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SEPARATING CURVE COMPLEX OF THE GENUS TWO SURFACE: QUASI-DISTANCE FORMULA AND HYPERBOLICITY

HAROLD SULTAN

ABSTRACT. We prove that the separating curve complex of $S_{2,0}$ satisfies a quasi-distance formula akin to the quasi-distance formulas for the marking and pants complexes of Howard A. Masur and Yair N. Minsky, and for the disk and arc complexes of Masur and Saul Schleimer. The proof uses basic properties of Farey graphs in conjunction with tools of Masur and Minsky and Masur and Schleimer. As a corollary, we provide an alternative proof that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic, a fact implicit in the work of Jeffrey Brock and Masur as well as explicit in recent work of Jiming Ma.

1. INTRODUCTION

In recent years, the curve complex and natural relatives thereof have been extensively featured in the geometric group theory literature. These natural combinatorial complexes have proven to be extremely useful tools with applications to a variety of settings including notably the study of mapping class groups and Teichmüller space. For a broad overview of the topics and close relations to various natural combinatorial complexes, see, for instance, [3]. In this context, we study the coarse geometry of the separating curve complex of the genus two surface.

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In studying the coarse geometry of natural combinatorial complexes, being able to approximate distance is of fundamental interest. A prominent idea in [7] is that distance in a combinatorial complex can sometimes be approximated by summing over the distances in the subsurface projections to the curve complexes of certain subsurfaces. Specifically, the set of subsurfaces summed over, or *holes*, are defined by the property that every vertex of the combinatorial complex has nontrivial subsurface projections into their curve complexes. Such an approximation, when it exists, is called a *quasi-distance* formula. In fact, in their groundbreaking paper [6], Howard A. Masur and Yair N. Minsky develop a notion of hierarchies which, in particular, provides examples of quasi-distance formulas in the marking and pants complexes.

The ideas in this paper are similar to, as well as motivated by, work of Masur and Saul Schleimer in [7]. Using ideas implicit in [1], Masur and Schleimer, in [7], establish axioms which are proven to be sufficient for ensuring that a combinatorial complex satisfies a quasi-distance formula and is δ-hyperbolic. In particular, verification of the Masur–Schleimer axioms is used in [7] to prove that the disk complex and the arc complex satisfy quasi-distance formulas and are δ-hyperbolic. Unfortunately, as we will see, one of the Masur–Schleimer axioms fails in the case of $\mathcal{C}_{sep}(S_{2,0})$. Nonetheless, in this paper we are able to show by a direct argument that $\mathcal{C}_{sep}(S_{2,0})$ does in fact satisfy a quasi-distance formula. Furthermore, carefully considering the Masur–Schleimer argument, as a corollary we also obtain that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic. Specifically, we prove the following main theorem.

Theorem 1.0.1. *The combinatorial complex $\mathcal{C}_{sep}(S_{2,0})$ satisfies a quasi-distance formula. That is, if we let $\mathcal{NS}$ denote the set of all nonseparating essential subsurfaces of $S_{2,0}$, then there is a constant K_0 such that for all $k \geq K_0$ there exists constants $K(k)$ and $L(k)$ such that for all $\alpha, \beta \in \mathcal{C}_{sep}(S_{2,0})$, we have the quasi-isometric relation*

$$\frac{1}{K} \sum_{Y \in \mathcal{NS}} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_k - L \leq d_{\mathcal{C}_{sep}(S_{2,0})}(\alpha, \beta)$$

$$\leq K \textstyle\sum_{Y \in \mathcal{NS}} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_k + L,$$

where the threshold function $\{f(x)\}_k$ is defined to be $f(x)$ if $f(x) \geq k$, and 0 otherwise. Moreover, the combinatorial complex $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic.

It is well known that the curve complex $\mathcal{C}(S)$ is δ-hyperbolic; see [5]. On the other hand, the separating curve complex $\mathcal{C}_{sep}(S)$, in general, is not δ-hyperbolic. In fact, for all closed surfaces $S = S_{g,0}$ with genus $g \geq 3$, $\mathcal{C}_{sep}(S)$ contains natural quasi-isometric embeddings of $\mathbb{Z}^2$, an obstruction to hyperbolicity [8]. For $S_{2,0}$, however, there are no such natural non-trivial quasi-flats. Given this context, Schleimer conjectures that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic [8, Conjecture 2.48]. In this paper we provide a proof in the affirmative of this conjecture. To be sure, the fact that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic is implicit in [2], as well as explicit in [4].

2. Preliminaries

2.1. Coarse geometry.

When studying large scale geometry, in place of the usual notions of functions and isometries, it is often useful to consider the notions of coarsely well-defined maps and quasi-isometries. The latter are natural large scale generalizations of the former.

Definition 2.1.1. Given metric spaces X and Y, a map

$$f\colon X \to 2^Y \setminus \emptyset$$

is *coarsely well defined* if there exists a constant C such that for all $x \in X$, $diam_Y(f(x)) < C$.

Given a coarsely well-defined map $f\colon X \to 2^Y$, by abuse of notation, we will sometimes consider it as a map $f\colon X \to Y$, obtained by assigning to each $x \in X$ an arbitrarily selected element $y \in f(x)$.

Definition 2.1.2. Given metric spaces (X, d_X) and (Y, d_Y), a map

$$f\colon (X, d_X) \to (Y, d_Y)$$

is called a (K, L) *quasi-isometric embedding of* X *into* Y if there exist constants $K \geq 1$ and $L \geq 0$ such that for all $x, x' \in X$, the following inequality holds:

$$K^{-1} d_X(x, x') - L \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + L. \tag{2.1.1}$$

If, in addition, the map f is *roughly onto*, i.e., a fixed neighborhood of the image is the entire codomain, f is called a *quasi-isometry.* Two metric spaces are called *quasi-isometric* if there exists a quasi-isometry between them. The special case of a quasi-isometric embedding with domain a line (segment, ray, or bi-infinite) is a *quasi-geodesic.*

Remark 2.1.3. To simplify notation, in place of equation (2.1.1), we sometimes write

$$d_X(x, x') \approx_{K,L} d_Y(y, y').$$

Similarly, we write $d_X(x, x') \lesssim_{K,L} d_Y(y, y')$ to imply $d_X(x, x') \leq K d_Y(y, y') + L$. When the constants K and L are not important, they may be omitted.

2.2. Curve complex and separating curve complex.

Let $S = S_{g,n}$ be a compact surface of genus g with n boundary components. The *complexity* of S, denoted $\xi(S)$, is a topological invariant defined to be $3g - 3 + n$. An isotopy class of a simple closed curve γ on S is *essential* if it does not have a representative bounding a disk or an annulus. We will only consider hyperbolic compact surfaces, and, to streamline the exposition, we will use the term "curve" to refer to a geodesic representative of an isotopy class of an essential simple closed curve.

A *multicurve* is a (possibly empty) set of curves which are pairwise disjoint. For any surface S with positive complexity, the *curve complex* of S, denoted $\mathcal{C}(S)$, is the simplicial complex obtained by associating to each curve a 0-cell, and, more generally, a k-cell to each multicurve consisting of $k + 1$ curves. In the special case of low complexity surfaces which do not admit disjoint curves, we relax the notion of adjacency to allow edges between vertices corresponding to curves that intersect minimally on the given surface. Along similar lines, given a curve $\gamma \subset S$, let $\mathcal{N}(\gamma)$ denote a regular annular neighborhood of γ. Then the *annular complex* $\mathcal{C}(\gamma)$ is defined as follows: Vertices correspond to isotopy classes, relative to the boundary, of arcs connecting the two boundary components of $\mathcal{N}(\gamma)$, and edges connect isotopy classes (relative to the boundary) of arcs which have representatives with disjoint interiors. The annular complex measures the twisting of arcs around the core curve γ and is quasi-isometric to $\mathbb{Z}$. In fact, for any fixed points x and y on different boundary components of $\mathcal{N}(\gamma)$ and for any arc $\alpha \in \mathcal{C}(\gamma)$, there exists an adjacent arc $\beta \in \mathcal{C}(\gamma)$ with endpoints x and y.

A (multi)curve $\gamma \subset S$ is said to be *separating* if $S \setminus \gamma$ consists of a disjoint union of at least two connected essential subsurfaces, and *nonseparating* otherwise. Given this distinction, we define the *separating curve complex*, denoted $\mathcal{C}_{sep}(S)$, to be the restriction of the curve complex to the subset of separating curves. For certain low complexity surfaces, such as $S_{2,0}$, $\mathcal{C}_{sep}(S)$, as defined, is a totally disconnected space. Accordingly, in such circumstances, we relax the definition of connectivity and define two separating curves to be connected by an edge if the curves intersect minimally on the given surface. More generally, in such low complexity situations, a set of $k + 1$ separating curves forms a k-simplex if the separating curves pairwise intersect minimally. For our purposes we will only be interested in the 1-skeleton of $\mathcal{C}_{sep}(S)$. In particular, for the case of $S_{2,0}$, two separating curves in $\mathcal{C}_{sep}(S_{2,0})$ are connected if and only if they intersect four times.

2.3. Essential subsurfaces, projections.

An *essential subsurface* Y of a surface S is a disjoint union of complexity at least one subsurface which is a union of (not necessarily all) complementary components of a multicurve. Note that due to the requirement that connected components of an essential subsurface have complexity at least one, annuli are not essential subsurfaces nor are they connected components of essential subsurfaces.

As multicurves are, in fact, sets of isotopy classes of essential simple closed curves, essential subsurfaces are only defined up to isotopy. However, in light of our assumption that curves always refer to geodesic representatives of their isotopy classes, we will similarly assume fixed representatives of all essential subsurfaces by assuming that all boundary components of essential subsurfaces are geodesics. An essential subsurface $Y \subseteq S$ is *proper* if $Y \subsetneq S$. Two essential surfaces $W, V \subset S$ are *disjoint* if they have empty intersection, and *intersecting* otherwise. We are implicitly using the fact that two geodesics on hyperbolic surfaces never form a bigon [3].

An essential subsurface $Y \subset S$ is called *separating* if the multicurve ∂Y contains a separating multicurve, and *nonseparating* otherwise. In particular, note that the entire surface is always a nonseparating essential subsurface.

Given a multicurve $\alpha \subset \mathcal{C}(S)$ and a connected essential subsurface $Y \subset S$ such that α intersects Y, we can define the projection of α to $2^{\mathcal{C}(Y)}$, denoted $\pi_{\mathcal{C}(Y)}(\alpha)$, to be the collection of vertices in $\mathcal{C}(Y)$ obtained by "surgering" the arcs of $\alpha \cap Y$ along ∂Y to obtain simple closed curves in Y. Specifically, the intersection $\alpha \cap Y$ consists of a (possibly empty) submulticurve $\beta \subset \alpha$ contained in Y, as well as a disjoint union of arc subsegments of α with the endpoints of the arcs on boundary components of Y. We define the projection $\pi_{\mathcal{C}(Y)}(\alpha) \subset \mathcal{C}(Y)$ to be the union of the submulticurve β in conjunction with all curves obtained by the following process. For an arc $\alpha \cap Y$, consider the union of the arc and the components of ∂Y incident to the endpoints of the arc. Then take a regular neighborhood of this union and define the subsurface projection $\pi_{\mathcal{C}(Y)}(\alpha)$ to include all essential curves in the boundary of this regular neighborhood. See Figure 1 for an example.

In [6], it is shown that subsurface projections are coarsely well defined. Note that the projection $\pi_{\mathcal{C}(Y)}$ is only defined on curves intersecting Y. To simplify notation, when measuring distance in the image subsurface complex, we write $d_{\mathcal{C}(Y)}(\alpha_1, \alpha_2)$ as shorthand for $d_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(\alpha_1), \pi_{\mathcal{C}(Y)}(\alpha_2))$. In particular, for this distance to be well defined, α_1 and α_2 must both intersect Y.

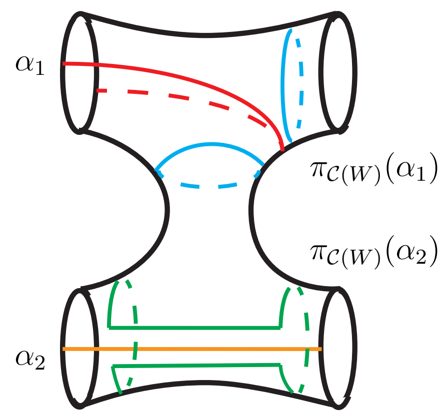

FIGURE 1. Examples of subsurface projections.

2.4. Combinatorial complexes.

A *combinatorial complex for* $\mathcal{MCG}(S)$, (for short, a *combinatorial complex* $\mathcal{G}(S)$), is any graph with vertices defined in terms of multicurves on a surface and edge relations defined in terms of upper bounds on intersection numbers between the corresponding multicurves. In addition, we will assume that combinatorial complexes admit an isometric action of the mapping class group $\mathcal{MCG}(S)$. Examples of combinatorial complexes of $\mathcal{MCG}(S)$ include the separating curve complex, the arc complex, the pants complex, and the marking complex, as well as many others.

A *hole for* $\mathcal{G}(S)$ is defined to be any connected essential subsurface or annulus such that every vertex of the combinatorial complex has non-trivial subsurface projection into it. For example, for the arc complex, holes are precisely all connected subsurfaces such that $\partial S \subset \partial Y$. On the other hand, for the complex $\mathcal{C}_{sep}(S_{2,0})$, holes are precisely the set of nonseparating essential subsurfaces.

The central idea in [7], which is also implicit in [1], is that distance in a combinatorial complex is approximated by summing over the distances in the subsurface projections to the curve complexes of holes. In particular, if a complex has disjoint holes, then the complex admits non-trivial quasi-flats, and hence cannot be δ-hyperbolic. Conversely, if a combinatorial complex has the property that no two holes are disjoint, then, assuming a couple of additional Masur–Schleimer axioms, the complex is δ-hyperbolic.

2.5. Marking complex and hierarchy paths.

A *complete marking* μ on S is a collection of *base curves* and *transverse curves* subject to the following conditions:

(1) The set of base curves $\{\gamma_1, ..., \gamma_n\}$ forms a top-dimensional simplex in $\mathcal{C}(S)$. Equivalently, $n = \xi(S)$.
(2) For each base curve γ_i, let S_{γ_i} denote the unique connected complexity one essential subsurface in the complement $S \setminus \gamma_i$. Then each base curve γ_i has a corresponding transversal curve t_i, transversely intersecting γ_i, such that t_i intersects γ_i once if S_{γ_i} is topologically $S_{1,1}$ and twice if S_{γ_i} is topologically $S_{0,4}$.

A complete marking μ is said to be *clean* if, in addition, each transverse curve t_i is disjoint from all other base curves γ_j. Two complete markings μ and μ' are *compatible* if they have the same base curves and, moreover, for all i, the distance in the annular complex $d_{\mathcal{C}(\gamma_i)}(t_i, t'_i)$ is minimal over all choices of t'_i. See [6] for technical details regarding the distance in annular complex. For our purposes, it suffices to use the fact that traveling in the annular complex is accomplished by taking an arc in a regular neighborhood of the annulus and Dehn twisting it about the core curve of the annulus. In [6], it is shown that there is a bound, depending only on the topological type of S, on the number of clean complete markings which are compatible with any given complete marking.

Let μ denote a clean complete marking with curve pair data (γ_i, t_i); then we define an *elementary move* to be one of the following two operations applied to the marking μ:

(1) *Twist*: For some i, we replace (γ_i, t_i) with (γ_i, t'_i) where t'_i is the result of one full or one half twist (when possible) of t_i around γ_i.
(2) *Flip*: For some i, we interchange the base and transversal curves. After a flip move, the resulting complete marking may no longer be clean, in which case, as part of the flip move, we then replace the non-clean complete marking with a compatible clean complete marking. Since there is a uniform bound on the number of clean complete markings which are compatible with any given complete marking, a flip move is coarsely well defined.

The *marking complex*, $\mathcal{M}(S)$, is defined to be the graph formed by taking clean complete markings of S to be vertices and connecting two vertices by an edge if they differ by an elementary move.

In [6], a 2-transitive family of quasi-geodesics in $\mathcal{M}(S)$, called *resolutions of hierarchies*, is developed. In broad strokes, hierarchies are defined inductively as a union of geodesics of multicurves in the curve complexes of essential subsurfaces or annuli, while resolutions of hierarchies are quasi-geodesics in the marking complex associated to a hierarchy. By abuse of notation, throughout this paper, we will refer to resolutions of hierarchies as hierarchies. The construction of hierarchies is technical, although, for our purposes, the following theorem recording some of their properties suffices.

Theorem 2.5.1 ([6, §4, p. 933]). *For $S = S_{g,n}$ and for all $\mu, \nu \in \mathcal{M}(S)$, there exists a hierarchy path $\rho = \rho(\mu, \nu) : [0, n] \to \mathcal{M}(S)$ with $\rho(0) = \mu$ and $\rho(n) = \nu$. Moreover, ρ is a quasi-geodesic with constants depending only on the topological type of S with the following properties:*

- [H1]: *The hierarchy ρ shadows a $\mathcal{C}(S)$ geodesic of multicurves, g_S, from a multicurve $a \subset base(\mu)$ to a multicurve $b \subset base(\nu)$, called the* main geodesic of the hierarchy. *That is, there is a monotonic map $\phi\colon \rho \to g_S$ such that for all i, $\phi(\rho(i)) \subset base(\rho(i))$.*
- [H2]: *There is a constant M_1 such that if an essential subsurface or an annulus $Y \subset S$ satisfies $d_{\mathcal{C}(Y)}(\mu, \nu) > M_1$, then there is a maximal connected interval $I_Y = [t_{Y,1}, t_{Y,2}]$ and a geodesic of multicurves, g_Y, in $\mathcal{C}(Y)$ from a submulticurve in $base(\rho(t_{Y,1}))$ to a submulticurve in $base(\rho(t_{Y,2}))$ such that for all $t_{Y,1} \leq t \leq t_{Y,2}$, ∂Y is a submulticurve in $base(\rho(t))$ and $\rho|_{I_Y}$ shadows the geodesic g_Y. Such a subsurface Y is called a* component domain *of ρ. By convention, the entire surface S is always considered a component domain.*

The next theorem contains a quasi-distance formula for $\mathcal{M}(S)$ which serves as both the motivation for, as well as an important ingredient in, proving Theorem 1.0.1.

Theorem 2.5.2 ([6, Theorem 6.12]). *For $S = S_{g,n}$, there is constant C_0 such that for all $c \geq C_0$, there exist constants $K(c)$ and $L(c)$ such that for all $\alpha, \beta \in \mathcal{M}(S)$, we have the quasi-isometry*

$$\sum_{Y \subseteq S} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_c \approx_{K,L} d_{\mathcal{M}(S)}(\alpha, \beta),$$

where the sums are over all connected essential subsurfaces Y or annuli.

Remark 2.5.3. Note that holes for the marking complex are precisely all essential subsurfaces Y or annuli. Hence, the sums in Theorem 2.5.2 are sums over all holes.

2.6. Farey graph.

The Farey graph is a classical graph which has vertices corresponding to elements of $\mathbb{Q} \cup \{\infty = \frac{1}{0}\}$ and edges between two rational numbers in lowest terms $\frac{p}{q}$ and $\frac{r}{s}$ if $|ps - qr| = 1$. The Farey graph can be drawn as an ideal triangulation of the unit disk as in Figure 2.

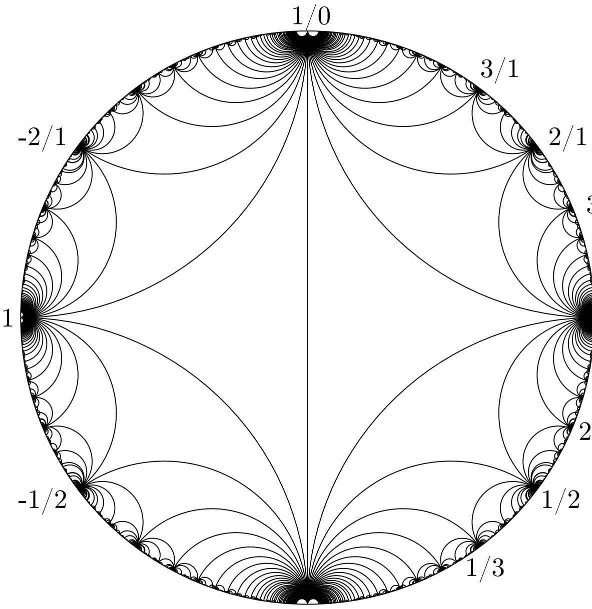

FIGURE 2. Farey Graph with some vertices labeled.

A nice feature of the Farey graph is the so-called *Farey addition property* which ensures that if rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ are connected in the Farey graph, then there is an ideal triangle in the Farey graph with vertices $\frac{p}{q}$, $\frac{r}{s}$, and $\frac{p+r}{q+s}$. The curve complexes $\mathcal{C}(S_{0,4})$ and $\mathcal{C}(S_{1,1})$ are isomorphic to the Farey graph. The isomorphism is given by sending the meridional curve of the surfaces to $\frac{1}{0}$, the longitudinal curve of the surfaces to $\frac{0}{1}$, and, more generally, the (p, q) curve to $\frac{p}{q}$.

3. Proof of Theorem 1.0.1

In subsection 3.1 we will show that $\mathcal{C}_{sep}(S_{2,0})$ has a quasi-distance formula as in Theorem 1.0.1. Then, in subsection 3.2, using the quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$, we show that the Masur–Schleimer proof for δ-hyperbolicity of a combinatorial complex found in [7] applies to $\mathcal{C}_{sep}(S_{2,0})$, thus proving that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic.

3.1. $\mathcal{C}_{sep}(S_{2,0})$ HAS A QUASI-DISTANCE FORMULA.

We begin by recalling a lemma from [7] which, in particular, ensures a quasi-lower bound for a quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$.

Lemma 3.1.1 ([7, Theorem 5.10])**.** *For $S = S_{g,n}$ and any combinatorial complex $\mathcal{G}(S)$, there is a constant C_0 such that for all $c \geq C_0$, there exists constants $K(c)$ and $L(c)$ such that for all $\alpha, \beta \in \mathcal{G}(S)$, we have the relationship*

$$\sum_{Y \text{ a hole for } \mathcal{G}(S)} \{d_{\mathcal{C}(Y)}(\alpha, \beta)\}_c \lesssim d_{\mathcal{G}(S)}(\alpha, \beta).$$

In light of Lemma 3.1.1, to prove a quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$, it suffices to obtain a quasi-upper bound on $\mathcal{C}_{sep}(S_{2,0})$ distance in terms of the sum of subsurface projections to holes. As motivated by [7], our approach for doing so will be by relating markings to separating curves and, more generally, by marking paths to paths in the separating curve complex. In the rest of this subsection, let $S = S_{2,0}$.

Let $\mu \in \mathcal{M}(S)$. Presently, we describe a coarsely well-defined mapping

$$\phi\colon\ \mathcal{M}(S) \to 2^{\mathcal{C}_{sep}(S)}.$$

If *base*(μ) contains a separating curve γ_i, then we define $\phi(\mu)$ to contain γ_i. On the other hand, if all three base curves of μ, (γ_1, γ_2, and γ_3), are nonseparating curves, then for any $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k \neq i$, denote the essential subsurface $S_{i,j} := S \setminus \gamma_i, \gamma_j \simeq S_{0,4}$. Note that $\mathcal{C}(S_{i,j})$ is a Farey graph containing the adjacent curves γ_k and t_k. Let o_k be a curve in $S_{i,j}$ such that γ_k, t_k, and o_k form a triangle in $\mathcal{C}(S_{i,j})$. Note that o_k is not uniquely determined by this condition; in fact, there are exactly two possibilities for o_k, which we denote $o_k^{\pm}$. Note that $d_{\mathcal{C}(S_{i,j})}(o_k^+, o_k^-) = 2$. Then, in this case (assuming that none of the base curves of μ are separating curves), define $\phi(\mu)$ to contain all the curves in the set $\{t_i o_i^{\pm}, t_j, o_j^{\pm}, t_k, o_k^{\pm}\}$ which are separating curves of S. The following lemma ensures that the mapping ϕ always has non-trivial image.

Lemma 3.1.2. *With the notation from above, assume base$(\mu) \cap \mathcal{C}_{sep}(S) = \emptyset$ and let γ_k, t_k, o_k form a triangle in the Farey graph $\mathcal{C}(S_{i,j})$. Then one and only one of the curves γ_k, t_k, or o_k is separating curves of S.*

Proof. The subsurface $S_{i,j}$ has four boundary components which glue up in pairs inside the ambient surface S. Any curve $\alpha \in \mathcal{C}(S_{i,j})$ gives rise to a partition of the four boundary components of $S_{i,j}$ into pairs given by pairing boundary components in the same connected component of $S_{i,j} \setminus \alpha$. In total, there are three different ways to partition the four boundary components of $S_{i,j}$ into pairs. Precisely one of the three partitions has the property that any curve that gives rise to the given partition is a separating curve. The partition corresponding to a (p, q)-curve on $S_{i,j}$ is entirely determined by $\frac{p \bmod 2}{q \bmod 2}$, with $\frac{0}{0} = \frac{1}{1}$. Using the Farey addition property, it follows that each triangle in the Farey graph has exactly one representative from each equivalence class. The lemma follows. □

The following theorem ensures that the mapping $\phi\colon \mathcal{M}(S) \to \mathcal{C}_{sep}(S)$ is coarsely well defined.

Theorem 3.1.3. *Using the notation from Lemma 3.1.2, let t_i and t_j be transversals which are separating curves. Then t_i and t_j are connected in the separating curve complex $\mathcal{C}_{sep}(S)$. Similarly, if t_i and o_j or o_i and o_j are separating curves, the same result holds.*

Proof. We will prove the first case; the similar statement follows from the same proof. Specifically, we will show that the separating curves t_i and t_j intersect four times. Up to action of $\mathcal{MCG}(S)$, there is only one picture for a marking μ which does not contain a separating base curve, as presented in Figure 3. Without loss of generality, we can assume $t_i = t_1$ and $t_j = t_2$. Notice that in the subsurface $S_{2,3}$, as in Figure 3, the base curve γ_1 corresponds to the meridional curve $\frac{1}{0}$ and, similarly, in the subsurface $S_{1,3}$, the base curve γ_2 also corresponds to the meridional curve $\frac{1}{0}$. Since t_1 is connected to γ_1 in the Farey graph $\mathcal{C}(S_{2,3})$, it follows that $t_1 \in \mathcal{C}(S_{2,3})$ is a curve of the form $\frac{n}{1}$ for some integer n. Similarly, $t_2 \in \mathcal{C}(S_{1,3})$ is a curve of the form $\frac{m}{1}$ for some integer m. As in the example of Figure 3, it is easy to draw representatives of the two curves t_1 and t_2 which intersect four times. □

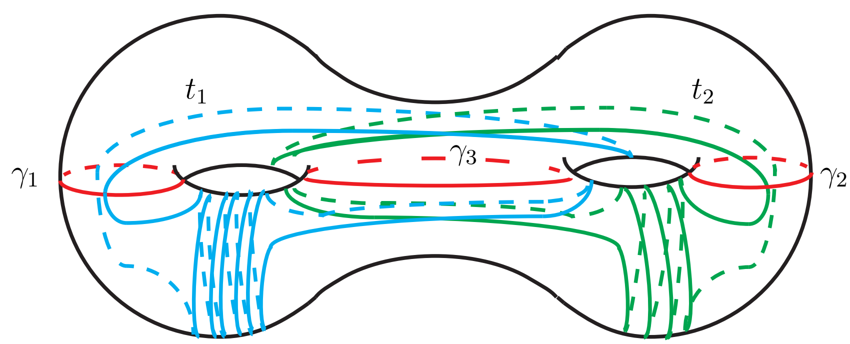

FIGURE 3. A marking μ on $S = S_{2,0}$ with no separating curves. Notice that $d_{\mathcal{C}_{sep}(S)}(t_1, t_2) = 1$.

The following lemma says that our coarsely well-defined mapping ϕ which associates a separating curve to a clean complete marking is natural with respect to elementary moves in the marking complex.

Lemma 3.1.4. *If $d_{\mathcal{M}(S)}(\mu,\nu) \leq 1$, then $\phi(\mu) \cap \phi(\nu) \neq \emptyset$.*

Proof. First, assume μ and ν differ by a twist move applied to the pair (γ_i, t_i). If μ has a separating base curve, then so does ν, as twists do not affect base curves. Then, by definition, this separating base curve is in the intersection $\phi(\mu) \cap \phi(\nu)$. On the other hand, if μ has no separating base curves, then by Lemma 3.1.2, either t_j or o_j, for $i \neq j$, is a separating curve. In either case, this separating curve is in the intersection $\phi(\mu) \cap \phi(\nu)$, and we are also done.

Next, assume μ and ν differ by a flip move applied to the pair (γ_i, t_i). Recall that after the flip move is performed, one must pass to a compatible clean marking. Specifically, if $\mu = \{(\gamma_i, t_i), (\gamma_j, t_j), (\gamma_k, t_k)\}$, then $\nu = \{(t_i, \gamma_i), (\gamma_j, t'_j), (\gamma_k, t'_k)\}$, where the transversals t'_j and t'_k are obtained by passing to a compatible clean marking if necessary. If γ_i, γ_j, or γ_k is a separating base curve, we are done. Finally, if none of the base curves of μ is a separating curve, then we are also done as, again by Claim 3.1.2, either t_j or o_j is a separating curve. In either case, this separating curve is in the intersection $\phi(\mu) \cap \phi(\nu)$. □

Considering our coarsely well-defined mapping $\phi\colon \mathcal{M}(S) \to \mathcal{C}_{sep}(S)$ in conjunction with Lemma 3.1.4, we have the following procedure for finding a path between any two separating curves. Given $\alpha, \beta \in \mathcal{C}_{sep}(S)$, complete the separating curves into clean complete markings μ and ν such that $\alpha \in base(\mu)$ and $\beta \in base(\nu)$. Then construct a hierarchy path ρ in $\mathcal{M}(S)$ between μ and ν. Applying the mapping ϕ to our hierarchy path ρ yields a path of separating curves in $\mathcal{C}_{sep}(S)$ between the separating curves α and β. By construction, the length of the obtained path in $\mathcal{C}_{sep}(S)$ between the separating curves α and β has length quasi-bounded above by the length of the marking path ρ. In fact, in the following corollary, we will use this procedure to obtain a quasi-upper bound on $\mathcal{C}_{sep}(S_{2,0})$ distance in terms of the sum of subsurface projection to holes. Note that together with Lemma 3.1.1, Corollary 3.1.5 gives a quasi-distance formula for $\mathcal{C}_{sep}(S)$, thus completing the proof of Theorem 1.0.1. Recall that the set of holes for $\mathcal{C}_{sep}(S)$ is precisely the set of all nonseparating essential subsurfaces or, equivalently, all essential subsurfaces whose boundary does not contain a separating curve.

Corollary 3.1.5. *For $S = S_{2,0}$, there is a constant K_0 such that for all $k \geq K_0$, there exists quasi-isometry constants such that for all $\alpha, \beta \in \mathcal{C}_{sep}(S)$,*

$$d_{\mathcal{C}_{sep}(S)}(\alpha,\beta) \lessapprox \sum_{Y \text{ a hole for } \mathcal{C}_{sep}(S)} \{d_{\mathcal{C}(Y)}(\alpha,\beta)\}_k.$$

Proof. As noted, we have a quasi-upper bound on $d_{\mathcal{C}_{sep}(S)}(\alpha,\beta)$ given by the length of a hierarchy path $\rho\colon [0,n] \to \mathcal{M}(S)$ such that $\alpha \in base(\rho(0))$ and $\beta \in base(\rho(n))$. In other words, by theorems 2.5.1 and 2.5.2, we already have a quasi-upper bound of the form

$$d_{\mathcal{C}_{sep}(S)}(\alpha,\beta) \lessapprox \sum_{\xi(Y)\geq 1, \text{ or } Y \text{ an annulus}} \{d_{\mathcal{C}(Y)}(\alpha,\beta)\}_k.$$

Hence, it suffices to show that for all component domains Y in the above sum which are not holes of $\mathcal{C}_{sep}(S)$, there is a uniform bound on $diam_{\mathcal{C}_{sep}(S)}(\phi(I_Y))$, where $I_Y = [t_{Y,1}, ..., t_{Y,m}]$ is as in property [H2] of Theorem 2.5.1. Since holes for $\mathcal{C}_{sep}(S)$ consist of all nonseparating essential subsurfaces, we can assume Y is either an annulus or a separating essential subsurface. Furthermore, since ϕ is coarsely well defined, it suffices to show that for any such Y, the intersection $\bigcap_{j=1}^m \phi(\rho(t_{Y,j}))$ is nonempty.

First, consider the case of Y an annulus. In this case, the subpath of ρ in the marking complex corresponding to I_Y consists of a sequence of clean complete markings $\rho(t_{Y,1}), ..., \rho(t_{Y,m})$, such that any adjacent markings $\rho(t_{Y,j}), \rho(t_{Y,j+1})$ differ by a twist move along a fixed base curve γ_i (which is the core of the annulus Y). Then, exactly as in the first paragraph of the proof of Lemma 3.1.4, it follows that $\bigcap_{j=1}^m \phi(\rho(t_{Y,j}))$ is nonempty, thus completing the proof for Y an annulus.

On the other hand, if Y is a separating essential subsurface or, equivalently, ∂Y contains a separating curve $\alpha \in \mathcal{C}_{sep}(S)$, then by property [H2] of Theorem 2.5.1, the separating curve α is contained as a base curve in all markings $\rho(t_{Y,j})$ for all $j \in \{1, ..., m\}$. In particular, $\alpha \in \bigcap_{j=1}^m \phi(\rho(t_{Y,j}))$. This completes the proof. □

3.2. $\mathcal{C}_{sep}(S_{2,0})$ IS δ-HYPERBOLIC.

In [7, §13], sufficient axioms are established for implying that a combinatorial complex admits a quasi-distance formula and, furthermore, is δ-hyperbolic. The first axiom is that no two holes for the combinatorial complex are disjoint. This is easily verified for $\mathcal{C}_{sep}(S_{2,0})$. The rest of the axioms are related to the existence of an appropriate marking path $\{\mu_i\}_{i=0}^N \subset \mathcal{M}(S)$ and a corresponding well-suited combinatorial path $\{\gamma_i\}_{i=0}^K \subset \mathcal{G}(S)$. In particular, there is a strictly increasing reindexing function $r : [J, K] \to [0, N]$ with $r(J) = 0$ and $r(K) = N$. In the event that one uses a hierarchy as a marking path, the rest of the axioms can be simplified as follows:

(1) (Combinatorial) There is a constant C_1 such that for all i,

$$d_{\mathcal{C}(Y)}(\gamma_i, \mu_{r(i)}) < C_1$$

for every hole Y and, moreover, $d_{\mathcal{G}(S)}(\gamma_i, \gamma_{i+1}) < C_1$.

(2) (Replacement) There is a constant C_2 such that
   [R1] if Y is a hole and $r(i) \in I_Y$, then there is a vertex $\gamma' \in \mathcal{G}(S)$ with $\gamma' \subset Y$ and $d_{\mathcal{G}(S)}(\gamma, \gamma') < C_2$;
   [R2] if Y is a non-hole and $r(i) \in I_Y$, then there is a vertex $\gamma' \in \mathcal{G}(S)$ with $\gamma' \subset Y$ or $\gamma' \subset S \setminus Y$ and $d_{\mathcal{G}(S)}(\gamma, \gamma') < C_2$.

(3) (Straight) There exist constants such that if for any subinterval $[p, q] \subset [0, K]$ with the property that $d_{\mathcal{C}(Y)}(\mu_{r(p)}, \mu_{r(q)})$ is uniformly bounded for all non-holes, then $d_{\mathcal{G}(S)}(\gamma_p, \gamma_q) \lesssim d_{\mathcal{C}(S)}(\gamma_p, \gamma_q)$.

Presently, we will show that in the case of the separating curve complex $\mathcal{C}_{sep}(S_{2,0})$, all of the above axioms, with the exception of [R2], hold. Let $\rho = \{\mu_i\}_{i=0}^n$ be a hierarchy path between two clean complete markings, each containing a separating base curve. Then, as in the proof of Corollary 3.1.5, define the combinatorial path $\{\gamma_i\}_{i=0}^K \subset \mathcal{C}_{sep}(S_{2,0})$ using the coarsely well-defined map $\phi\colon \mathcal{M}(S_{2,0}) \to \mathcal{C}_{sep}(S_{2,0})$ applied to the hierarchy ρ. Let the reindexing function r be defined by

$$r(i) = max_{[0,N]}\{j | \gamma_i \in \phi(\mu_j)\}.$$

Given this setting, the first clause of the combinatorial axiom is immediate from the definition of ϕ, while the "moreover" clause follows from Lemma 3.1.4 and the fact that ϕ is coarsely well defined. Similarly, the straight axiom follows from the properties of hierarchy paths of Theorem 2.5.1. Replacement axiom [R1] also holds, for if Y is a hole, then ∂Y contains at most two nonseparating curves. Then, for all markings $\mu \in I_Y$, $base(\mu)$ contains the at most two nonseparating curves ∂Y. Let γ_i be a base curve of μ not in ∂Y. Then, by Lemma 3.1.2, one of γ_i, t_i, or o_i, is a separating curve properly contained in Y. Lemma 3.1.2 ensures that exactly one of the three curves γ_i, t_i, or o_i is a separating curve. On the other hand, [R2] fails because if Y is an essential subsurface which is a nonhole, then perforce by topological considerations, $\partial Y \in \mathcal{C}_{sep}(S_{2,0})$. In this case, there cannot exist any separating curve properly contained in either Y or $S_{2,0} \setminus Y$.

Nonetheless, while the Masur–Schleimer axioms fail due to the failure of [R2], Masur and Schleimer's proof that a combinatorial complex satisfying the Masur-Schleimer axioms is δ-hyperbolic carries through in the case of $\mathcal{C}_{sep}(S_{2,0})$. In §14 of [7], Masur and Schleimer prove that a combinatorial complex satisfying the Masur–Schleimer axioms satisfies a quasi-distance formula. Then, in §20 of [7], they go on to prove that, in addition, a combinatorial complex satisfying the Masur–Schleimer axioms is also δ-hyperbolic. To be sure, in §20, in the course of proving δ-hyperbolicity, the replacement axiom is not, in fact, ever directly needed. Instead, the replacement axiom is only used in §14 to prove the existence of a quasi-distance formula, and then this formula, in turn, is used in §20, along with other axioms, to prove δ-hyperbolicity. Consequently, since, in this paper, we have obtained a direct proof of the existence of a quasi-distance formula for $\mathcal{C}_{sep}(S_{2,0})$, it follows from the argument in §20 of [7] that $\mathcal{C}_{sep}(S_{2,0})$ is δ-hyperbolic.

3.3. A quasi-distance formula for $\mathcal{C}_{sep}(S)$ in general?

Considering the arguments in section 3.1, naïve consideration suggests appropriate modifications may provide a proof of a quasi-distance formula for $\mathcal{C}_{sep}(S)$ in general. However, this is certainly not immediate. Specifically, an explicit construction in [9] implies that, for high enough genus, there exist clean complete markings of closed surfaces which are arbitrarily far (with respect to elementary moves) from any clean complete marking containing a separating base or transversal curve. This is in stark contrast to the situation in $\mathcal{C}_{sep}(S_{2,0})$, for which we make strong use of the fact that any clean complete marking is distance at most one from a clean complete marking containing a separating base or transversal curve.

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