 0$ such that $B(X;\delta) \subset \mathcal{U}$ where $B(X;\delta) = \{Y : d(X,Y) < \delta\}$.

Proposition 3.4. *The topology induced by the metric $d_B^{\omega_C}$ on $\mathcal{CA}$ is not the discrete topology.*

Proof. In order to show that the topology in question is not the discrete topology, it suffices to find a single space $X \in \mathcal{CA}$ which is not open.

We show that all singletons are not open. Let X be any space with $\operatorname{conn}(X) = n-1$ and let $\{X\} = \mathcal{U}$. We wish to show that there does not exist $\delta > 0$ such that $B(X;\delta) \subset \mathcal{U}$. Suppose such a δ exists and choose $m \in \mathbb{N}$ such that $\frac{1}{m-1} < \delta$. Then the diagrams

$$\begin{array}{ccc} S^{m-1} & & \\ \downarrow{\scriptstyle *} & & \\ X & \longrightarrow & X \vee S^m \end{array}$$

and

$$\begin{array}{ccc} S^{m} & & \\ \downarrow{\scriptstyle i_2} & & \\ X \vee S^m & \longrightarrow & X \end{array}$$

show that $\ell(X, X \vee S^m) \leq \frac{1}{m-1}$ and $\ell(X \vee S^m, X) \leq \frac{1}{m}$, which implies that $d_B(X, X \vee S^m) \leq \frac{1}{m-1}$. Hence, $X \vee S^m \in B(X,\delta)$ since $d_B(X, X \vee S^m) \leq \frac{1}{m-1} < \delta$, but $X \vee S^m \notin \mathcal{U}$ so $\mathcal{U} = \{X\}$ is not open in $\mathcal{CA}$; thus, the metric d_B does not induce the discrete topology. □

Proposition 3.5. *The function* $\omega_C \colon (\mathcal{CA}, d_B) \to \mathbb{R}^{\geq 0}$ *is continuous.*

Proof. We first note that since $\omega_C(X) = \frac{1}{1+\operatorname{conn}(X)}$, ω_C is onto the set $J = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$. Hence, it suffices to show that ω_C is continuous at any nonzero singleton and any open set around 0. We first show that the inverse image of any nonzero singleton is open. Let $a \in J$, $a \neq 0$, and $X \in \omega_C^{-1}(a)$ so that $\operatorname{conn}(X) = n-1$ and $a = \frac{1}{n} = \omega_C(X)$.

Now let $B = B(X, \frac{1}{n+1})$ and let $Y \in B$. We wish to show that $B \subseteq \omega_C^{-1}(a)$, i.e., that $\operatorname{conn}(Y) = \operatorname{conn}(X)$. Assume by way of contradiction that $\operatorname{conn}(Y) \neq \operatorname{conn}(X)$. Since $Y \in B$,

$$d_B(X,Y) = \max\{\ell(X,Y), \ell(Y,X)\} < \frac{1}{n+1}.$$

Without loss of generality, assume $d_B(X,Y) = \ell^\omega(X,Y)$. We consider the cases when $\operatorname{conn}(X) > \operatorname{conn}(Y)$ and $\operatorname{conn}(X) < \operatorname{conn}(Y)$.

(a) Assume $\operatorname{conn}(X) > \operatorname{conn}(Y)$. By Theorem 3.3, $\frac{1}{\operatorname{conn}(Y)} \leq \ell(X,Y)$. But then $\frac{1}{n-1} = \frac{1}{\operatorname{conn}(X)} < \frac{1}{\operatorname{conn}(Y)} \leq \ell(X,Y) \leq \frac{1}{n+1}$, which is a contradiction.

(b) Assume $\operatorname{conn}(X) < \operatorname{conn}(Y)$. By [12, Corollary 18], $\frac{1}{\operatorname{conn}(X)+1} \leq \ell^\omega(X,Y)$ so that $\frac{1}{n} = \frac{1}{\operatorname{conn}(X)+1} \leq \ell(X,Y) < \frac{1}{n+1}$, which is, again, a contradiction.

Hence, $\mathrm{conn}(Y) = \mathrm{conn}(X)$ so that $B \subseteq \omega^{-1}(a)$ and $\omega^{-1}(a)$ is open.

Finally, let $O(k) = \{\frac{1}{n} : n \geq k\} \cup \{0\}$ be an open set around 0 and let $\mathcal{U} = \omega_C^{-1}(O(k))$. We wish to show that $\mathcal{U}$ is open in $(\mathcal{CA}, d_B)$. Let $X \in \mathcal{U}$. Then $k - 1 \leq \mathrm{conn}(X) = t \leq \infty$. But we saw in Proposition 3.4 how to find a ball B around any element with connectivity t such that every element in B has connectivity t. This open ball is clearly contained in $\mathcal{U}$ so that $\mathcal{U}$ is open in $(\mathcal{CA}, d_B)$. Thus, the inverse image of any open set in J is open and ω_C is continuous. □

Corollary 3.6. *The function* $\mathrm{kl}^{\omega_C} \colon (\mathcal{CA}, d_B) \to \mathbb{R}^{\geq 0}$ *is continuous.*

Proof. By [12, Corollary 19], $\omega_C(X) = \mathrm{kl}^{\omega_C}(X)$ and the result follows. □

We now prove that cl^{ω_C} is also a continuous function.

Proposition 3.7. *The function* $\mathrm{cl}^{\omega} \colon (\mathcal{CA}, d_B) \to \mathbb{R}^{\geq 0}$ *is continuous.*

Proof. Let $\epsilon > 0$ be given and choose $\delta < \frac{\epsilon}{2}$. Let X and Y be any spaces such that $\max\{\ell(X,Y), \ell(Y,X)\} = d_B(X,Y) < \delta$, and approximate $\ell^{\omega}(X,Y)$ and $\ell^{\omega}(Y,X)$ by maps f_1 and f_2, respectively, such that $\ell(f_1) \leq \ell(X,Y) + \frac{\delta}{2}$ and $\ell(f_2) \leq \ell(Y,X) + \frac{\delta}{2}$. By [11, Corollary 16.b)], $\mathrm{cl}^{\omega_C}(Y) - \mathrm{cl}^{\omega_C}(X) \leq \ell(f_1)$ and $\mathrm{cl}^{\omega_C}(X) - \mathrm{cl}^{\omega_C}(Y) \leq \ell(f_2)$. Thus,

$$\begin{aligned} |\mathrm{cl}^{\omega_C}(X) - \mathrm{cl}^{\omega_C}(Y)| &\leq \max\{\ell(f_1), \ell(f_2)\} \\ &\leq \max\{\ell(X,Y), \ell(Y,X)\} + \frac{\delta}{2} \\ &= d_B(X,Y) + \delta \\ &< \delta + \delta \\ &< \epsilon. \end{aligned}$$

□

Proposition 3.8. *Let* $\mathcal{U}_n = \{X \in \mathcal{CA} : \mathrm{conn}(X) = n\}$. *Then* $\mathcal{U}_n$ *is closed.*

Proof. Since $\mathcal{U}_n = \{X \in \mathcal{CA} : \mathrm{conn}(X) = n\} = \{X \in \mathcal{CA} : \omega(X) = \frac{1}{n+1}\} = \omega_C^{-1}\{\frac{1}{n+1}\}$, the preimage of a singleton $\mathcal{U}_n$ is closed. □

Corollary 3.9. *The topological space* $\mathcal{CA}$ *is disconnected with infinitely many components.*

4. Questions

Question 4.1. Proposition 2.2.7 asserts that if $\mathcal{F}$ is closed under suspensions, then $|\mathrm{kl}^{\omega}_{\mathcal{F}}(X) - \mathrm{kl}^{\omega}_{\mathcal{F}}(Y)| \leq d_S(X,Y)$. Is the same true if we replace d_S with d_A? What about the special case when $\omega(A) = 1$?

Question 4.2. We have used cohomological information about spaces X and Y in an ad hoc manner to show that $2 \leq \ell(X,Y)$ in this paper. Is there a way to come up with stronger lower bound estimates for $\ell(X,Y)$ based on cohomological information of X and Y or some other invariant? The literature is rich with such estimates when $X \equiv *$. See for example chapters 2 and 3 of [4].

Question 4.3. As noted in the introduction, Mikhael Gromov developed a theory of distance between metric spaces [7]. Because the use of the Riemannian metric structure on the metric space was so integral to Gromov's definition of a distance between spaces, it is not clear how to generalize it to arbitrary topological spaces which do not necessarily have a Riemannian metric. Is there a way to do this and if so, what can Gromov's theory tell us about topological spaces? Is there any connection with the above metrics and the Gromov–Hausdorff metric?

Question 4.4. The metric structure on CW complexes has the potential to turn homotopy problems into problems about metric spaces. Questions concerning relationships between topological invariants can be posed in terms of the kind of topology induced on the space of CW complexes. For example, Donald Stanley [14] showed that for any positive integer n, there exists a space $X(n)$ such that $\mathrm{cat}(X(n)) < \mathrm{cl}(X(n))$. Define a weight function $\omega(X)\frac{\mathrm{cat}(X)}{\mathrm{cl}(X)}$. Studying properties of the metric space induced by ω could lead to an alternative proof of Stanley's result. The same could be done with other invariants. How much can they differ? Different topologies could infer the existence of different kinds of values.

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References

[1] Martin Arkowitz, Donald Stanley, and Jeffrey Strom, *The A-category and A-cone length of a map*, in Lusternik-Schnirelmann Category and Related Topics (South Hadley, MA, 2001). Ed. O. Cornea, G. Lupton, J. Oprea, and D. Tanré. Contemp. Math., 316. Providence, RI: Amer. Math. Soc., 2002. 15–33.

[2] ———, *The cone length and category of maps: Pushouts, products and fibrations*, Bull. Belg. Math. Soc. Simon Stevin **11** (2004), no. 4, 517–545.

[3] G. C icortaş, *Categorical sequences and relative Lusternik-Schnirelmann category*, An. Univ. Oradea Fasc. Mat. **17** (2010), no. 2, 107–112.

[4] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann Category.* Mathematical Surveys and Monographs, 103. Providence, RI: American Mathematical Society, 2003.

[5] M. Cuvilliez and Y. Félix, *Approximations to the $\mathcal{F}$-killing length of a space*, in Lusternik-Schnirelmann Category and Related Topics (South Hadley, MA, 2001). Ed. O. Cornea, G. Lupton, J. Oprea, and D. Tanré. Contemp. Math., 316. Providence, RI: Amer. Math. Soc. 2002. 73–78.

[6] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73.

[7] ———, *Metric Structures for Riemannian and non-Riemannian Spaces*. Based on the 1981 French original. With appendices by M. Katz, P. Pansu, and S. Semmes. Trans. Sean Michael Bates. [Reprint of the 2001 English edition.] Modern Birkhäuser Classics. Boston, MA: Birkhäuser Boston, Inc., 2007.

[8] E. Macías-Virgós and M. J. Pereira-Sáez, *An upper bound for the Lusternik-Schnirelmann category of the symplectic group*. Available at arXiv:1202.5570 [math.AT].

[9] Rob Nendorf, Nick Scoville, and Jeff Strom, *Categorical sequences*, Algebr. Geom. Topol. **6** (2006), 809–838.

[10] John Oprea and Jeff Strom, *On Fox's m-dimensional category and theorems of Bochner type*, Topology Appl. **159** (2012), no. 5, 1448–1461.

[11] Nicholas A. Scoville, *Mapping cone sequences and a generalized notion of cone length*, JP J. Geom. Topol. **11** (2011), no. 3, 209–233.

[12] ———, *Lusternik-Schnirelmann category and the connectivity of X*, Algebr. Geom. Topol. **12** (2012), no. 1, 435–448.

[13] C. Sormani, *Friedmann cosmology and almost isotropy*, Geom. Funct. Anal. **14** (2004), no. 4, 853–912.

[14] Donald Stanley, *Spaces and Lusternik-Schnirelmann category n and cone length $n+1$*, Topology **39** (2000), no. 5, 985–1019.

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