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COUNTABLE IRRESOLVABLE SPACES AND CARDINAL INVARIANTS

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Abstract. Answering a question of Marion Scheepers, we show that the cardinal invariant $\mathfrak{d}$ is a lower bound on $\mathfrak{irr}$, the minimal weight (equivalently, minimal π-weight) of a countable regular irresolvable space. We consider related cardinal invariants such as $\mathfrak{r}_{\mathsf{scat}}$, the reaping number of the quotient algebra $\mathcal{P}(\mathbb{Q})$ mod the ideal of scattered subsets of the rationals, and prove that $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ implies that $\mathfrak{irr} = \omega_1$.

1. Introduction

All topological spaces considered are regular and *crowded*, i.e., have no isolated points. A topological space X is said to be *irresolvable* provided there are no disjoint dense subsets Y, $W \subseteq X$. Otherwise, X is *resolvable.*

It is easy to see that $\mathbb{Q}$ is a resolvable space. It follows, due to a well-known theorem of Sierpiński, that every countable first countable crowded regular space is resolvable. So, if X is a countable regular irresolvable space, $w(X)$ should be uncountable. In fact, the same is true for countable regular spaces with countable π-weight.

Marion Scheepers [6] defines the *irresolvability number* as follows:

$$\mathfrak{irr} = \min\{\pi w((\omega, \tau)) : \tau \subseteq \mathcal{P}(\omega) \textit{ is an irresolvable } T_3 \textit{ topology on } \omega\}.$$

It is folklore that $\mathfrak{r} \leq \mathfrak{irr} \leq \mathfrak{i}$ (see [3], [6]), where $\mathfrak{r}$ denotes the *reaping number* (the minimal size of a *reaping* (or *unsplittable*) family, i.e., the

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minimal size of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for any $X \in [\omega]^\omega$, there is an $R \in \mathcal{R}$ such that $R \subseteq X$ or $R \cap X = \emptyset$), and $\mathfrak{i}$ is the minimal size of a *maximal independent* family (see [3], [5]). In [6], Scheepers asks whether the equality $\mathfrak{r} = \mathfrak{irr}$ is provable in ZFC. We will show that the *dominating number* $\mathfrak{d}$ (see [5]) is also a lower bound for $\mathfrak{irr}$; hence, in particular, it is relatively consistent with ZFC to have $\mathfrak{r} < \mathfrak{irr}$. We also consider related cardinal invariants such as the reaping number of the quotient algebra $\mathcal{P}(\mathbb{Q})$ mod the ideal of scattered subsets of the rationals, and compare them to the cardinal invariants already mentioned.

2. The Cardinal Invariant $\mathfrak{irr}$ and Other Cardinals

The following proposition tells us that we can replace "π-weight" with "weight" in the definition of $\mathfrak{irr}$.

Proposition 2.1. *The irresolvability number* $\mathfrak{irr}$ *is equal to the minimum weight of a countable irresolvable* T_3 *space.*

Proof. It is clear that $\mathfrak{irr}$ is less or equal to the minimum weight of a countable irresolvable T_3 space.

Let τ be an irresolvable T_3 topology on ω of minimal π-weight and let $\mathcal{B}$ be a π-base witnessing the minimality of $\pi w((\omega, \tau))$. We claim that there is an $X \subseteq \omega$ and a topology τ' on X such that (X, τ') is irresolvable T_3 and has weight at most $\mathfrak{irr}$. Let $X = \bigcup \mathcal{B}$, and for each $U \in \mathcal{B}$ and each pair of distinct points $x, y \in U$, pick $W_x(U, x, y)$ and $W_y(U, x, y)$ disjoint clopen sets such that $x \in W_x(U, x, y)$ and $y \in W_y(U, x, y)$, and $W_x(U, x, y), W_y(U, x, y) \subseteq U$. Now consider the following family of sets:

$$\begin{aligned}\mathcal{B}' = \{&W_x(U,x,y), W_y(U,x,y) : U \in \mathcal{B},\, x, y \in U,\, x \neq y\} \cup \\ &\{X \setminus W_x(U,x,y), X \setminus W_y(U,x,y) : U \in \mathcal{B},\, x, y \in U,\, x \neq y\}.\end{aligned}$$

Finally, let τ' be the topology on X generated by $\mathcal{B}'$. Then (X, τ') is a countable irresolvable T_3 space of weight at most $\mathfrak{irr}$. $\square$

Recall that a set $X \subseteq \mathbb{Q}$ is *scattered* if every non-empty $Y \subseteq X$ has an isolated point. The collection of all scattered subsets of $\mathbb{Q}$ forms a proper ideal which will be denoted by scat, and the family $\mathcal{P}(\mathbb{Q}) \setminus \mathsf{scat}$ of scat-positive sets will be denoted by scat^+.

Definition 2.2. A family $\mathcal{R} \subseteq \mathsf{scat}^+$ is called *scattered-reaping* if, for every $X \in \mathsf{scat}^+$, there is a $Y \in \mathcal{R}$ such that $Y \subseteq X$ or $X \cap Y = \emptyset$. The *scattered-reaping number*, which we denote by $\mathfrak{r}_{\mathsf{scat}}$, is defined as the minimum size of a scattered-reaping family

$$\begin{aligned}\mathfrak{r}_{\mathsf{scat}} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathsf{scat}^+ \\ (\forall X \in \mathsf{scat}^+)(\exists Y \in \mathcal{R})(Y \subseteq X \vee Y \cap X = \emptyset)\}.\end{aligned}$$

It is easy to see that $\mathfrak{r}_{\mathsf{scat}}$ is equal to $\mathfrak{r}(\mathcal{P}(\mathbb{Q})/\mathsf{scat})$—the reaping number of the Boolean algebra $\mathcal{P}(\mathbb{Q})/\mathsf{scat}$.

Proposition 2.3. $\mathfrak{r}_{\mathsf{scat}} \leq \mathfrak{irr}$.

Proof. Let τ be an irresolvable topology on ω, and let $\mathcal{M} \preccurlyeq H(\theta)$ be a countable elementary submodel with $\tau \in \mathcal{M}$ and $\mathcal{B}$ a π-basis for τ. Take $\mathcal{B}' = \tau \cap \mathcal{M}$. Due to Sierpiński's theorem, $\mathcal{B}'$ generates a topology τ' which is homeomorphic to the topology of $\mathbb{Q}$, so we can assume that $\mathsf{scat} = \mathsf{scat}_{(\omega,\tau')}$. Note that every non-empty $U \in \tau$ is a scat-positive set. Let $X \in \mathsf{scat}^+$. Since (ω, τ) is irresolvable, one of X and $\omega \setminus X$ has non-empty τ-interior. If $int_\tau(X)$ is not empty, we get $U \in \mathcal{B}$ such that $U \subseteq X$. If $int_\tau(\omega \setminus X) \neq \emptyset$, we get basic open $U \in \mathcal{B}$ such that $U \cap X = \emptyset$. So, $\mathcal{B}$ is a scattered-reaping family. $\square$

Lemma 2.4. *For every $A \in \mathsf{scat}^+$, there is a crowded closed nowhere-dense set B such that $B \subseteq A$.*

Proof. Let $A \in \mathsf{scat}^+$. Without loss of generality, A is a crowded set. For each $n \in \omega$, let $\{B_{n,m} : m \in \omega\}$ be a local basis of clopen sets at n. Recursively, construct an increasing sequence $\{F_m : m \in \omega\}$ of finite subsets of A and an increasing sequence of clopen sets $\{U_m : m \in \omega\}$ satisfying the following:

(a) For all $n \in \omega$, there is m such that $n \in F_m$ or $n \in U_m$.
(b) For all $m \in \omega$, $F_m \cap U_m = \emptyset$.
(c) For all $m \in \omega$, all $k \in F_m$, and all $i > m$, $B_{k,i} \cap (F_i \setminus \{k\}) \neq \emptyset$.

Suppose both sequences have been successfully constructed. Put $F = \bigcup_{n\in\omega} F_n$. The clause (c) ensures that F is a crowded set, while (a) and (b) tell us that F is closed. Since every F_n is a subset of A, we have $F \subseteq A$. If F is not nowhere dense, replace it by any of its closed crowded nowhere dense subsets.

In order to carry out the construction, let $k_0 = \min(A)$ and $F_0 = \{k_0\}$. Pick a clopen set U_0 such that $\{i : i < k_0\} \subseteq U_0$ and $k_0 \notin U_0$. Now, suppose that F_m and U_m have been defined. Then $F_m \subseteq A \setminus U_m$ and $A \setminus U_m$ is crowded. For each $k \in F_m$, pick $n_k \in B_{k,m+1} \cap (A \setminus U_m)$, and let $F_{m+1} = F_m \cup \{n_k : k \in F_m\}$. Finally, let $j = \min(A \setminus (F_{m+1} \cup U_m))$, and pick a clopen set V such that $j \in V$ and $V \cap F_{m+1} = \emptyset$, and let $U_{m+1} = U_m \cup V$. Obviously (a), (b) and (c) are satisfied. $\square$

Proposition 2.5. $\mathfrak{d} \leq \mathfrak{r}_{\mathsf{scat}}$.

Proof. Let $\mathcal{F} \subseteq \mathsf{scat}^+$ be a collection of crowded sets of cardinality less than $\mathfrak{d}$. We can assume that $\mathcal{F}$ contains a countable basis for the topology (of the rationals). We will find a set $Y \in \mathsf{scat}^+$ such that for all $X \in \mathcal{F}$, both $X \cap Y$ and $X \setminus Y$ are in scat^+. By Lemma 2.4, we can assume that each $X \in \mathcal{F}$ is a crowded closed nowhere dense set. For each $n \in \omega$, let C_n be the set of all $X \in \mathcal{F}$ such that $n \in X$. Note that C_n has size less than $\mathfrak{d}$. Recursively, construct two sequences $\{A_n : n \in \omega\}$, $\{B_n : n \in \omega\}$ of subsets of ω such that

(i) $A_0 = B_0 = \emptyset$;
(ii) for all n, $A_n \cap B_n = \emptyset$;
(iii) for all n, $A_n, B_n \in \mathsf{scat}$;
(iv) for all n, $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$;
(v) for all n, $n \in \bar{A}_{n+1} \cap \bar{B}_{n+1}$;
(vi) for all n and for all $X \in C_n$, $A_{n+1} \cap X \neq \emptyset$ and $B_{n+1} \cap X \neq \emptyset$.

It is clear from the construction that $A = \bigcup_{n\in\omega} A_n$ and $B = \bigcup_{n\in\omega} B_n$ are disjoint dense sets, and item (vi) implies that for all $X \in \mathcal{F}$, $A \cap X$ and $A \setminus X \supseteq X \cap B$ are both infinite. We prove below that these intersections are actually scattered-positive sets.

Suppose that both A_n and B_n have been defined. If $n \in \bar{A}_n \cap \bar{B}_n$, put $A_{n+1} = A_n$ and $B_{n+1} = B_n$. If $n \notin \bar{A}_n$, let $\{W_m : m \in \omega\}$ be a partition of $\omega \setminus (\bar{A}_n \cup \bar{B}_n \cup \{n\})$ into clopen sets. Note that none of them has n in its closure, so for all $X \in C_n$ and all $k \in \omega$, $X \nsubseteq W_k$. Moreover, for infinitely many $k \in \omega$, $W_k \cap X$ is infinite (otherwise, X would be a scattered set with n as its unique limit point). For each $X \in C_n$, let $\tilde{X}(k)$ be the minimum $i \geq k$ such that $X \cap W_i \neq \emptyset$. Then define the following function:

$$f_X(i) = \min(X \cap W_{\tilde{X}(i)}) + 1.$$

Since $|C_n| < \mathfrak{d}$, there is an increasing function f which is not dominated by $\{f_X : X :\in C_n\}$. Having fixed such f, let

$$A_{n+1} = A_n \cup \bigcup_{k\in\omega} \big(W_k \cap f(k)\big).$$

It is easily seen that A_{n+1} satisfies (i)–(vi). Now, consider $W'_k = W_k \setminus A_{n+1}$. Note that again, for all $X \in C_n$, there are infinitely many $k \in \omega$ such that $X \cap W'_k$ is infinite. Let $\tilde{X}'(k)$ be the minimum $i \geq k$ such that $X \cap W'_i \neq \emptyset$. Define a new family of functions $\{g_X : X \in C_n\}$ as follows:

$$g_X(i) = \min(X \cap W'_{\tilde{X}'(i)}) + 1.$$

Again, fix an increasing function $g : \omega \to \omega$ which is not dominated by $\{g_X : X \in C_n\}$ and let

$$B_{n+1} = B_n \cup \bigcup_{k\in\omega} \Big(W'_k \cap g(k)\Big).$$

Then B_{n+1} satisfies (i)–(vi).

Finally, we prove that for all $X \in \mathcal{F}$, both $A \cap X$ and $X \cap B$ are scat-positive sets. Let $X \in \mathcal{F}$ be an arbitrary set. We prove $A \cap X \in \mathsf{scat}^+$;

the proof for B follows a similar argument. It is enough to prove that for all n, each $m \in A_n \cap X$ greater than n is a limit point of $A \cap X$. This follows from the construction: Since for such m, we have $X \in C_m$, the set $\bigcup_{k\in\omega} \big(W_k \cap f(k)\big)$ in the definition of A_{m+1} has infinite intersection with X, but $\bigcup_{k\in\omega} \big(W_k \cap f(k)\big)$ has m as its unique limit point, so in consequence, $X \cap \bigcup_{k\in\omega} \big(W_k \cap f(k)\big)$ has m as a limit point, which in turn implies that m is a limit point of $A \cap X$.

Making $Y = A$, we are done. □

Corollary 2.6. $\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{r}_{\mathsf{scat}} \leq \mathfrak{irr}$.

Let us turn our attention to the question of Scheepers. It is well known that in the *Miller model* (see [2], [4]) $\mathfrak{r} < \mathfrak{d}$. In particular, in this model, $\mathfrak{r} < \mathfrak{r}_{\mathsf{scat}} = \mathfrak{irr}$ holds.

Corollary 2.7. *It is relatively consistent with* ZFC *that* $\mathfrak{r} < \mathfrak{irr}$.

3. A Diamond for $\mathfrak{r}_{\mathsf{scat}}$

It is well known that for many non-Borel cardinal invariants there is a Borel cardinal invariant such that its associated $\diamondsuit$-principle implies the former to be equal to ω_1 (see [5]). Some examples of this phenomena are the cases of $\mathfrak{b}$ and $\mathfrak{a}$, $\mathfrak{r}$ and $\mathfrak{u}$, and $\mathfrak{r}_{\mathbb{Q}}$[1] and $\mathfrak{i}$ (see [1], [5]). This section is devoted to proving that the relation between $\mathfrak{r}_{\mathsf{scat}}$ and $\mathfrak{irr}$ has the same flavor.

Definition 3.1. $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ is the following statement:

> For every Borel function[2] $F : 2^{<\omega_1} \to \mathsf{scat}^+$, there is a $g : \omega_1 \to \mathsf{scat}^+$ such that for all $f \in 2^{\omega_1}$, the set $\{\alpha \in \omega_1 : g(\alpha) \subseteq F(f \restriction \alpha) \vee F(f \restriction \alpha) \cap g(\alpha) = \emptyset\}$ is stationary.

The function g, given by $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$, is called a $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$-guessing sequence for F.

Theorem 3.2. $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ *implies* $\mathfrak{irr} = \omega_1$.

Proof. By a suitable coding, we will assume that the domain of our F is the set of all ordered pairs $(A, \vec{I})$, where $\vec{I} = \langle I_\beta : \beta < \alpha \rangle \subseteq \mathcal{P}(\omega)$ is a sequence of length $\alpha \in \omega_1$ and A is a subset of ω. Define F as follows:

- If $\{I_\beta : \beta < \alpha\} \cup \{\omega \setminus I_\beta : \beta \in \alpha\}$ is not a subbasis for a topology homeomorphic to the usual topology on $\mathbb{Q}$, then $F(A, \vec{I}) = \mathbb{Q}$.

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[1] $\mathfrak{r}_{\mathbb{Q}} = \mathfrak{r}(\mathcal{P}(\mathbb{Q})/\mathsf{nwd})$ is the reaping number of the Boolean algebra $\mathcal{P}(\mathbb{Q})/\mathsf{nwd}$, where nwd denotes the ideal of nowhere dense subsets of the rationals.

[2] A function F from $2^{<\omega_1}$ to a metric space X is *Borel* if all of its restrictions to levels 2^α are Borel. Here we consider scat^+ as a subspace of $\mathcal{P}(\mathbb{Q})$ endowed with the product topology.

- If $\{I_\beta : \beta < \alpha\} \cup \{\omega \setminus I_\beta : \beta \in \alpha\}$ is a subbasis for a topology homeomorphic to the usual topology on $\mathbb{Q}$ and A is scattered relative to this topology, then $F(A, \vec{I}) = \mathbb{Q}$.
- If $\{I_\beta : \beta < \alpha\} \cup \{\omega \setminus I_\beta : \beta \in \alpha\}$ is a subbasis for a topology homeomorphic to the usual topology on $\mathbb{Q}$ and A is not scattered relative to this topology, pick $h_{\vec{I}} : \omega \to \mathbb{Q}$, a recursive homeomorphism, and define $F(A, \vec{I}) = h_{\vec{I}}[A]$.

Here, the homeomorphism $h_{\vec{I}}$ depends (in a recursive, or Borel, way) only on $\vec{I}$; in particular, it is the same homeomorphism for all pairs $(A, \vec{I})$ with the same second coordinate.

Now, let $g : \omega_1 \to \mathsf{scat}^+$ be a $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$-guessing sequence and recursively define a family of subbases as follows:

(1) Let $\mathcal{B}_0 = \langle U_n : n \in \omega \rangle$ be a basis for the usual topology on $\mathbb{Q}$.
(2) Suppose we have defined $\mathcal{B}_\beta$ for all $\beta < \alpha$. If α is a limit ordinal, then make $\mathcal{B}_\alpha = \bigcup_{\beta<\alpha} \mathcal{B}_\beta$. For $\alpha = \beta + 1$, look at $g(\beta) \in \mathsf{scat}^+$. By Lemma 2.4, there is a perfect nowhere-dense set B_β contained in $g(\beta)$. Also $\langle U_\gamma : \gamma < \alpha \rangle$ generates a topology homeomorphic to the usual topology on $\mathbb{Q}$, so in the definition of F, we make use of the recursive homeomorphism $h_{\mathcal{B}_\alpha}$. Let $U_\alpha = h^{-1}_{\mathcal{B}_\alpha}[B_\alpha]$. It is not hard to see that $\mathcal{B}_\alpha = \mathcal{B}_\beta \cup \{U_\alpha\} \cup \{\omega \setminus U_\alpha\}$ generates a topology homeomorphic to the rationals.

We claim that $\{U_\alpha : \alpha \in \omega_1\}$ generates an irresolvable T_3 topology τ_{ω_1} on ω. Since we are making each U_α clopen, then the topology we get is 0-dimensional. Let us see that it is irresolvable. That means, for every $A \subseteq \omega$, either A or $\omega \setminus A$ has non-empty interior in τ_{ω_1}. We only have to worry about the sets $A \in \mathsf{scat}^+_{\tau_{\omega_1}}$ (if $A \in \mathsf{scat}_{\tau_{\omega_1}}$, then obviously $\omega \setminus A$ has non-empty interior in τ_{ω_1}). Pick one of such A. Then, in particular, $A \in \mathsf{scat}^+$. If g guesses $(A, \langle U_\alpha : \alpha \in \omega_1 \rangle)$ at γ, then $h_{\langle U_\alpha : \alpha < \gamma \rangle}[A] \supseteq g(\gamma)$ or $h_{\langle U_\alpha : \alpha < \gamma \rangle}[A] \cap g(\gamma) = \emptyset$. So, either A or $\omega \setminus A$ has non-empty interior in τ_{ω_1}. In the former case, we have $A \supseteq U_\gamma$, and in the latter case $A \cap U_\gamma = \emptyset$, so it is not possible for A to be both dense and codense. By Proposition 2.1, we are done. □

4. Related Facts and Questions

In [1], it is proved that $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{r}_{\mathbb{Q}} \leq \mathfrak{i}$.

Proposition 4.1. $\mathfrak{r}_{\mathsf{scat}} \leq \mathfrak{r}_{\mathbb{Q}}$.

Proof. Let $\mathcal{R}$ be a Dense($\mathbb{Q}$)-reaping family and let $\mathcal{B}$ be a basis for the usual topology on $\mathbb{Q}$. The following family witnesses scattered-reaping:

$$\mathcal{R}_S = \{A \cap U : A \in \mathcal{R} \wedge U \in \mathcal{B}\}.$$ □

The following diagram summarizes some of the results related with those presented here.

$$\begin{array}{ccccc} \mathsf{cof}(\mathcal{M}) & \longrightarrow & \mathfrak{r}_{\mathbb{Q}} & \longrightarrow & \mathfrak{i} \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{d} & \longrightarrow & \mathfrak{r}_{\mathsf{scat}} & \longrightarrow & \mathfrak{irr} \\ & & \uparrow & & \\ & & \mathfrak{r} & & \end{array}$$

Some inequalities are folklore knowledge. The inequalities $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{r}_{\mathbb{Q}} \leq \mathfrak{i}$ were proved in [1]. We have the following questions concerning some of the cardinal invariants in the above diagram.

(1) Is $\mathfrak{r}_{\mathsf{scat}} = \mathfrak{r}_{\mathbb{Q}}$?
(2) Is $\mathfrak{r}_{\mathsf{scat}} = \max\{\mathfrak{d}, \mathfrak{r}\}$?
(3) Is there a model where $\mathfrak{r}_{\mathsf{scat}} < \mathfrak{irr}$?
(4) Is $\mathfrak{irr} = \mathfrak{i}$?
(5) Is $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{r}_{\mathsf{scat}}$?
(6) Are $\mathsf{cof}(\mathcal{M})$ and $\mathfrak{irr}$ provably comparable?
(7) Are $\mathfrak{r}_{\mathbb{Q}}$ and $\mathfrak{irr}$ provably comparable?

Recently, the authors have noticed that the inequality $\max\{\mathfrak{d}, \mathfrak{r}\} < \mathfrak{irr}$ is consistent with ZFC. The proof will appear elsewhere.

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