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A NOTE ON F-SPACES

SWAPAN KUMAR GHOSH

Abstract. An abstract ring in which every finitely generated ideal is principal is called an F-*ring*. A space X is called an F-*space* if $C(X)$ is an F-ring. Equivalently, X is an F-space if and only if for each $f \in C(X)$, the sets pos$f = \{x : f(x) > 0\}$ and neg$f = \{x : f(x) < 0\}$ are completely separated. In an earlier paper, we initiated the subring $C_{\mathcal{P}}(X)$ of $C(X)$ as a natural generalization of the ring $C_K(X)$ of all real-valued continuous functions on X with compact support and also locally-$\mathcal{P}$ spaces where $\mathcal{P}$ is an ideal of closed subsets of a space X. In this paper, we show that if X is a locally-$\mathcal{P}$ normal space, then X is an F-space if and only if, for each $f \in C_{\mathcal{P}}(X)$, the sets pos$f$ and neg$f$ are completely separated. We also show that if X is a locally compact F-space, then $C_K(X)$ is an F-ring.

1. Introduction

Throughout, X will stand for a completely regular Hausdorff topological space and $C(X)$ denotes the ring of all real-valued continuous functions on X. A space X is called an F-*space* if every finitely generated ideal in $C(X)$ is principal. It is well known that X is an F-space if and only if for every $f \in C(X)$, the sets pos$f = \{x : f(x) > 0\}$ and neg$f = \{x : f(x) < 0\}$ are completely separated [4, 14.25]. F-spaces were studied in detail by Leonard Gillman and Melvin Henriksen in 1956 [3], and several conditions, both topological and algebraic, were proved equivalent for a space to be an F-space. In 1989, some algebraic characterizations of F-spaces were given by Charles W. Neville [6].

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Let $\mathcal{P}$ be a family of closed subsets of X satisfying the following two conditions:

(i) If $A, B \in \mathcal{P}$, then $A \cup B \in \mathcal{P}$.
(ii) If $A \in \mathcal{P}$ and $B \subseteq A$ with B closed in X, then $B \in \mathcal{P}$; i.e., $\mathcal{P}$ is an ideal of closed sets in X.

For any $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f, $\mathrm{coz}f = X - Z(f)$ is the cozero-set of f, and $\mathrm{supp}f = cl_X(X - Z(f))$ is the support of f. Let $\Omega(X)$ be the set of all ideals of closed sets in X. In 2010, S. K. Acharyya and I [1] initiated the ring $C_{\mathcal{P}}(X)$ for each $\mathcal{P} \in \Omega(X)$ as $C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X - Z(f)) \in \mathcal{P}\}$ and we called X a *locally-$\mathcal{P}$ space* if, for each $x \in X$, there is an open neighborhood U of x such that $cl_X U \in \mathcal{P}$. For any $\mathcal{P} \in \Omega(X)$, it is clear that $C_{\mathcal{P}}(X)$ is an ideal (possibly improper) of $C(X)$.

Two very important ideals of $C(X)$ are $C_K(X)$ and $C_\psi(X)$; $C_K(X)$ contains all functions in $C(X)$ with compact support and $C_\psi(X)$ contains all functions in $C(X)$ with pseudocompact support. Both of these ideals are of the form $C_{\mathcal{P}}(X)$. Let $\mathcal{K}(X)$ be the family of all compact subsets of X. If $\mathcal{P} = \mathcal{K}(X)$, then $C_{\mathcal{P}}(X)$ coincides with $C_K(X)$. Also, X is locally-$\mathcal{K}(X)$ if and only if X is locally compact. Let $\mathcal{B}(X)$ be the ideal of all closed bounded (relatively pseudocompact) subsets of X. (A subset A of X is called *bounded* or *relatively pseudocompact* if each function in $C(X)$ is bounded on A.) If $\mathcal{P} = \mathcal{B}(X)$, then obviously, $C_{\mathcal{P}}(X)$ coincides with $C_\psi(X)$ since any bounded support in a topological space is pseudocompact [5, Theorem 2.1]. Again, X is locally-$\mathcal{B}(X)$ if and only if X is locally bounded (i.e., each point of X has an open neighborhood with bounded closure).

In this paper, we continue the study of the ring $C_{\mathcal{P}}(X)$ and establish that if X is a locally-$\mathcal{P}$ normal space, then X is an F-space if and only if, for every $f \in C_{\mathcal{P}}(X)$, the sets $\mathrm{pos}f = \{x : f(x) > 0\}$ and $\mathrm{neg}f = \{x : f(x) < 0\}$ are completely separated. We also show that if X is a locally-$\mathcal{P}$ normal space and if $C_{\mathcal{P}}(X)$ is an F-ring, then X is an F-space. We do not know whether $C_{\mathcal{P}}(X)$ is an F-ring when X is a locally-$\mathcal{P}$ F-space. However, we establish that if X is a locally compact F-space, then $C_K(X)$ is an F-ring.

An ideal I of $C(X)$ is called free if $\bigcap_{f\in I} Z(f) = \emptyset$. If X is locally-$\mathcal{P}$, it is easy to see that $\{Z(f) : f \in C_{\mathcal{P}}(X)\}$ is a base for the closed sets of X, and therefore $\cap_{f\in C_{\mathcal{P}}(X)} Z(f) = \emptyset$. Hence, $C_{\mathcal{P}}(X)$ is a free ideal of $C(X)$. Also, if $C_{\mathcal{P}}(X)$ is free, then it is clear that X is locally-$\mathcal{P}$. Thus, we have the following theorem.

Theorem 1.1. *$C_{\mathcal{P}}(X)$ is a free ideal of $C(X)$ if and only if X is locally-$\mathcal{P}$.*

Choosing $\mathcal{P} = \mathcal{K}(X)$, we get the following as an immediate consequence.

Corollary 1.2. *$C_K(X)$ is a free ideal of $C(X)$ if and only if X is locally compact.*

If I is an ideal of $C(X)$ (or $C_{\mathcal{P}}(X)$) and $Z[I] = \{Z(f) : f \in I\}$, then the members of $Z[I]$ are called the *zero-sets of* I. An ideal I of $C(X)$ is called a *z-ideal* if $Z(f) = Z(g)$ and $f \in I$ imply that $g \in I$. For each $p \in X$, let O_p denote the set of all f in $C(X)$ for which $Z(f)$ is a neighborhood of p. It is clear that O_p is a z-ideal of $C(X)$ for each p.

Let us now recall the following result [4, 3.2(b)].

Theorem 1.3. *In a completely regular Hausdorff space, every neighborhood of a point contains a zero-set neighborhood of that point.*

We note that $O_p \cap C_{\mathcal{P}}(X)$ is an ideal of $C_{\mathcal{P}}(X)$. Let us now prove the following lemmas.

Lemma 1.4. *Let $f \in C_{\mathcal{P}}(X)$. If $posf$ and $negf$ are completely separated, then, for each $p \in X$, there is a zero-set of $O_p \cap C_{\mathcal{P}}(X)$ on which f does not change sign.*

Proof. Let pos$f$ and neg$f$ be completely separated and $p \in X$. Then cl_Xpos$f \cap cl_X$neg$f = \emptyset$. Thus, $p \notin cl_X$pos$f \cap cl_X$neg$f$. We assume that $p \notin cl_X$neg$f$. So by Theorem 1.3, there is a zero-set neighborhood U of p such that $U \cap$neg$f = \emptyset$. Let $U = Z(g)$ and $g \in C(X)$. So $g \in O_p$, and thus $fg \in O_p$. Also, $fg \in C_{\mathcal{P}}(X)$ since $f \in C_{\mathcal{P}}(X)$. Therefore, $fg \in O_p \cap C_{\mathcal{P}}(X)$. So $Z = Z(fg)$ is a zero-set of $O_p \cap C_{\mathcal{P}}(X)$. Let $x \in Z$; i.e., $x \in Z(fg) = Z(f) \cup Z(g)$. Then either $x \in Z(f)$ or $x \in Z(g)$. Now if $x \in Z(f)$, then $x \notin$ neg$f$. Again, if $x \in Z(g) = U$, then also $x \notin$ neg$f$ as $U \cap$neg$f = \emptyset$. Hence, $Z \cap$neg$f = \emptyset$. Consequently, f does not change its sign on Z. □

Lemma 1.5. *Let I be an ideal of $C_{\mathcal{P}}(X)$ such that $Z(f) \in Z[I]$ and $f \in C_{\mathcal{P}}(X)$ imply that $f \in I$. If, for each $f \in C_{\mathcal{P}}(X)$, there is a zero-set of I on which f does not change sign, then I is a prime ideal of $C_{\mathcal{P}}(X)$.*

Proof. Assume that for each $f \in C_{\mathcal{P}}(X)$, there is a zero-set of I on which f does not change sign. Let $gh \in I$ where $g, h \in C_{\mathcal{P}}(X)$. Clearly, $|g| - |h| \in C_{\mathcal{P}}(X)$. By assumption, there is a zero-set Z of I such that $|g| - |h|$ does not change sign on Z. Let us assume that $(|g| - |h|)(x) \geq 0$ for each $x \in Z$. We claim that $Z \cap Z(gh) \subseteq Z(h)$. So let $y \in Z \cap Z(gh)$. Then either $g(y) = 0$ or $h(y) = 0$. If $h(y) = 0$, then $y \in Z(h)$. Also, if $g(y) = 0$, then $(|g|)(y) = 0$, and hence $h(y) = 0$ since $(|g| - |h|)(y) \geq 0$. So $y \in Z(h)$. Hence, $Z \cap Z(gh) \subseteq Z(h)$. Now $gh \in I$ shows that $Z(gh) \in Z[I]$ and also

$Z \in Z[I]$. Thus, $Z \cap Z(gh) \in Z[I]$, and hence there is an l in I with $Z(l) = Z \cap Z(gh)$. Also, $l \in I$ shows that $hl \in I$. Now since $Z \cap Z(gh) \subseteq Z(h)$, we conclude that $Z(h) = Z(h) \cup (Z \cap Z(gh)) = Z(h) \cup Z(l) = Z(hl)$. So $Z(h) \in Z[I]$ since $hl \in I$. Therefore, $h \in I$. Consequently, I is a prime ideal of $C_{\mathcal{P}}(X)$. □

Lemma 1.6. *Let X be a locally-$\mathcal{P}$ space. For $p \in X$, if $O_p \cap C_{\mathcal{P}}(X)$ is a prime ideal of $C_{\mathcal{P}}(X)$, then O_p is a prime ideal of $C(X)$.*

Proof. Let O_p be not prime in $C(X)$. Then there are $f, g \in C(X)$ such that $f \notin O_p$ and $g \notin O_p$, but $fg \in O_p$. Since X is locally-$\mathcal{P}$, by Theorem 1.1, $C_{\mathcal{P}}(X)$ is a free ideal of $C(X)$ and so there is an $l \in C_{\mathcal{P}}(X)$ such that $p \notin Z(l)$. Let $h_1 = fl$ and $h_2 = gl$. Then $h_1, h_2 \in C_{\mathcal{P}}(X)$ and also $h_1h_2 \in C_{\mathcal{P}}(X)$ and $h_1h_2 = fgl^2 \in O_p$. So $h_1h_2 \in O_p \cap C_{\mathcal{P}}(X)$. We claim that $p \in cl_X(X - Z(h_1))$. In fact, if $p \notin cl_X(X - Z(h_1))$, then there is an open neighborhood U of p satisfying $U \cap (X - Z(h_1)) = \emptyset$. Then $U \subseteq Z(h_1) = Z(fl) = Z(f) \cup Z(l)$. Thus, $U - Z(l) \subseteq Z(f)$. Also, $p \in U - Z(l)$ since $p \notin Z(l)$. Now $U - Z(l)$ is open and $p \in U - Z(l) \subseteq Z(f)$, and therefore $f \in O_p$, a contradiction. Thus, $p \in cl_X(X - Z(h_1))$, and hence $h_1 \notin O_p$. So $h_1 \notin O_p \cap C_{\mathcal{P}}(X)$. Similarly, $h_2 \notin O_p \cap C_{\mathcal{P}}(X)$. This shows that $O_p \cap C_{\mathcal{P}}(X)$ is not a prime ideal of $C_{\mathcal{P}}(X)$. □

2. F-Spaces and the Ring $C_{\mathcal{P}}(X)$

A space X is called an *F'-space* if, for all $f \in C(X)$, the sets $cl_X \text{pos} f$ and $cl_X \text{neg} f$ are disjoint [3, Definition 8.12]. Equivalently, X is an F'-space if and only if for each $p \in X$ the ideal O^p is prime, [3, Theorem 8.13]. Obviously, every F-space is an F'-space. Also, if X is a normal F'-space, then X is an F-space (last part of Theorem 2.1). Theorem 2.1 shows that in the family of all locally-$\mathcal{P}$ normal spaces X, the ring $C_{\mathcal{P}}(X)$ determines when X is an F-space.

Theorem 2.1. *Let X be a locally-$\mathcal{P}$ normal space. Then X is an F-space if and only if, for each $f \in C_{\mathcal{P}}(X)$, the sets pos$f$ and neg$f$ are completely separated.*

Proof. Necessity is trivial. To prove the sufficiency, we first show that O_p is a prime ideal of $C(X)$ for each $p \in X$. So let $p \in X$. Suppose that $Z(g) \in Z[O_p \cap C_{\mathcal{P}}(X)]$ where $g \in C_{\mathcal{P}}(X)$. Then there is $h \in O_p \cap C_{\mathcal{P}}(X)$ such that $Z(h) = Z(g)$. Since $h \in O_p$ and O_p is a z-ideal of $C(X)$, we conclude that $g \in O_p$. Thus, $g \in O_p \cap C_{\mathcal{P}}(X)$. Now by assumption, for each $f \in C_{\mathcal{P}}(X)$, pos$f$ and neg$f$ are completely separated. Hence, by Lemma 1.4, for each $f \in C_{\mathcal{P}}(X)$, there is a zero-set of $O_p \cap C_{\mathcal{P}}(X)$ on which f does not change sign. Consequently, by Lemma 1.5, $O_p \cap C_{\mathcal{P}}(X)$ is a prime ideal of $C_{\mathcal{P}}(X)$. Hence, by Lemma 1.6, O_p is a prime ideal of $C(X)$. We now use normality of X to prove that X is an F-space. If possible, let X be not an F-space. So pos$f$ and neg$f$ are not completely separated for some $f \in C(X)$. Since X is normal, $cl_X \text{pos} f \cap cl_X \text{neg} f \neq \emptyset$. Choose $p \in cl_X \text{pos} f \cap cl_X \text{neg} f$. Obviously, f changes its sign on every neighborhood of p. Define $g = f \vee 0$ and $h = f \wedge 0$. Then $g \notin O_p$ and $h \notin O_p$, while $gh = 0 \in O_p$. So O_p is not prime, a contradiction. Hence, X is an F-space. □

Corollary 2.2. (a) *Let X be a locally compact normal space. Then X is an F-space if and only if for each $f \in C_K(X)$, the sets pos$f$ and neg$f$ are completely separated.*

(b) *Let X be a locally bounded normal space. Then X is an F-space if and only if for each $f \in C_\psi(X)$, the sets pos$f$ and neg$f$ are completely separated.*

Proof. For (a), Choose $\mathcal{P} = \mathcal{K}(X)$. For (b), choose $\mathcal{P} = \mathcal{B}(X)$. □

For $f, g \in C_{\mathcal{P}}(X)$, let $(f)_{C_{\mathcal{P}}}$ be the principal ideal of $C_{\mathcal{P}}(X)$ generated by f. Also, let $(f, g)_{C_{\mathcal{P}}}$ be the ideal of $C_{\mathcal{P}}(X)$ generated by f and g. We now prove the following theorem.

Theorem 2.3. *Let X be a locally-$\mathcal{P}$ normal space. If $C_{\mathcal{P}}(X)$ is an F-ring, then X is an F-space.*

Proof. Let $f \in C_{\mathcal{P}}(X)$. If $C_{\mathcal{P}}(X)$ is an F-ring, then there exists an $l \in C_{\mathcal{P}}(X)$ such that $(f, |f|)_{C_{\mathcal{P}}} = (l)_{C_{\mathcal{P}}}$. So there are $g, h, s, t \in C_{\mathcal{P}}(X)$ such that $f = gl$, $|f| = hl$, and $sf + t|f| = l$. Hence, $(sg + th)l = l$. Let $y \in \text{pos} f$. Then $f(y) = |f(y)|$, and hence $g(y)l(y) = h(y)l(y)$. Also, $f(y) = g(y)l(y)$. So $l(y) \neq 0$. Thus, $g(y) = h(y)$, and therefore $y \in Z(g - h)$. Since $l(y) \neq 0$, we also have $s(y)g(y) + t(y)h(y) = 1$. So $y \in Z(sg + th - 1)$. Hence, $\text{pos} f \subseteq Z(g - h) \cap Z(sg + th - 1)$. Similarly, we get $\text{neg} f \subseteq Z(g + h) \cap Z(sg + th - 1)$. Now $Z(g - h) \cap Z(sg + th - 1)$ and $Z(g + h) \cap Z(sg + th - 1)$ are disjoint zero-sets containing pos$f$ and neg$f$, respectively. So pos$f$ and neg$f$ are completely separated. The proof now follows from Theorem 2.1. □

Corollary 2.4. (a) *Let X be a locally compact normal space. If $C_K(X)$ is an F-ring, then X is an F-space.*

(b) *Let X be a locally bounded normal space. If $C_\psi(X)$ is an F-ring, then X is an F-space.*

Proof. For (a), Choose $\mathcal{P} = \mathcal{K}(X)$. For (b), choose $\mathcal{P} = \mathcal{B}(X)$. □

The converse of the Theorem 2.3 is true when $\mathcal{P} = \mathcal{K}(X)$, i.e., when $C_{\mathcal{P}}(X) = C_K(X)$. We do not require the normality of X to prove this. Before going to the proof, we state the following theorem which follows as a corollary of Theorem 10.3 in [7].

Theorem 2.5. *If X is a locally compact Hausdorff space, then the ideal $C_K(X)$ of $C(X)$ is pure; i.e., for each $f \in C_K(X)$, there is $g \in C_K(X)$ such that $f = fg$.*

Remark 2.6. If $f \in C_K(X)$, by Theorem 2.5, there is $g \in C_K(X)$ such that $f = fg$. Hence, $fC(X) = fC_K(X)$.

We now prove the converse of Theorem 2.3 when $C_{\mathcal{P}}(X) = C_K(X)$.

Theorem 2.7. *Let X be a locally compact F-space. Then $C_K(X)$ is an F-ring.*

Proof. Let $f_1, f_2, ..., f_n \in C_K(X)$. Since X is an F-space, the ideal of $C(X)$ generated by $f_1, f_2, ..., f_n$ is a principal ideal of $C(X)$. So there is an $l \in C(X)$ such that $f_1C(X) + f_2C(X) + ... + f_nC(X) = lC(X)$. By Remark 2.6, $f_1C_K(X) + f_2C_K(X) + ... + f_nC_K(X) = lC(X)$. Evidently, $l \in C_K(X)$. Hence, again by Remark 2.6, $lC(X) = lC_K(X)$, and thus $f_1C_K(X) + f_2C_K(X) + ... + f_nC_K(X) = lC_K(X)$. Consequently, $C_K(X)$ is an F-ring. □

At the end of the paper we offer an alternative proof of Theorem 2.7. Let us recall the following two theorems for completely regular spaces [4, 3.11(a),(c)].

Theorem 2.8. *In a completely regular space, any two disjoint closed sets, one of which is compact, are completely separated.*

Theorem 2.9. *Every compact set in a completely regular space is C-embedded.*

The following theorem gives a sufficient condition for a subspace of an F-space to be an F-space [4, 14.26].

Theorem 2.10. *Every C^*-embedded subspace of an F-space is an F-space.*

Let us now prove the following lemma.

Lemma 2.11. *Let X be locally compact and $f_1, f_2, ..., f_n \in C_K(X)$. Then there is a cozero-set C of X such that $cl_X(X - Z(f_i)) \subseteq C$ for all $i = 1, 2, ..., n$ and cl_XC is compact.*

Proof. Let $f = f_1^2 + f_2^2 + ... + f_n^2$. Then $f \in C_K(X)$. Since $cl_X(X - Z(f_i) \subseteq cl_X(X - Z(f))$ for all $i = 1, 2, ..., n$, to complete the proof it is enough to find a cozero-set C of X such that $cl_X(X - Z(f)) \subseteq C$ and cl_XC is compact. Since X is locally compact, by Corollary 1.2, $C_K(X)$ is a free ideal of $C(X)$. So, for each $x \in cl_X(X - Z(f))$, we can select an $f_x \in C_K(X)$ with $x \notin Z(f_x)$. Since $cl_X(X - Z(f))$ is compact and since

$\{X - Z(f_x) : x \in cl_X(X - Z(f))\}$ is an open cover of $cl_X(X - Z(f))$, there are $f_{x_1}, f_{x_2}, ..., f_{x_n} \in C_K(X)$ such that $cl_X(X-Z(f)) \subseteq \bigcup_{i=1}^n (X - Z(f_{x_i}))$. Let $f_0 = f_{x_1}^2 + f_{x_2}^2 + ... + f_{x_n}^2$. Then $f_0 \in C_K(X)$, and also $X - Z(f_0) = \bigcup_{i=1}^n (X - Z(f_{x_i}))$. Let $C = X - Z(f_0)$. Then $cl_X C$ is compact and $cl_X(X - Z(f)) \subseteq C$. □

We now give an alternative proof of Theorem 2.7.

Alternative proof of Theorem 2.7. Consider any finitely generated ideal $(f_1, f_2, ..., f_n)_{C_K}$ of $C_K(X)$ where $f_1, f_2, ..., f_n \in C_K(X)$. Using Lemma 2.11, we select a cozero-set C of X such that $cl_X(X-Z(f_i)) \subseteq C$ for all $i = 1, 2, ..., n$ and $cl_X C$ is compact. Let $B = cl_X C$. Now, Theorem 2.9 tells us that B is C-embedded in X. So, from Theorem 2.10, we conclude that B is an F-space. Consider the finitely generated ideal $(f_1|_B, f_2|_B, ..., f_n|_B)$ of $C(B)$. Since B is an F-space, there is $g \in C(B)$ such that the principal ideal (g) of $C(B)$ coincides with the ideal $(f_1|_B, f_2|_B, ..., f_n|_B)$. Since $cl_X(X-Z(f_i)) \subseteq C$ for all $i = 1, 2, ..., n$, it is evident that $cl_B(B-Z(g)) \subseteq C$. Note that $cl_B(B - Z(g))$ and $X - C$ are two disjoint closed sets in X and $cl_B(B - Z(g))$ is compact. Using Theorem 2.8, we can select a function $k \in C(X)$ satisfying $k(cl_B(B-Z(g))) = \{1\}$ and $k(X-C) = \{0\}$. Since $cl_X(X - Z(k)) \subseteq cl_X C = B$, we find that $k \in C_K(X)$. Again, since B is C-embedded in X, we can extend g into a function l in $C(X)$. Let $h = kl$. Then $h(X-C) = \{0\}$ and h agrees with g on C. Now $h \in C_K(X)$ since $k \in C_K(X)$.

To complete the proof, we now check that $(f_1, f_2, ..., f_n)_{C_K} = (h)_{C_K}$. Choose $i \in \{1, 2, ..., n\}$. Since $(f_1|_B, f_2|_B, ..., f_n|_B) = (g)$, there is $t_i \in C(B)$ for which $f_i|_B = t_i g$. As B is C-embedded in X, t_i can be extended to a function l_i in $C(X)$. We claim that $f_i = kl_i h$. In fact, if $x \in X - C$, then $f_i(x) = 0$, and also $h(x) = 0$. If $x \in C - (cl_B(B - Z(g)))$, then also $f_i(x) = 0$ and $h(x) = 0$ since $g(x) = 0$. Again, if $x \in cl_B(B - Z(g))$, then $k(x) = 1$. Hence, $f_i(x) = k(x)l_i(x)h(x)$ since $f_i|_B = t_i g$. Thus, $f_i = kl_i h$, and therefore $f_i \in (h)_{C_K}$ as $kl_i \in C_K(X)$. Consequently, $(f_1, f_2, ..., f_n)_{C_K} \subseteq (h)_{C_K}$. In a similar way we can check that $(h)_{C_K} \subseteq (f_1, f_2, ..., f_n)_{C_K}$. Hence, $(f_1, f_2, ..., f_n)_{C_K} = (h)_{C_K}$ and the proof is complete. □

At the end of the paper, we give an example to show that the condition of normality is essential in Corollary 2.4(a). In 1983, Alan Dow [2, Example 1.10] showed that there is a locally compact F'-space U which is not an F-space and which is an open subset of a compact F-space.

Example 2.12. We start with the locally compact F'-space U which is not an F-space and which is an open subset of a compact F-space, say Y. Since U is an F'-space but not an F-space, it is clear that U is not normal.

We now show that $C_K(U)$ is an F-ring. Consider any finitely generated ideal $(f_1, f_2, ..., f_n)_{C_K}$ of $C_K(U)$ where $f_1, f_2, ..., f_n \in C_K(U)$. Since U is locally compact, using Lemma 2.11, we can select a cozero-set C of U such that $cl_U(X - Z(f_i)) \subseteq C$ for all $i = 1, 2, ..., n$ and $cl_U C$ is compact. Let $B = cl_U C$. Since B is a compact subset of Y and Y is an F-space, B is also an F-space. It is now evident that we can replace X by U in the alternative proof of Theorem 2.7 to conclude that $C_K(U)$ is an F-ring.

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