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TOPOLOGY PROCEEDINGS

Volume 45, 2015

Pages 53–61

http://topology.nipissingu.ca/tp/

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by

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Electronically published on February 14, 2014

Topology Proceedings

Web: http://topology.auburn.edu/tp/
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E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 45 (2015)
Pages 53-61

E-Published on February 14, 2014

A REMARK ON SCREENABLENESS IN COUNTABLE PRODUCTS

JIANJUN WANG

Abstract. In this paper, we present that the product$Y\times\prod_{n\in\omega}X_n$ is screenable if Y is a hereditarily screenable space and $\{X_n : n\in\omega\}$ is a countable collection of Čech-scattered screenable spaces. And we obtain a group of equivalent conditions for hereditarily screenableness in countable products.

1. Introduction

As a generalization of C-scattered spaces, Čech-scattered spaces introduced by Hohti and Yun [7] play the fundamental role in study of paracompactness of countable products. And they showed that if $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact (Lindeöf) spaces, then the product $\prod_{n\in\omega} X_n$ is paracompact (Lindeöf). In 2005, Aoki and Tanaka [2] extended the Hohti and Ziqiu's results by proving that if Y is a perfect paracompact space (hereditarily Lindeöf), and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact (Lindeöf) spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindeöf).

Recently there has been a renewed interest in studying weaker topological spaces to its countable products. The authors [8, 9] have respectively proved that subparacompact space, weakly submetacompactness have an analogous result with Aoki and Tanaka.

2010 *Mathematics Subject Classification.* Primary 54B10; Secondary 54D15, 54D20.

Key words and phrases. Screenableness, hereditarily screenableness, Čech-scattered.

Supported by the Nature Scientific Fund Committee of China, grant No.11201053.

Bing [4] defined a space to be screenable if every open cover has a σ-disjoint refinement. And Greever [6] showed that countably paracompact screenable space is paracompact. However, Balogh [3] constructed a normal, screenable, nonparacompact space in ZFC. And Peiyong [11] demonstrated that there is a first countable regular screenable space X such that for each $n \in \omega$, X^n is screenable, but X^ω is not screenable. In 2012, the authors [10] showed that the product $\prod_{n\in\omega} X_n$ is screenable if $\{Xn : n \in \omega\}$ is a countable collection of Čech-scattered screenable spaces. Obviously, perfect screenable space is hereditarily screenable. It is natural to pose the following question:

Question 1. Let Y be a hereditarily screenable space. Is the product $Y \times \prod_{n\in\omega} X_n$ screenable if $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered screenable spaces?

The purpose of this paper is to give an affirmative answer to Question 1 for screenableness. And then a group of equivalent conditions for hereditarily screenable spaces is obtained.

Throughout this paper, assume that each space is Tychonoff. ω denotes the set of natural numbers.

2. Preliminaries

In the rest of this section, we state some notation and basic facts. Undefined terminology can be found in Engelking [5]. A space X is *scattered* if every nonempty closed subset S has an isolated point s. And a space X is said to be C-*scattered* (Čech-*scattered*) if in every nonempty closed subset S of X, there exists a point $s \in S$ which has a compact (Čech-complete) neighborhood in S. Evidently, all of the scattered spaces, locally compact spaces and C-scattered spaces are Čech-scattered.

Let X be a space. For a subset S of X, $|S|$ $(\overline{S})$ denotes its cardinality (closure). Assume that S is closed. Put

$$S^* = \{x \in S : x \text{ has no Čech-complete neighborhood in } S\}.$$

Let $S^0{=}S$, $S^{(\alpha+1)}{=}(S^{(\alpha)})^*$, and $S^{(\alpha)}{=}\cap_{\beta<\alpha} S^{(\beta)}$ for a limit ordinal α. Note that each $S^{(\alpha)}$ is closed in X. Furthermore, a space X is Čech-scattered if and only if $X^{(\alpha)}{=}\emptyset$ for some ordinal α. Obviously, a Čech-scattered space is hereditary for its closed (open) subspace. A closed subset S of X is called *topped* if $S \cap S^{(\alpha)}$ is nonempty Čech-complete and $S \cap S^{(\alpha+1)}{=}\emptyset$ for some ordinal α, and denotes $S \cap S^{(\alpha)}$ by $Top(S)$. For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. Let $rank(x){=}\alpha$. Then, there is an open neighborhood base $\mathcal{V}$ of x in X such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped in X and $\alpha(\overline{V}){=}rank(x)$. A collection $\mathcal{V}$ of subsets of X is a refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$ and $\cup\mathcal{V}{=}\cup\mathcal{U}$.

Definition 2.1. A space X is said to be hereditarily screenable if every subspace of X is screenable.

By the definition, it is easy to check that a space X is hereditarily screenable if and only if every open subspace of X is screenable.

To complete our proof, the following lemmas will be needed.

Lemma 2.2. [8] *The product $X \times Y$ is Čech-scattered if X and Y are Čech-scattered spaces.*

Lemma 2.3. [5] *A Tychonoff space X is Čech-complete if and only if there exists a countable family $\{\mathcal{A}_i\}_{i\in\omega}$ of open covers of the space X with the property that any family $\mathcal{F}$ of closed subsets of X, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_i$ for $i \in \omega$, has nonempty intersection.*

Note that the intersection $\cap\mathcal{F}$ is countable compact in Lemma 2.3. The following lemmas play an important role in the study of our main result which is due to [10].

Lemma 2.4. *If X is a Čech-scattered screenable space, then for every open cover $\mathcal{U}$ of X, there exists a σ-disjoin open cover $\mathcal{V}=\cup_{n\in\omega}\mathcal{V}_n$ of X such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped and is contained in some element of $\mathcal{U}$.*

3. Main Results

Now we show the main result in this note.

Theorem 3.1. *If Y is a hereditarily screenable space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered screenable spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is screenable.*

Proof. Using the proof of [Theorem, 1], we may assume that for each $n \in \omega$, $X_n=X$ and $Top(X)=\{a\}$ for some $a \in X$. Let $\mathcal{B}$ be a base of $Y \times X^\omega$, consisting of all sets of the form $V=\widetilde{V} \times \prod_{n\in\omega} V_n$, where $\widetilde{V}$ is an open subset of Y and for each $n \in \omega$, V_n is open in X, and there is an $n \in \omega$ such that for each $i < n$, $\overline{V_i}$ is topped and for each $i \geq n$, $V_i=X$. Let $n(V)=\inf\{i : V_j=X, \text{ for } j \geq i\}$.

Let $\mathcal{G}$ be an arbitrary open covering of $Y \times X^\omega$, which is closed under finite unions. Let $\mathcal{G}'=\{\mathcal{B} : V \in \mathcal{B}, V \subset G$ for some $G \in \mathcal{G}\}$. We are going to find a σ-disjoint open refinement of subcover $\mathcal{G}'$.

Let $V=\widetilde{V} \times \prod_{n\in\omega} V_n \in \mathcal{B}$. Then for each $i < n(V)$, $\overline{V_i}$ is topped, i.e, $Top(\overline{V_i})$ is Čech-complete. By Lemma 2.3, there is a sequence $\{\mathcal{W}_{i,m}(V) : m \in \omega\}$ of open covers of $Top(\overline{V_i})$, such that if $\mathcal{F}$ is a collection of nonempty closed subsets of $Top(\overline{V_i})$ with the finite intersection property such that for each $m \in \omega$, there are $F_m \in \mathcal{F}$ and $W_m \in \mathcal{W}_{i,m}(V)$ with $F_m \subset W_m$, then the intersection $\cap\mathcal{F}$ is nonempty. Next, define $\mathcal{C}$ as follows:

(*) $(V, \mathcal{W}_{i,m}(V)) \in \mathcal{C}$, $m \in \omega$, if $V{=}\widetilde{V} \times \prod_{i\in\omega} V_i \in \mathcal{B}$ and $\mathcal{W}_{i,m}(V)$ is an open cover of $Top(\overline{V_i})$, satisfying the conditions described above.

For each $m \in \omega$, let $(V, \mathcal{W}_{i,m}(V)) \in \mathcal{C}$. In the case that $i < n(V)$, let $m{=}1$. Then for each $W \in \mathcal{W}_{i,1}(V)$, there is an open subset $W^{'}$ of $\overline{V_i}$ such that $W{=}W^{'} \cap Top(\overline{V_i})$. Moreover, $\{W^{'} : W \in \mathcal{W}_{i,1}(V)\} \cup \{\overline{V_i} - Top(\overline{V_i})\}$ covers $\overline{V_i}$ and hence, it follows from Lemma 2.4 that there is an open covering $\mathcal{A}_i(V){=}\cup_{j<\omega}\mathcal{A}_{i,j}(V)$ of V_i such that

(i) for each $j \in \omega$, $\mathcal{A}_{i,j}(V)$ is disjoint,

(ii) for each $A \in \mathcal{A}_{i,j}(V)$, $j \in \omega$, $\overline{A}$ is topped and contained in some member of $\{W^{'} : W \in \mathcal{W}_{i,m}(V)\} \cup \{\overline{V_i} - Top(\overline{V_i})\}$.

In the case that $i{=}n(V)$, we can also take a σ-disjoint open covering $\mathcal{A}_{n(V)}(V)$ of V_i such that for each $A \in \mathcal{R}_i(V)$, $\overline{A}$ is topped. And there is a proper member $A_0 \in \mathcal{A}_{n(V)}(V)$ with $a \in A_0$ and for each $A^* \in \mathcal{A}_{n(V)}(V) - \{A_0\}$, $a \notin A^*$.

Let $\mathcal{R}(V)_j{=}\prod_{i\leq n(V)} \mathcal{A}_{i.j}(V)$. Then $\mathcal{R}(V){=}\cup_{j\in\omega}\mathcal{R}_j(V)$ is a σ-disjoint open covering of $\prod_{i\leq n(V)} V_i$. Fix an $R{=}\prod_{i\leq n(V)} R_i \in \mathcal{R}(V)$ with $Top(\overline{R}) \cap Top(\prod_{i\leq n(V)} \overline{V_i}) \neq \emptyset$. Then, $Top(\overline{R_i}) \cap Top(\overline{V_i}) \neq \emptyset$ for each $i \leq n(R)$. And then $Top(\overline{R_i}) \cap Top(\overline{V_i}){=}\overline{R_i} \cap Top(\overline{V_i}) = Top(\overline{R_i})$.

Hence, by (ii), $Top(\overline{R_i}) \subset W$ for some $W \in \mathcal{W}_{i,1}(V)$. For $R \in \mathcal{R}(V)$, let $P(R){=}R \times X \times \cdots$. Then $Top(\overline{P(R)}){=}Top(\overline{R}) \times \{a\} \times \cdots$. Let $R_{\widetilde{V}}{=}\widetilde{V} \times P(R){=}\widetilde{V} \times \prod_{i\leq n(V)} R_i \times X \times \cdots$. Then for each $y \in \widetilde{V}$, let $\widehat{R}_y{=}\{y\}{\times}Top(\overline{P(R)})$. Namely, $\widehat{R}_y$ is Čech-complete. Define $\widehat{R}_y$ satisfying $(**)$ as follows:

$(**)$ if there are some basic open subsets O_1, O_2 and O_3 in $Y \times X^\omega$ and some $G \in \mathcal{G}^{'}$ such that $\widehat{R}_y \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset O_3 \subset \overline{O_3} \subset G$.

We say that R holds $(**)$ if there exists a $y \in \widetilde{V}$ such that $\widehat{R}_y$ holds $(**)$. Fix a $y \in \widetilde{V}$. Assume that $\widehat{R}_y$ satisfies the condition $(**)$. Let $k(R, y){=}\inf\{n(O_1) : O_1, O_2 \text{ and } O_3$ are some basic open subsets in $Y{\times}X^\omega$ with $n(O_1){=}n(O_2){=}n(O_3)$ such that $\widehat{R}_y \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset O_2 \subset O_3 \subset \overline{O_3} \subset G$ for some $G \in \mathcal{G}^{'}\}$. Then, we can take some basic open subsets $O_1(R, y){=}\widetilde{O_1(R, y)} \times \prod_{i\in\omega} O_1(R, y)_i$, $O_2(R, y){=}\widetilde{O_2(R, y)} \times \prod_{i\in\omega} O_2(R, y)_i$ and $O_3(R, y){=}\widetilde{O_3(R, y)} \times \prod_{i\in\omega} O_3(R, y)_i$ in $Y \times X^\omega$ and some $G(R, y) \in \mathcal{G}^{'}$ such that

(1) (a) $\widehat{R}_y \subset O_1(R, y) \subset \overline{O_1(R, y)} \subset O_2(R, y) \subset \overline{O_2(R, y)} \subset O_3(R, y) \subset \overline{O_3(R, y)} \subset G(R, y)$.

(b) $k(R,y)=n(O_1(R,y))$.

Let $r(R,y)=\max\{n(V)+1,k(R,y)\}$. Define

$$H(R,y)=\widetilde{H(R,y)}\times\prod_{i<r(R,y)}P(R)_i\cap O_3(R,y)_i\times X\times\cdots=\widetilde{H(R,y)}\times\prod_{i\in\omega}H(R,y)_i.$$

By the definition of $H(R,y)$, we assume that

(2) (a) for $i\in\omega$ with $k(R,y)\le i<r(R,y)$, let $H(R,y)_i=P(R)_i$,

(b) for $i\in\omega$ with $i<k(R,y)$ and $i<n(V)$, let $H(R,y)_i=P(R)_i\cap O_3(R,y)_i$,

(c) for $i\in\omega$ with $n(V)\le i<k(R,y)$, let $H(R,y)_i=\{a\}$,

(d) in the case that $r(R,y)=n(V)+1$, let $H(R,y)_i=X$ for each $i\ge n(V)+1$; in the case that $r(R,y)=k(R,y)>n(V)+1$, let $H(R,y)_i=X$ for $i\ge k(R,y)$.

(e) $\widetilde{H(R,y)}=\cap_{i=1}^3\widetilde{O_i(R,y)}$.

Then, $H(R,y)\in\mathcal{B}$ with $\widehat{R}_y\subset H(R,y)$, and contained in some member of $\mathcal{G}$ and for each $i\in\omega$, $\overline{H(R,y)_i}$ is topped.

For each $k\in\omega$, let $\mathcal{H}(R,k)=\{\widetilde{H(R,y)}\cap\widetilde{V}\ :\ k(R,y)\le k\}$. For $k\in\omega$, let $L(R,k)=\{y\in\widetilde{V}\ :\ k(R,y)\le k\}$. Clearly, for each $k\in\omega$, $L(R,k)=\cup\mathcal{H}(R,k)$. By the hereditarily screenableness of Y, there exists a collection $\mathcal{L}(R,j)=\cup_{j\in\omega}\mathcal{L}_j(R,k)$ of open subsets in Y such that

(3) (a) $L(R,k)=\cup\mathcal{L}(R,k)$,

(b) $\mathcal{L}(R,k)$ refines $\mathcal{H}(R,k)$,

(c) each $\mathcal{L}_j(R,k)$ is disjoint in Y.

For each $L\in\mathcal{L}_j(R,k)$, there exists a $y(L)\in L(R,k)$ such that $L\subset H\widetilde{(R,y(L))}\cap\widetilde{V}$. Then $k(R,y(L))\le k$. Now, define $E(R,L)$ as follows:

$$E(R,L)=L\times\prod_{i\in\omega}H(R,y(L))_i.$$

Then $E(R,L)\in\mathcal{B}$ and $L\times Top(\overline{P(R)})\subset E(L,R)$. By a similar manner as in the proof of [Theorem 3.6, 10] and definitions of $P(R)$,$E(R,L)$, we can obtain that the following collection $\mathcal{V}(R,L)=\cup_{n\in\omega}\mathcal{V}_n(R,L)$ satisfies the following:

(4) $\mathcal{V}(R,L)$ is σ-disjoint in $Y\times X^\omega$ and $L\times P(R)=E(R,L)\cup(\cup\mathcal{V}(R,L))$.

Let $\mathcal{L}(R)=\cup_{k\in\omega}\mathcal{L}_j(R,k)$. Furthermore, let

$\mathcal{E}(V,R)=\{E(R,L):L\in\mathcal{L}(R)\}$,

$\mathcal{V}(V,R)=\cup\{\mathcal{V}(R,L):L\in\mathcal{L}(R)\}$.

When R does not satisfy $(**)$ or $Top(\overline{R}) \cap Top(\prod_{i\leq n(V)} \overline{V_i})=\emptyset$, let $\mathcal{E}(V,R)=\{\emptyset\}$, $\mathcal{V}(V,R)=\{V^*\}$, where $V^*=R\times X\times\cdots$. We can also take a proper sequence $\{\mathcal{W}_{i,m}(V^*) : m\in\omega\}$ such that $(V^*,\mathcal{W}_{i,m}(V^*))\in\mathcal{C}$, $m\in\omega$, as the ones described before.

Again, let

$\mathcal{E}_i(V)=\cup\{\mathcal{E}(V,R) : R\in\mathcal{R}_i(V)\}$,

$\mathcal{V}_i(V)=\cup\{\mathcal{V}(V,R) : R\in\mathcal{R}_i(V)\}$.

Then the following statements are straightforward from the above.

(5) (a) $\mathcal{E}(V)=\cup_{i\in\omega}\mathcal{E}_i(V)$ is a σ-disjoint collection of basic open subsets of $Y\times X^\omega$ such that every member of $\mathcal{E}(V)$ is contained in some member of $\mathcal{G}$,

(b) $\mathcal{V}(V)=\cup_{i\in\omega}\mathcal{V}_i(V)$ is a σ-disjoint collection of basic open subsets of $Y\times X^\omega$,

(c) $V=\cup\mathcal{E}(V)\cup(\cup\mathcal{V}(V))$,

for each $V^*=\widetilde{V^*}\times\prod_{i\in\omega}V_i^*\in\mathcal{V}(V,R)$, $R=\prod_{i\leq n(V)}R_i\in\mathcal{R}(V)$,

(d) $n(V^*)>n(V)$ and for each $i\in\omega$, $\alpha(\overline{V_i^*})\leq\alpha(\overline{V_i})$,

(e) $(V^*,\mathcal{W}_{i,m}(V^*))\in\mathcal{C}$ such that for each $i\leq n(V)$, if $\alpha(\overline{V_i^*})=\alpha(\overline{V_i})$, then $Top(\overline{V_i^*})\subset Top(\overline{R_i})$ and for each $m\in\omega$, $\mathcal{W}_{i,m}(V^*)=\{W\cap\overline{V_i^*} : W\in\mathcal{W}_{i,m+1}(V)\}$,

(f) if R satisfies $(**)$ and $V^*=L\times\prod_{i\in\omega}V_i^*$ for some $L\in\mathcal{L}(R)$, with $k(R,y(L))<n(V)$, then there is an $i<k(R,y(L))$ such that $\alpha(\overline{V_i^*})<\alpha(\overline{V_i})$.

Now, proceeding by induction on $n\in\omega$, we define two families $\mathcal{E}_n$ and $\mathcal{V}_n$ as follows. Let $\mathcal{E}_0=\{\emptyset\}$, $\mathcal{V}_0=\{V(0)\}$, where $V(0)=Y\times X^\omega$. Put $\mathcal{W}_{i,m}=\{\{a\}\}$ for each $i,m\in\omega$. Now assume that $n=m$. Both of the families $\mathcal{E}_n$ and $\mathcal{V}_n$ of basic open subsets of $Y\times X^\omega$, satisfy the following:

(6) (a) $\mathcal{E}_n=\cup\{\mathcal{E}(V) : V\in\mathcal{V}_{n-1}\}$ is a σ-disjoint collection of basic open subsets of $Y\times X^\omega$ such that every member of $\mathcal{E}_n$ is contained in some member of $\mathcal{G}$,

(b) $\mathcal{V}_n=\cup\{\mathcal{V}(V) : V\in\mathcal{V}_{n-1}\}$ is a σ-disjoint collection of basic open subsets of $Y\times X^\omega$,

for each $V=\widetilde{V}\times\prod_{i\in\omega}V_i\in\mathcal{V}_{n-1}$, $V^*=\widetilde{V^*}\times\prod_{i\in\omega}V_i^*\in\mathcal{V}(V,R)$, $R=\prod_{i\leq n(V)}R_i\in\mathcal{R}(V)$,

(c) $(V,\mathcal{W}_{i,m}(V))\in\mathcal{C}$

(d) $V=\cup\mathcal{E}(V)\cup(\cup\mathcal{V}(V))$,

(e) $n(V^*)>n(V)$,

(f) for each $i\in\omega$, $\alpha(\overline{V_i^*})\leq\alpha(\overline{V_i})$,

(g) $(V^*,\mathcal{W}_{i,m}(V^*))\in\mathcal{C}$ such that for each $i\leq n(V)$, if $\alpha(\overline{V_i^*})=\alpha(\overline{V_i})$, then $Top(\overline{V_i^*})\subset Top(\overline{R_i})$ and for each $m\in\omega$, $\mathcal{W}_{i,m}(V^*)=\{W\cap\overline{V_i^*} : W\in\mathcal{W}_{i,m+1}(V)\}$,

(h) $Y(V,R)=\{y \in \widetilde{V} : \widehat{R}_y \text{satisfies}(**)\}$ for $R \in \mathcal{R}(V)$ and $\mathrm{Y}(n-1)=\cup\{Y(V,R) : V \in \mathcal{V}_{n-1}, R \in \mathcal{R}(V)\}$.

(i) if $y \in Y(V,R)$, $R \in \mathcal{R}(V)$ with $k(R,y) < n(V)$, then there is an $i < k(R,y)$ such that $\alpha(\overline{V_i^*}) < \alpha(\overline{V_i})$.

By the above constructions, we infer that the families $\mathcal{E}_{n+1}$ and $\mathcal{V}_{n+1}$ satisfy the consequents of (6) (a) $\sim$ (i). Let $\mathcal{E}=\cup_{n\in\omega}\mathcal{E}_n$. We shall show that $\mathcal{E}$ is a σ-disjoint open refinement of $\mathcal{G}'$. By (6) (a), (b) and the induction, $\mathcal{E}$ is a σ-disjoint collection of open sets in $Y \times X^\omega$. It suffices to show that $\mathcal{E}$ covers $Y \times X^\omega$. To show this, assume the contrary. Let $(y,(x_k)) \in Y \times X^\omega - \cup\mathcal{E}$. By (5) and (6) repeatedly, there are some collections $\{R(m) : m \geq 1\}$, $\{V(m) : m \geq 1\}$, where $V(0)=Y \times X^\omega$, $\{y(m) : m \geq 1\}$ satisfying for each $m \geq 1$,

(7) (a) $(y,(x_k)) \in V(m)=\widetilde{V(m)} \times \prod_{i\in\omega} V(m)_i \in \mathcal{V}(V(m-1), R(m))$, and $R(m)=\prod_{i\leq n(V(m-1))} R(m)_i \in \mathcal{R}(V(m-1))$, $y(m-1) \in Y(m-1)$,

(b) $n(V(m)) > n(V(m-1))$ and $\alpha(\overline{V(m)_i}) \leq \alpha(\overline{V(m-1)_i})$,

(c) for each $i \leq n(V(m-1))$, if $\alpha(\overline{V(m)_i})=\alpha(\overline{V(m-1)_i})$, then $Top(\overline{V(m)_i}) \subset Top(\overline{R(m-1)_i})$ and for each $j \in \omega$, $\mathcal{W}_{i,j}(V(m))=\{W \cap \overline{V(m)_i} : W \in \mathcal{W}_{i,j+1}(V(m-1))\}$,

(d) if $\widehat{R(m-1)}_{y(m-1)}$ satisfies $(**)$ with $k(R(m-1), y(m-1)) < n(V(m-1))$, then there is an $i < k(R(m-1), y(m-1))$ such that $\alpha(\overline{V(m)_i}) < \alpha(\overline{V(m-1)_i})$.

Fix an $i \in \omega$. By (7) (b), $n(V(m)) > n(V(m-1))$ for each $m \geq 1$. Then by (10) (b) again, there exists an $m_i^* \in \omega$ such that for each $m \geq m_i^*$, $i < n(V(m))$ and $\alpha(\overline{V(m)_i})=\alpha(\overline{V(m_i^*)_i})$. Moreover, by (7) (c), $Top(\overline{V(m)_i}) \subset Top(\overline{R(m-1)_i})$ for $m \geq m_i^*$. Then there is a sequence $\{W(m-1) : m \geq m_i^*\}$ of open subsets of X such that for each $m \geq m_i^*$, $W(m-1) \in \mathcal{W}_{i,m-m_i^*+1}(V(m_i^*-1))$ and $Top(\overline{R(m-1)_i}) \subset W(m-1)$.

Let $K_i=\cap_{m\geq m_i^*} Top(\overline{V(m)_i})$. Then $K_i \subset \cap_{m\geq m_i^*} Top(\overline{R(m-1)_i})$. It follows from Lemma 2.3 that K_i is nonempty and compact. And then, let $K=\{y\} \times \prod_{i\in\omega} K_i$. Clearly, K is compact. Hence, by Wallace theorem in Engelking [5], $K \subset G$ for some $G \in \mathcal{G}$. Define $p=\inf\{n(Z) : K \subset Z \subset \overline{Z} \subset G\}$, where $Z=\widetilde{W} \times \prod_{i\in\omega} Z_i$ is an open subset of $Y \times X^\omega$. Then, there exists an $m_0 \in \omega$ such that $p < n(V(m_0))$. Let $m_1=\max\{m_i^* : i < p\}$. And let $m^*=\max\{m_0, m_1\}$. Then, we infer that $p < n(V(m^*)) < n(V(m^*))$ and for each $i < p$, $m_i^* \leq m^*$ and $Top(\overline{V(m^*)_i}) \subset Z_i$. Furthermore, $Top(\overline{R(m^*)_i}) \subset V_i$. Then $\widehat{R(m^*)}_y \subset V$ and hence, $\widehat{R(m^*)}_y$ satisfies $(**)$. Then, by (7) (d), $k(R(m^*), y(m^*))=k(R(m^*), y) \leq p < n(V(m^*))$. Thus there is an $i < k(R(m^*), y(m^*))$ such that $\alpha(\overline{V(m^*+1)_i}) < \alpha(\overline{V(m^*)_i})$, which is a contradiction.

In view of above, the product $Y \times X^\omega$ is screenable. And hence the proof is completed. □

Consequently, the following result is obvious.

Proposition 3.2. *If Y is a hereditarily screenable space and $\{X_n : n \in \omega\}$ is a countable collection of C-scattered screenable spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is screenable.*

For a inverse system $\{X_n, \pi^n_m, \omega\}$ and its limit S, let π_n be the projection from S into X_n for each $n \in \omega$. Denoted it by $\varprojlim\{X_n, \pi^n_m, \omega\}$ (for the detailed definition of inverse limit, see [2.5, 5]).

In 1998, Zhu [11] investigated that if every finite subproduct of a countable product $S=\prod_{n\in\omega} X_n$ is hereditarily screenable, then so is S. Next, using the properties of the Cartesian product being homeomorphic to the limit space of an inverse system which is induced by Engelking [5], we shall obtain a same result with Zhu.

Lemma 3.3. *Let $S=\varprojlim\{X_n, \pi^n_m, \omega\}$. If each space X_n is hereditarily screenable, then so is S.*

Proof. Let $\mathcal{G}=\{G_\xi : \xi \in \Xi\}$ be an open cover of subspace G of S. For each $n \in \omega$ and $\xi \in \Xi$, let $V_{n,\xi}=\cup\{V : V$ is open in X_n and $\pi_n^{-1}(V) \subset G_\xi\}$ and put $V_n=\cup\{V_{n,\xi} : \xi \in \Xi\}$. Then $\{\pi_n^{-1}(V_n) : n \in \omega\}$ is an open cover of S with $\pi_n^{-1}(V_n) \subset \pi_{n+1}^{-1}(V_{n+1})$ for each $n \in \omega$. For each $n \in \omega$, since X_n is hereditarily screenable, there exists a σ-disjoint covering $\mathcal{O}_n=\cup_{i\in\omega}\mathcal{O}_{n,i}$ of V_n such that for each $O \in \mathcal{O}_n$, there exists a $\xi \in \Xi$ such that $O \subset V_{n,\xi}$. For each $n, i \in \omega$, put $\mathcal{H}_{n,i}=\{\pi_n^{-1}(V_n) \cap \pi_n^{-1}(O_{n,i,\xi}) : \xi \in \Xi\}$. Then, it is easy to check that $\mathcal{H}=\cup_{n\in\omega} \cup_{i\in\omega} \mathcal{H}_{n,i}$ is a σ-disjoint open refinement of $\mathcal{G}$.

Thus S is hereditarily screenable. □

Theorem 3.4. *Let $S=\prod_{i\in\omega} X_i$. Then the following are equivalent.*

(1) S is hereditarily screenable.

(2) The product $\prod_{i\in\sigma} X_i$ is hereditarily screenable for each $\sigma \in [\omega]^{<\omega}$.

(3) The product $\prod_{i<n} X_i$ is hereditarily screenable for each $n \in \omega$.

Proof. And (1)⇒(2)⇒(3) hold trivially. Now, we infer that (3)⇒(1):

For each $n \in \omega$, let $\sigma_n=\{0, 1, \cdots, n\}$. Then, $\Sigma=\{\sigma_n : n \in \omega\}$ is directed by the relation $\subset$. Define $X_{\sigma_n}=\prod_{i<n} X_i$ for each $n \in \omega$. For each $i, j \in \omega$ with $i \leq j$, let $\pi^{\sigma_i}_{\sigma_j} : X_{\sigma_j} \to X_{\sigma_i}$ be the projection map. Denote $S'=\varprojlim\{X_{\sigma_i}, \pi^{\sigma_i}_{\sigma_j}\}$. Then, it is easy to check that S' is homeomorphism with S. It follows from Lemma 3.3 that S' is hereditarily screenable, and hence so is S. □

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