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TOPOLOGY
PROCEEDINGS

Volume 45, 2015

Pages 227–231

http://topology.nipissingu.ca/tp/

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Electronically published on November 9, 2014

Topology Proceedings

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E-mail: topolog@auburn.edu
ISSN: 0146-4124

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TOPOLOGY PROCEEDINGS
Volume 45 (2015)
Pages 227-231

E-Published on November 9, 2014

THE FREUDENTHAL COMPACTIFICATION AS AN INVERSE LIMIT

MICHAEL G. CHARALAMBOUS

Abstract. We describe a construction of the Freudenthal compactification of a locally compact connected space via inverse limits.

Introduction and definitions

A space is called rim–compact if it has a base consisting of open sets B with compact boundary, $\mathrm{bd}(B)$. A subset A of a space X is said to be zero–dimensionally embedded in X, if X has a base consisting of open sets B with $\mathrm{bd}(B) \cap A = \emptyset$. Every rim–compact space X has a compactification Y such that the remainder $Y \setminus X$ is zero–dimensionally embedded in Y. With respect to the usual order of compactifications of X, there is a maximal such compactification, FX, the Freudenthal compactification of X. FX is the unique compactification of X where two closed subsets E and F of X have disjoint closures iff E and F are separated in X by a compact set of X. For the relevant information the reader is referred to the books [3] and [1].

Outside general topology, the Freudenthal compactification has applications in manifold theory, group theory and graph theory. Here the space X is locally compact, σ–compact and connected, sometimes even locally connected, and the points of the *Freudenthal remainder* $FX \setminus X$ are called *ends*. See [4] for some historical comments and definitions.

The quasi–component of a point $x \in X$ is the intersection of all clopen sets of X that contain x. A subset B of X is called bounded if its closure, $\mathrm{cl}(B)$, is compact.

2010 *Mathematics Subject Classification.* Primary 54B35, 54D35, 54D40; Secondary 54D05, 54D45.

Key words and phrases. Rim–compact, locally compact, compact, separable metrizable spaces, compactifications, remainders, inverse limits.

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1. The Freudenthal compactification

From now on X denotes a locally compact, non–compact, connected space.

Let B be an open set of X. $Q(B,X)$ will denote the set whose points are the points of B together with all quasi–components of $X \setminus B$. Let $q_B : X \to Q(B,X)$ be the function that extends the identity on B and sends $x \in X \setminus B$ to its quasi–component. It is readily seen that q_B respects finite intersections of open sets G of X such that $G \setminus B$ is clopen in $X \setminus B$. Thus, sets of the form $q_B(G)$, where G is open in X and $G \setminus B$ is clopen in $X \setminus B$, form a base for a topology on $Q(B,X)$, from now on the topology of $Q(B,X)$. Evidently, for every basic open set $q_B(G)$, $q_B^{-1}(q_B(G)) = G$ so that $q_B : X \to Q(B,X)$ is continuous and $q_B : B \to B$ is a homeomorphism. Note that $Q(B,X)$ is T_1: if k,l are distinct quasi–components of $X \setminus B$, there is a clopen set H of $X \setminus B$ such that $k \subset H$ and $l \not\subset H$. Then $q_B(B \cup H)$ is a basic open set containing k but not l. Suppose now that D is another open set of X with $B \subset D$. Then the function $q_{D,B} : Q(D,X) \to Q(B,X)$ that extends the restriction $q_B|D$ and sends a quasi–component of $X \setminus D$ to the quasi–component of $X \setminus B$ that contains it, satisfies $q_{D,B} \circ q_D = q_B$ and is therefore continuous and surjective. For open sets $B \subset D \subset G$, it is readily verified that $q_{D,B} \circ q_{G,D} = q_{G,B}$.

Lemma 1.1. *Let B be a non–empty bounded open subset of X. Then $Q(B,X)$ is compact and connected and $Q(B,X) \setminus B$ is zero–dimensionally embedded in $Q(B,X)$. As regards weight, $w(Q(B,X)) \le w(\mathrm{cl}(B)) \le w(X)$.*

Proof. Let $k \notin B$ be a point outside a closed set F of $Q(B,X)$. There is an open set G of X such that $G \setminus B$ is clopen in $X \setminus B$, $k \in q_B(G)$ and $q_B(G) \cap F = \emptyset$. This means $k \subset G$ and $G \cap q_B^{-1}(F) = \emptyset$. Note that $\mathrm{bd}(B) \cap G$ is compact. Let H be an open set of X with $\mathrm{bd}(B) \cap G \subset H \subset \mathrm{cl}(H) \subset G$. Let $U = (G \setminus B) \cup H = (G \setminus \mathrm{cl}(B)) \cup H$ and $V = X \setminus \mathrm{cl}(U)$. It is readily checked that $q_B(U)$ and $q_B(V)$ are disjoint basic open neighbourhoods of k and F, respectively, and $Q(B,X) \setminus B \subset q_B(U) \cup q_B(V)$. Hence the boundaries of $q_B(U)$ and $q_B(V)$ are contained in B. It readily follows that the T_1 space $Q(B,X)$ is Hausdorff and $Q(B,X) \setminus B$ is zero–dimensionally embedded in $Q(B,X)$.

Suppose a quasi–component k of $X \setminus B$ fails to intersect the compact $\mathrm{bd}(B)$. Then some clopen set of $X \setminus B \neq X$ that contains k would not intersect $\mathrm{bd}(B)$, and X would be disconnected. It follows that $Q(B,X) = q_B(\mathrm{cl}(B)) = q_B(X)$ and $Q(B,X)$ is compact and connected. Because the restriction $q_B|\,\mathrm{cl}(B) : \mathrm{cl}(B) \to Q(B,X)$ is perfect, $w(Q(B,X)) \le w(\mathrm{cl}(B)) \le w(X)$. □

Evidently, the restrictions $q_B|\,\mathrm{cl}(B) : \mathrm{cl}(B) \to Q(B,X)$ and $q_B|S : S \to Q(B,X) \setminus B$ are quotient maps, for any compact S with $\mathrm{bd}(B) \subset S \subset X \setminus B$. As such, they define the topologies of $Q(B,X)$ and $Q(B,X) \setminus B$.

Let $\mathcal{B}$ be a cover of X that is closed under finite unions and consists of non–empty, bounded open sets. Write $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ where $\alpha = \beta$ iff $B_\alpha = B_\beta$. Write Q_α instead of $Q(B_\alpha, X)$ and q_α instead of q_{B_α}. Direct A by defining $\alpha \leq \beta$ to mean $B_\alpha \subset B_\beta$. We have an inverse system $(\mathrm{Q}_\alpha, \pi^\beta_\alpha, A)$ of compact spaces and surjective maps, where for $\alpha \leq \beta$, $\pi^\beta_\alpha = q_{B_\beta, B_\alpha} : \mathrm{Q}_\beta \to \mathrm{Q}_\alpha$. Let Q be the limit space of the system and $\pi_\alpha : Q \to Q_\alpha$ the canonical projection.

Proposition 1.2. *Q is the Freudenthal compactification of X and has the same weight as X.*

Proof. The maps $q_\alpha : X \to Q_\alpha$ induce a map $q : X \to Q$ such that $\pi_\alpha \circ q = q_\alpha$. By Lemma 1.1, Q is compact. Because each q_α embeds B_α in Q_α as an open set, q embeds $X = \bigcup_{\alpha \in A} B_\alpha$ in Q as an open set. Because each q_α is surjective, $q(X)$ is dense in Q. Thus, Q is a compactification of X.

Consider closed sets E, F of X. If they are separated by a compact set K in X, there are disjoint open sets G, H of X such that $E \subset G$, $F \subset H$ and $X \setminus K = G \cup H$. Let B be a bounded open set of the locally compact space X such that $K \subset B \subset X \setminus (E \cup F)$. There is some $\alpha \in A$ with $\mathrm{cl}(B) \subset B_\alpha$. None of the open sets $q_\alpha(G)$, $q_\alpha(H)$ and B, which cover Q_α, intersects both $q_\alpha(E)$ and $q_\alpha(F)$. It follows that $q_\alpha(E)$ and $q_\alpha(F)$ have disjoint closures in Q_α, and $q(E)$ and $q(F)$ have disjoint closures in Q. Conversely, if $q(E)$ and $q(F)$ have disjoint closures in the compact space Q, then for some $\alpha \in A$, $q_\alpha(E) = \pi_\alpha(q(E))$ and $q_\alpha(F) = \pi_\alpha(q(F))$ have disjoint closures in Q_α. By Lemma 1.1, $Q_\alpha \setminus B_\alpha$ is zero–dimensionally embedded in the compact space Q_α. Hence we can find an open set U of Q_α such that $\mathrm{cl}(q_\alpha(E)) \subset U \subset \mathrm{cl}(U) \subset Q_\alpha \setminus \mathrm{cl}(q_\alpha(F))$ and $K = \mathrm{bd}(U) \subset B_\alpha$. Clearly, the compact set $\pi_\alpha^{-1}(K)$ of $q(X)$ separates $q(E)$ and $q(F)$. This shows that Q is the Freudenthal compactification of X.

We could have chosen $\mathcal{B}$ to be a base of X with cardinality $w(X)$. As A and $\mathcal{B}$ have the same cardinality and, by Lemma 1.1, $w(Q_\alpha) \leq w(X)$ for each $\alpha \in A$, we have $w(Q) \leq w(X) \leq w(Q)$. Hence $w(Q) = w(X)$. □

More generally, if $Q(\emptyset, X)$ is compact, for any rim–compact space X, then its Freudenthal compactification has the same weight as X. See [1, Theorem 3.15] for a proof.

Remark 1.3. The compact remainder $Z = Q \setminus q(X)$ consists of points $k = (k_\alpha)$ where each k_α is a quasi–component of $X \setminus B_\alpha$ and $k_\beta \subset k_\alpha$ for $\alpha \leq \beta$. It is not difficult to see that $Z_\alpha = \pi_\alpha(Z)$ consists of all unbounded components of $X \setminus B_\alpha$ and if $p_\alpha^\beta : Z_\beta \to Z_\alpha$ denotes the restriction of π_α^β, then we have an inverse system $(Z_\alpha, p_\alpha^\beta, A)$ whose limit is the Freudenthal remainder Z. As noted in the introduction, any zero–dimensionally embedded remainder of X is the perfect image of Z. Note also that a zero–dimensional remainder of a locally compact space is zero–dimensionally embedded.

In the recent paper [2], *metrizable remainder* means remainder in a metrizable compactification. The authors look for a zero–dimensional metrizable remainder Z of a non–compact, locally compact and connected separable metrizable space X such that any other zero–dimensional metrizable remainder of X is the continuous image of Z. Fixing a metric yielding bounded balls and considering concentric open balls B_m and spheres S_m of radius m, they construct an inverse sequence $(Z_m, p_m^n, \mathbb{N})$ and its limit Z, where $\mathbb{N}$ is the set of positive integers. Note that $q_{B_n}|S_n : S_n \to Y_n = Q(B_n, X) \setminus B_n$, whose fibers are the traces on S_n of the quasi–components of $X \setminus B_n$, coincides with their quotient map $\phi_n : S_n \to Y_n$. $Z_n \subset Y_n$ is defined to consist of all unbounded quasi–components of $X \setminus B_n$, and p_m^n sends an unbounded quasi–component of $X \setminus B_n$ to the unique quasi–component of $X \setminus B_m$ that contains it. From the last paragraph, Z is what is wanted because Z is the remainder of X in its Freudenthal compactification, which is metrizable because X is connected. Unaware of this, the authors unnecessarily proceed to construct a metrizable compactification in which Z is a remainder of X.

Remark 1.4. Another paper we have located where inverse limits are used in connection with the Freudenthal compactification is [5]. There X is locally compact, connected and locally connected. For every bounded open set U of X, the set A_U of unbounded components of $X \setminus \mathrm{cl}(U)$ is finite. For $V \subset \mathrm{cl}(V) \subset U$, $\pi_{U,V} : A_U \to A_V$ sends a member of A_U to the unique member of A_V that contains it. This results in an inverse system of finite discrete spaces with limit a compact space B and canonical projections $\pi_U : B \to U$. The topology on $FX = X \cup B$ is generated by the open sets of X together with the sets of the form $\pi_U^{-1}(k) \cup k$, where k is an unbounded component of $X \setminus \mathrm{cl}(U)$.

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