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## PENNER SEQUENCES AND ASYMPTOTICS OF MINIMUM DILATATIONS FOR SUBFAMILIES OF THE MAPPING CLASS GROUP

by

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**PENNER SEQUENCES AND  
ASYMPTOTICS OF MINIMUM DILATATIONS  
FOR SUBFAMILIES OF THE MAPPING CLASS GROUP**

ERIKO HIRONAKA

**ABSTRACT.** Let  $\mathcal{F}_m \subset \text{Mod}(S_m)$  be a collection of subsets of the mapping class group of a compact oriented surface  $S_m$  of genus  $g_m$ , where  $g_m$  is unbounded. We say  $\mathcal{F} = \bigcup_m \mathcal{F}_m$  admits asymptotically small dilatations if there exists a sequence  $\phi_m \in \mathcal{F}_m$  of pseudo-Anosov elements so that  $\lambda(\phi_m)^{g_m}$  is bounded. In this paper, we describe Penner's construction for producing sequences of pseudo-Anosov mapping classes whose normalized dilatations converge and apply the construction to the setting of handlebody mapping class groups and mapping classes with trivial homological dilatation.

**1. INTRODUCTION**

Let  $S_g$  be a closed oriented surface of genus  $g$ , and let  $\text{Mod}(S_g)$  be its mapping class group. Robert C. Penner [16] shows that for each genus  $g$ , the minimum dilatation  $\delta_g$  of pseudo-Anosov mapping classes in  $\text{Mod}(S_g)$  satisfies

$$(1.1) \quad \log \delta_g \asymp \frac{1}{g}.$$

There are several naturally defined subgroups and subcollections of the mapping class group for which this asymptotic behavior on minimum dilatations does not hold (see, for example, [4], [3], [18]). Let  $g_m$  be a strictly monotone increasing sequence of integers  $g_m \geq 2$ . A collection

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$\mathcal{F} = \bigcup_m \mathcal{F}_{g_m}$  of subsets  $\mathcal{F}_{g_m} \subset \text{Mod}(S_g)$  admits asymptotically small dilatation pseudo-Anosov maps if there is a sequence  $\phi_{g_m} \in \mathcal{F}_{g_m}$  of pseudo-Anosov mapping classes that satisfy

$$\log(\lambda(\phi_{g_m})) \asymp \frac{1}{g_m}.$$

In this paper, we describe a generalization of Penner's construction of small dilatation mapping classes and apply it to find two "naturally defined" collections of mapping classes that admit asymptotically small dilatation pseudo-Anosov maps.

Our first example is the handlebody subgroups. A mapping class  $\phi$  on a surface  $S$  is a *handlebody mapping class* if there is an identification of  $S$  with the boundary of a handlebody  $H$  so that  $\phi$  extends to  $H$ . Howard Masur [12] shows that the limit set of the handlebody subgroup has measure zero in Thurston's sphere of measured foliations. Thus, these subgroups are *small* in this sense. On the other hand, the following theorem shows that the handlebody subgroups are *large* in the sense of the range of dilatations of pseudo-Anosov elements.

**Theorem 1.1.** *Let  $\mathcal{H}_g \subset \text{Mod}(S_g)$  be the set of handlebody mapping classes on a genus  $g$  surface. Then  $\mathcal{H}_g$  admits asymptotically small dilatation pseudo-Anosov maps.*

Our second example is the collection of mapping classes with trivial homological dilatation. In [4], Benson Farb, Christopher J. Leininger, and Dan Margalit prove that the set of dilatations of pseudo-Anosov mapping classes in the Torelli subgroup of  $\text{Mod}(S_g)$  is bounded from below by a constant greater than one. Thus, mapping classes that act trivially on first homology do not admit small dilatations. If we look, however, at mapping classes whose action on first homology has spectral radius equal to one, the behavior of minimum dilatations is different.

**Theorem 1.2.** *Let  $\mathcal{F}_g \subset \text{Mod}(S_g)$  be the subcollections of mapping classes whose homological dilatation equals one. Let  $g_m = 2m$  range over the even numbers  $\geq 2$ . Then  $\mathcal{F}_{g_m}$  admits asymptotically small dilatation pseudo-Anosov maps.*

Dilatations of pseudo-Anosov mapping classes  $\phi_g \in \text{Mod}(S_g)$ , where  $g$  is the genus of  $S_g$ , are bounded from below by the following inequality [16] (see also [14, p. 44])

$$\frac{\log(2)}{12g - 12} \leq \log(\lambda(\phi_g)).$$

Thus, to prove statements like Theorem 1.1 and Theorem 1.2, it suffices to find a sequence of pseudo-Anosov mapping classes  $\phi_{g_m} : S_{g_m} \rightarrow S_{g_m}$ ,

so that the *genus-normalized dilatation*

$$L_{\text{genus}}(S_m, \phi_m) = \lambda(\phi_{g_m})^{g_m}$$

is bounded.

In [16], Penner developed techniques for constructing sequences of this kind which are sometimes known as *Penner sequences* (see also [2], [18], [19]). Penner sequences have the following useful properties:

- (i)  $\phi_m$  is pseudo-Anosov,
- (ii) the genus of  $S_m$  is linear in  $m$ , and
- (iii)  $L_{\text{genus}}(S_m, \phi_m)$  is bounded.

We define generalized Penner sequences in section 2 and use this in section 3 to prove Theorem 1.1 and Theorem 1.2 using explicit constructions. Section 4 contains further questions about small dilatation mapping classes.

## 2. GENERALIZED PENNER SEQUENCES

In this section, we define a generalization of Penner's example in [16], which we will use in section 3. This generalization is a special case of the ones studied in [19] and [8].

Let  $S$  be a compact surface of finite type with negative topological Euler characteristic. A *simple closed curve* on  $S$  is the image of an embedded circle on  $S$ . A *relative closed curve* is the image of an interval on  $S$  whose endpoints lie on the boundary of  $S$ . A simple closed or relative closed curve on  $S$  is *essential* if it does not bound a disk on  $S$  and it is not homotopic to a curve on the boundary of  $S$ . A *multi-curve* on  $S$  is a finite union of pairwise disjoint essential closed curves on  $S$ . A *relative multi-curve* on  $S$  is a finite union of pairwise disjoint closed and relative closed curves on  $S$ .

A pair of multi-curves  $a$  and  $b$  *fills*  $S$  if

- (i)  $a$  and  $b$  intersect minimally, and
- (ii) the complementary components of  $a \cup b$  are either open disks or boundary parallel annuli.

Given a multi-curve  $a$ , let  $\delta_a$  be the composition of right Dehn twists centered at the components of  $a$ . (See, for example, [5] for definition and properties of Dehn twists.)

Let  $a$  and  $b$  be multi-curves, let  $c$  be a simple closed curve that is disjoint from  $b$ , and assume that the pair of multi-curves  $a$  and  $b \cup c$  fills  $S$ . Let  $d \subset S$  be a relative closed multi-curve. Assume the following:

- (i)  $c$  is connected,
- (ii)  $d$  is disjoint from  $a$  and  $b$ ,
- (iii)  $S \setminus d$  is connected, and

(iv) the algebraic intersection of  $c$  and  $d$  is zero.

We call  $d$  the *cutting curve*.

**Lemma 2.1.** *The mapping class  $\phi : S \rightarrow S$  defined by  $\phi = \delta_c \delta_a^{-1} \delta_b$  is pseudo-Anosov.*

*Proof.* This follows from Penner's semi-group criterion [15]. □

Let  $\alpha : \pi_1(S) \rightarrow \mathbb{Z}$  be the map sending loops on  $S$  to their algebraic intersection with  $d$ . By composing with the quotient map  $\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ , we have regular  $m$ -cyclic coverings  $\rho_m : S_m \rightarrow S$ . Let  $\Sigma$  be the surface with boundary obtained by cutting  $S$  along  $d$  and let  $\Sigma_0$  be the closure of a lift of the interior of  $\Sigma$  to  $S_m$ .

Let  $d^+$  and  $d^-$  be the loci on the boundary of  $\Sigma$  corresponding to the two sides of  $d$  in  $S$  and let  $d_0^\pm$  be their lifts to  $\Sigma_0$ . Then there is a generator  $r_m$  of the group of deck transformations of  $S_m$  over  $S$  so that  $r_m(d_0^-) = d_0^+$ . Let  $\Sigma_i = r_m^i(\Sigma_0)$  and  $d_i^\pm = r_m^i(d_0^\pm)$ . Then  $S_m$  is the union

$$S_m = \bigcup_{i=0}^{m-1} \Sigma_i,$$

where  $\Sigma_i$  is attached to  $\Sigma_{i+1}$  by gluing  $d_i^+$  to  $d_{i+1}^-$ , the sum “ $i+1$ ” being taken modulo  $m$ .

By construction, each of  $a$ ,  $b$ , and  $c$  has  $m$  disjoint lifts in  $S_m$ . Let  $a^{(0)}$  and  $b^{(0)}$  be the lifts of  $a$  and  $b$  that are strictly contained in  $\Sigma_0$ , and let  $c^{(0)}$  be the lift of  $c$  that intersects  $\Sigma_0$ , but does not intersect  $\Sigma_{m-1}$ . Define the *Penner sequence*  $\phi_m : S_m \rightarrow S_m$  associated to  $(S, a, b, c, d)$  to be the mapping class

$$\phi_m = r_m \delta_{c^{(0)}} \delta_{a^{(0)}}^{-1} \delta_{b^{(0)}}.$$

**Theorem 2.2.** *Let  $(S_m, \phi_m)$  be a Penner sequence associated to  $(S, a, b, c, d)$ . Then*

- (1) *the topological Euler characteristic of  $S_m$  satisfies*

$$\chi(S_m) = m\chi(S),$$

- (2) *the mapping class  $\phi_m$  is pseudo-Anosov,*  
 (3) *the  $\chi$ -normalized dilatations  $L_\chi(S_m, \phi_m) = \lambda(\phi_m)^{|\chi(S_m)|}$  form a convergent sequence*

$$\lim_{m \rightarrow \infty} L_\chi(S_m, \phi_m) = L_\chi(S, \phi),$$

and

- (4) *the genus normalized dilatations  $L_{\text{genus}}(S_m, \phi_m)$  are bounded.*

*Proof.* The topological Euler characteristic of  $\Sigma$  is given by

$$\chi(\Sigma) = \chi(S) - \chi(d).$$

The covering  $S_m$  contains  $m$  copies of  $d$  and  $m$  copies of the interior of  $\Sigma$ , and hence

$$\chi(S_m) = m\chi(d) + m(\chi(S) - \chi(d)) = m\chi(S).$$

The mapping classes  $\phi_m$  lie in the Penner semigroup generated by negative Dehn twists on  $a$  and positive Dehn twists on  $b \cup c$ . By Penner's semigroup criterion, it follows that  $\phi_m$  is pseudo-Anosov [15], proving (2). William P. Thurston's fibered face theory [17] gives a correspondence between pseudo-Anosov mapping classes on surfaces and rational points on fibered faces of hyperbolic 3-manifolds. In [8], it is shown that each Penner sequence  $(S_m, \phi_m)$  corresponds to a convergent sequence on a fibered face whose limit is the point associated to  $(S, \phi)$ . By a result of David Fried [6] (see also [13], [14]), the normalized dilatation extends to a continuous (and convex) function on fibered faces, implying (3).

Let  $r_m$  be the number of boundary components of  $S_m$ . Then, since  $r_m \geq 0$ , we have  $|\chi(S_m)| = 2g_m + r_m - 2 \geq 2g_m - 2$ . Thus,

$$L_{\text{genus}}(S_m, \phi_m) = \lambda(\phi_m)^{g_m} \leq \lambda(\phi_m)^{|\chi(S_m)|/2+1} = L(S_m, \phi_m)^{\frac{1}{2}} \lambda(\phi_m).$$

Since, by Theorem 2.2(3),  $\lambda(\phi_m)$  and  $L(S_m, \phi_m)$  are bounded,  $L_{\text{genus}}(S_m, \phi_m)$  is bounded.  $\square$

**Remark 2.3.** That the normalized dilatations  $L_\chi(S_m, \phi_m)$  are bounded was proved for a special case in [16] and generalized in [19]. Theorem 2.2 is stronger because it also gives information about the limiting value of  $L(S_m, \phi_m)$ .

Let  $(S_m, \phi_m)$  be a Penner sequence. Let  $\bar{S}_m$  be the closed surface obtained by filling each boundary component of  $S_m$  with a disk. Let  $\bar{\phi}_m$  be the mapping class on  $\bar{S}_m$  induced by  $\phi_m$ . We call  $(\bar{S}, \bar{\phi}_m)$  the closures of  $(S_m, \phi_m)$ .

By the construction, the multi-curves  $a$  and  $b \cup c$  divide  $S$  into disk or boundary parallel annular regions bounded by polygons whose sides alternate between lying on  $a$  and lying on  $b \cup c$ , and each polygon bounding a disk is even-sided with at least four sides.

**Proposition 2.4.** *Let  $(S, a, b, c, d)$  have the additional property that each boundary parallel disk in the complement of  $a \cup b \cup c$  is bounded by a polygon with at least four sides. Then the closures  $(\bar{S}_m, \bar{\phi}_m)$  are pseudo-Anosov mapping classes with*

$$\lambda(\bar{S}_m, \bar{\phi}_m) = \lambda(S_m, \phi_m),$$

*and the genus-normalized dilatations  $L_{\text{genus}}(\bar{S}_m, \bar{\phi}_m)$  are bounded.*

*Proof.* For the covering  $S_m$  two things can happen locally. If a complementary component is homeomorphic to a disk, then the number of sides of the bounding polygon stays the same. If the complementary component is a boundary parallel annulus, the number of sides of the polygon either stays the same or increases. It follows (see [15]) that  $\phi_m$  cannot have any one-pronged boundary components, and hence  $(\bar{S}_m, \bar{\phi}_m)$  is pseudo-Anosov with the same dilatation as  $(S_m, \phi_m)$  (see, for example, [9, Lemma 2.6]).

Since the genus of  $S_m$  and of  $\bar{S}_m$  are the same, we have

$$L_{\text{genus}}(S_m, \phi_m) = L_{\text{genus}}(\bar{S}_m, \bar{\phi}_m).$$

The rest follows from Theorem 2.2(4).  $\square$

### 3. APPLICATIONS

Here, we prove the theorems stated in the introduction using families of examples that satisfy the conditions of Theorem 2.2 and Proposition 2.4.

#### 3.1. HANDLEBODY MAPPING CLASSES.

We now construct Penner sequences consisting of handlebody mapping classes and prove Theorem 1.1.

Let  $(S, a, b, c, d)$  be the surface and curves shown in Figure 1. Then  $S$  is a closed genus-2 surface and  $b$  is the empty curve. Let  $p : S \rightarrow H$  be the inclusion of  $S$  as the boundary of the genus-2 handlebody. Then  $(S, a, \emptyset, c, d)$  defines a Penner sequence  $\phi_m : S_m \rightarrow S_m$  with no boundary components.

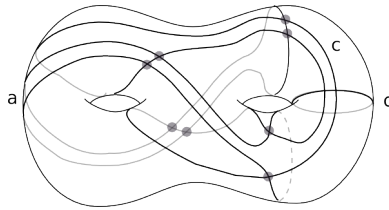


FIGURE 1. Pair of multi-curves  $a$  and  $c$  and cutting curve  $d$  on the surface  $S$ .

By Theorem 2.2, we have

$$\log \lambda(\phi_m) \asymp \frac{1}{m}.$$

Figure 2 gives a picture of the mapping classes  $\phi_m : S_m \rightarrow S_m$ . One observes that  $S_m$  has genus  $g = m + 1$ , and  $\phi_m$  is a union of Dehn

twists that bound disks in the interior of the handlebody. This proves Theorem 1.1.

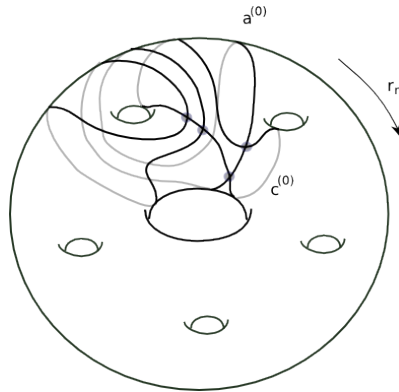


FIGURE 2. The mapping class  $(S_m, \phi_m)$ .

Using the theory of fibered faces, it is possible to compute the dilatations of the mapping classes in these examples explicitly.

By studying the action of  $\phi$  on the curves  $a$  and  $c$  (and the associated train track defined in the proof of Penner’s semi-group criterion), one can find the dilatation  $\lambda(\phi)$  as the largest eigenvalue of

$$\begin{bmatrix} 1 & 8 \\ 8 & 65 \end{bmatrix}.$$

Thus,  $\lambda(\phi)$  is the largest root of the characteristic polynomial

$$x^2 - 66x + 1 = 0.$$

By Theorem 2.2, we have

$$\lim_{m \rightarrow \infty} L_\chi(\bar{S}_m, \bar{\phi}_m) = \lambda(\phi)^2 \approx (65.98)^2 \approx 4353.99.$$

**Remark 3.1.** Using the McMullen polynomial [14], one can also find the dilatations of each of the mapping classes  $(\bar{S}_m, \bar{\phi}_m)$ : the dilatations of  $\phi_m$  is the largest root of

$$x^{2m} - 16x^{m+1} - 34x^m - 16x^{m-1} + 1 = 0.$$

(See [8] for more detailed descriptions of the computational techniques.)



**3.2. MAPPING CLASSES WITH HOMOLOGICAL DILATATION EQUAL TO ONE.**

Consider the surface and curves shown in Figure 3 where  $a = a_1 \cup a_2$ . (The base example shown in Figure 3 also appears in [5, Figure 14.1] and [11, Lemma 5.1].) Then  $(S, a, b, c, d)$  satisfies the conditions of Theorem 2.2 and gives rise to Penner sequence  $(S_m, \phi_m)$  where  $S_m$  has genus  $2m$ .

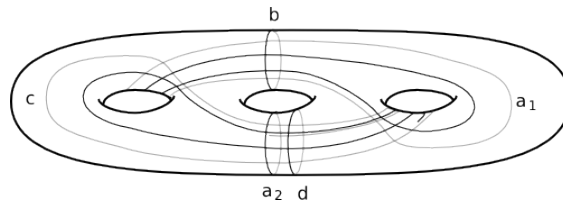


FIGURE 3. Example generating a Penner sequence with trivial homological dilatation.

The mapping classes  $\delta_{a_2}^{-1}\delta_b$ ,  $\delta_{a_1}$ , and  $\delta_c$  are all elements of the Torelli subgroup of  $\text{Mod}(S_3)$ , as are their lifts to  $S_m$  (see Figure 4).

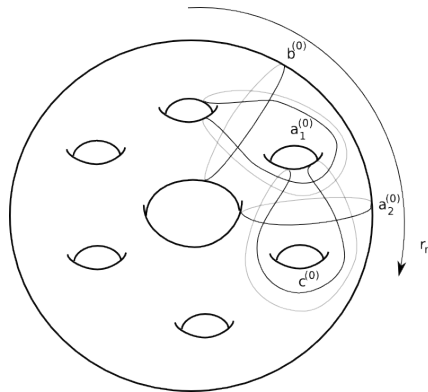


FIGURE 4. Penner sequence of pseudo-Anosov mapping classes with trivial homological dilatation on even genus surfaces.

The rotation  $r_m$  is a rotation by “two clicks” and has order  $m = g_m/2$ . Since  $\phi_m$  is a composition of an element in the Torelli group and a rotation

$$\phi_m = r_m \delta_{c^{(0)}} \delta_{a_1^{(0)}}^{-1} \delta_{a_2^{(0)}}^{-1} \delta_{b^{(0)}}$$

it has trivial homological dilatation. This completes the proof of Theorem 1.2.

#### 4. FURTHER QUESTIONS

Let  $\mathcal{P} = \bigcup_g \mathcal{P}(S_g)$ , where  $\mathcal{P}(S_g)$  is the set of pseudo-Anosov mapping classes on the closed surface  $S_g$  of genus  $g$ . The smallest known accumulation point for the genus-normalized dilatation

$$L_{\text{genus}}(\phi_g) = \lambda(\phi_g)^g$$

equals

$$\mu = \gamma_0^2,$$

where  $\gamma_0$  is the golden mean (see [7], [1], [10]).

**Question 4.1.** Is  $\mu$  the smallest accumulation point for  $L_{\text{genus}}$  on  $\mathcal{P}$ ?

The results of this paper lead to the following more specific questions.

**Question 4.2.** What is the smallest accumulation point for genus-normalized dilatation restricted to handlebody mapping classes or to mapping classes with trivial homological dilatation?

For example, one would expect the smallest accumulation point for handlebody subgroups to relate to the geometry of fibered 4-dimensional manifolds, and perhaps this further restriction gives a lower bound for normalized dilatation that is higher than  $\mu$ .

**Question 4.3.** Can the smallest accumulation point for genus-normalized dilatations of pseudo-Anosov mapping classes be achieved as the limit of dilatations for closures of a generalized Penner sequence?

So far there is no known example of a Penner sequence whose genus-normalized dilatations converge to  $\mu$ .

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