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by

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LOCALLY CONVEX S-COMPACTIFICATIONS

H. BOUSTIQUE, B. LOSERT, AND G. RICHARDSON

Abstract. Properties of a convergence semigroup S acting continuously on a locally convex convergence space X are investigated. Sufficient conditions are given for which the action can be continuously extended to a locally convex compactification of X. Local convexity is also extended to the generalized quotient of X and S.

1. Introduction and Preliminaries

Several articles have been written on the topic of a topological group or semigroup acting on a topological space. The authors have studied this notion in the larger category of convergence spaces. The category of convergence spaces has more desirable properties than the category of topological spaces. For example, quotient maps are both productive and hereditary in the category of convergence spaces, and these properties are particularly useful when forming the generalized quotient space. The primary focus of this work is to give sufficient conditions which guarantee that the action of a convergence monoid on a locally convex convergence space can be continuously extended to a locally convex compactification. Without convexity, this problem was investigated in [8]. It is shown in [7] that each locally convex convergence space possesses a locally convex compactification (no action considered). Basic results pertaining to a convergence monoid acting continuously on a convergence space (without convexity) can be found in [2] and [3].

Let X be a set, let $\mathbf{P}(X)$ be the power set of X, and let $\mathbf{F}(X)$ be the set of all filters on X. For each $x \in X$, $\dot{x}$ denotes the fixed ultrafilter on X whose base is $\{ \{ x \} \}$. Define a partial order $\geq$ on $\mathbf{F}(X)$ as follows:

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Given $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, $\mathcal{F} \geq \mathcal{G}$ (read "$\mathcal{F}$ is finer than $\mathcal{G}$") if and only if $\mathcal{G} \subseteq \mathcal{F}$. With this partial order, the least upper bound of $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, which we denote as $\mathcal{F} \vee \mathcal{G}$, exists provided that $F \cap G \neq \emptyset$ for each $F \in \mathcal{F}$ and $G \in \mathcal{G}$; it is the smallest filter containing both $\mathcal{F}$ and $\mathcal{G}$.

Definition 1.1. A pair (X, q) is called a ***convergence space*** whenever X is a set and $q : \mathbf{F}(X) \to \mathbf{P}(X)$ obeys:

(CS1) $x \in q(\dot{x})$ for all $x \in X$;
(CS2) $\mathcal{G} \geq \mathcal{F}$ implies $q(\mathcal{F}) \subseteq q(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$;
(CS3) $x \in q(\mathcal{F})$ implies $x \in q(\mathcal{F} \cap \dot{x})$ for all $x \in X$ and $\mathcal{F} \in \mathbf{F}(X)$.

A function q obeying (CS1) through (CS3) is called a ***convergence structure*** on X. The notation $x \in q(\mathcal{F})$ is read as "$\mathcal{F}$ q-converges to x" or as "$\mathcal{F}$ converges to x" and is usually written as $\mathcal{F} \xrightarrow{q} x$ or $\mathcal{F} \to x$. Most of the time we will not need to make explicit reference to the convergence structure so we will normally write (X, q) as X and $\mathcal{F} \xrightarrow{q} x$ as $\mathcal{F} \to x$.

A function $f : X \to Y$ between two convergence spaces is ***continuous*** if for all $x \in X$ and $\mathcal{F} \in \mathbf{F}(X)$, $\mathcal{F} \to x$ implies $f^{\to}(\mathcal{F}) \to f(x)$. Here, $f^{\to}(\mathcal{F})$ denotes the filter on Y whose base is $\{ f(F) \mid F \in \mathcal{F} \}$. Given two convergence structures p and q on a set X, we say that q is ***finer*** than p, denoted $q \geq p$, whenever the identity mapping $\mathrm{id}_X : (X, q) \to (X, p)$ is continuous. We write Conv for the category of convergence spaces and continuous functions.

Definition 1.2. Let $(X, \leq)$ be a partially ordered set and let $A \subseteq X$. Define

$$\mathrm{i}(A) = \{ x \in X \mid a \leq x \text{ for some } a \in A \},$$
$$\mathrm{d}(A) = \{ x \in X \mid x \leq a \text{ for some } a \in A \},$$
$$\mathrm{c}(A) = \mathrm{i}(A) \cap \mathrm{d}(A).$$

The set $\mathrm{c}(A)$ is called the ***convex hull*** of A, and if $A = \mathrm{c}(A)$, then A is said to be ***convex***. If $\mathcal{F} \in \mathbf{F}(X)$, then $\mathrm{c}\,\mathcal{F}$ denotes the filter on X whose base is $\{ \mathrm{c}(F) \mid F \in \mathcal{F} \}$; $\mathcal{F}$ is called a ***convex filter*** provided that $\mathcal{F} = \mathrm{c}\,\mathcal{F}$. Let $\hat{\mathbf{F}}(X)$ denote the set of all convex filters on X. A filter $\mathcal{F} \in \hat{\mathbf{F}}(X)$ is called a ***maximal convex filter*** and is said to be ***maximally convex*** whenever $\mathcal{F} \subseteq \mathcal{G} \in \hat{\mathbf{F}}(X)$ implies that $\mathcal{F} = \mathcal{G}$. Given $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, we will write $\mathcal{F} \lesssim \mathcal{G}$ whenever $\mathrm{i}\,\mathcal{F} \subseteq \mathcal{G}$ and $\mathrm{d}\,\mathcal{G} \subseteq \mathcal{F}$. Here, $\mathrm{i}\,\mathcal{F}$ (resp. $\mathrm{d}\,\mathcal{F}$) is the filter on X whose base is $\{ \mathrm{i}(F) \mid F \in \mathcal{F} \}$ (resp. $\{ \mathrm{d}(F) \mid F \in \mathcal{F} \}$). Note that $\lesssim$ is a partial order on $\hat{\mathbf{F}}(X)$.

The triple $(X, \leq, q)$ is called a ***locally convex space*** if $(X, \leq)$ is a poset, $(X, q) \in |\mathsf{Conv}|$, and if for all $x \in X$ and $\mathcal{F} \in \mathbf{F}(X)$, $\mathcal{F} \to x$ implies $\mathrm{c}\,\mathcal{F} \to x$. We write LCS for the category whose objects consist of all the locally convex spaces and whose morphisms are all the continuous increasing functions.

Lemma 1.3. *For any poset X, the following facts are true:*

(i) $\mathrm{c}(\mathcal{F} \times \mathcal{G}) = \mathrm{c}\,\mathcal{F} \times \mathrm{c}\,\mathcal{G}$ *for all* $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$.

(ii) $\mathcal{F} \in \mathbf{F}(X)$ *is maximally convex if and only if whenever A and B are convex subsets of X such that $A \cup B \in \mathcal{F}$, either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

(iii) $\mathcal{F} \in \mathbf{F}(X)$ *is maximally convex if and only if $\mathcal{F} = \mathrm{c}\,\mathcal{G}$ for some ultrafilter $\mathcal{G}$ on X. In particular, for any $x \in X$, $\mathcal{F} \times \dot{x}$ is maximally convex whenever $\mathcal{F}$ is maximally convex.*

(iv) *If Y is another poset and $f : X \to Y$ is an increasing function, then $f^{\rightarrow}(\mathrm{i}\,\mathcal{F}) \geq \mathrm{i}\,f^{\rightarrow}(\mathcal{F})$, $f^{\rightarrow}(\mathrm{d}\,\mathcal{F}) \geq \mathrm{d}\,f^{\rightarrow}(\mathcal{F})$, $f^{\rightarrow}(\mathrm{c}\,\mathcal{F}) \geq \mathrm{c}\,f^{\rightarrow}(\mathcal{F})$, and $\mathrm{c}\,f^{\rightarrow}(\mathcal{F})$ is maximally convex whenever $\mathcal{F}$ is maximally convex.*

Proof. Verification of (ii) and the last part of (iv) appear in [7]. Justification of (iii) is given here. If $\mathcal{F}$ is maximally convex and $\mathcal{G}$ is an ultrafilter on X containing $\mathcal{F}$, then $\mathcal{F} \subseteq \mathrm{c}\,\mathcal{G}$ and thus $\mathcal{F} = \mathrm{c}\,\mathcal{G}$. Conversely, assume that $\mathcal{F} = \mathrm{c}\,\mathcal{G}$. Suppose that A, B are convex subsets of X such that $A \cup B \in \mathcal{F} \subseteq \mathcal{G}$. Then either $A \in \mathrm{c}\,\mathcal{G}$ or $B \in \mathrm{c}\,\mathcal{G}$. Since $\mathrm{c}\,\mathcal{G} = \mathcal{F}$, according to (ii), $\mathcal{F}$ is maximally convex. The other parts are straightforward to prove. □

Order convergence of a net was defined by Birkhoff [1]. Ward [11], Kent [6], and Preuss [10] studied this concept from the filter perspective. It is shown in Proposition 1.5 below that order convergence is locally convex. Let $(X, \leq)$ be a complete lattice. For each $A \subseteq X$, define

$$A^- = \{\, x \in X \mid x \leq a \text{ for each } a \in A \,\},$$
$$A^+ = \{\, x \in X \mid a \leq x \text{ for each } a \in A \,\}.$$

A filter $\mathcal{F}$ on X ***order converges to*** x, written $\mathcal{F} \xrightarrow{0} x$, provided that $\inf \bigcup_{F \in \mathcal{F}} F^+ = \sup \bigcup_{F \in \mathcal{F}} F^- = x$.

Lemma 1.4 (Proposition 2.2 in [10])**.** *Assume that $(X, \leq)$ is a complete lattice and $\mathcal{B}$ is a filter base for a filter $\mathcal{F}$ on X. Then $\mathcal{F} \xrightarrow{0} x$ if and only if $x = \inf \bigcup_{B \in \mathcal{B}} B^+ = \sup \bigcup_{B \in \mathcal{B}} B^-$.*

Proposition 1.5. *Let $(X, \leq)$ be a complete lattice and let 0 denote order convergence as defined above. Then $(X, \leq, 0) \in |\mathsf{LCS}|$.*

Proof. Lemma 1.4 can be used to show that $\dot{x} \xrightarrow{0} x$ and $\mathcal{F} \xrightarrow{0} x$ implies that $\mathcal{F} \cap \dot{x} \xrightarrow{0} x$. Next, assume that $\mathcal{G} \geq \mathcal{F} \xrightarrow{0} x$. Since $\mathcal{F} \subseteq \mathcal{G}$, it follows that $\bigcup_{F \in \mathcal{F}} F^+ \subseteq \bigcup_{G \in \mathcal{G}} G^+$ and $\bigcup_{F \in \mathcal{F}} F^- \subseteq \bigcup_{G \in \mathcal{G}} G^-$, hence

$$\inf \bigcup_{G \in \mathcal{G}} G^+ \leq \inf \bigcup_{F \in \mathcal{F}} F^+ = x = \sup \bigcup_{F \in \mathcal{F}} F^- \leq \sup \bigcup_{G \in \mathcal{G}} G^-.$$

Since $\sup \bigcup_{G\in\mathcal{G}} G^- \leq \inf \bigcup_{G\in\mathcal{G}} G^+$, it follows that $\mathcal{G} \xrightarrow{0} x$ and that $(X, 0) \in |\mathsf{Conv}|$.

It remains to show that $(X, 0)$ is locally convex. First, it is shown that if $A \subseteq X$, then $\mathrm{c}(A)^+ = A^+$. Since $A \subseteq \mathrm{c}(A)$, $\mathrm{c}(A)^+ \subseteq A^+$. To prove the converse, first fix $y \in A^+$. Then $z \leq y$ for each $z \in A$, and if $r \in \mathrm{c}(A) = \mathrm{i}(A) \cap \mathrm{d}(A) \subseteq \mathrm{d}(A)$, then $r \leq a$ for some $a \in A$, and thus $r \leq a \leq y$. It follows that $y \in \mathrm{c}(A)^+$ and hence $A^+ = \mathrm{c}(A)^+$. Likewise $\mathrm{c}(A)^- = A^-$. If $\mathcal{F} \xrightarrow{0} x$, then

$$\inf \bigcup_{F\in\mathcal{F}} \mathrm{c}(F)^+ = \inf \bigcup_{F\in\mathcal{F}} F^+ = x = \sup \bigcup_{F\in\mathcal{F}} F^- = \sup \bigcup_{F\in\mathcal{F}} \mathrm{c}(F)^-,$$

and thus $\mathrm{c}\,\mathcal{F} \xrightarrow{0} x$. Therefore $(X, \leq, 0) \in |\mathsf{LCS}|$. ☐

Definition 1.6. The triple $(S, \cdot, p)$ is called a ***convergence monoid*** if:

(CM1) $(S, \cdot)$ is a monoid with identity e,
(CM2) $(S, p) \in |\mathsf{Conv}|$, and
(CM3) the binary operation $(s, t) \mapsto s \cdot t$ is continuous.

Let CM denote the category of convergence monoids and continuous homomorphisms. Let $X \in |\mathsf{Conv}|$, $S \in |\mathsf{CM}|$, and $\lambda : X \times S \to X$. Consider the following conditions on λ:

(A1) $\lambda(x, e) = x$ for each $x \in X$.
(A2) $\lambda(\lambda(x, s), t) = \lambda(x, s \cdot t)$ for all $x \in X$ and all $s, t \in S$.
(A3) λ is continuous.

If λ satisfies (A1) and (A2), then λ is an ***action*** of S on X, and if in addition it satisfies (A3), then λ is a ***continuous action***.

Definition 1.7. Let CA be the category whose objects consist of all triples (X, S, λ), where $X \in |\mathsf{LCS}|$, $S \in |\mathsf{CM}|$ and λ is a continuous action such that $\lambda(\cdot, s)$ is an increasing function for each fixed $s \in S$, and the morphisms are all pairs $(f, k) : (X, S, \lambda) \to (Y, T, \mu)$ such that:

(C1) $f : X \to Y$ is a morphism in LCS,
(C2) $k : S \to T$ is a morphism in CM, and
(C3) $\mu \circ (f \times k) = f \circ \lambda$.

Definition 1.8. A ***compactification*** of an object (X, S, λ) in CA is an object (Y, S, μ) in CA such that:

(COM1) $Y \in |\mathsf{LCS}|$ is compact,
(COM2) $f : X \to Y$ is a dense embedding, and
(COM3) (f, id_S) is a morphism in CA.

Throughout the remainder of this work, S will denote a convergence monoid, X a locally convex convergence space, and λ a continuous action of S on X such that $\lambda(\cdot, s)$ is an increasing function for each fixed $s \in S$. We will call an object (Y, S, μ) in CA an ***S-space*** and we will call a compactification of X in CA an ***S-compactification*** of X.

Lemma 1.9. *Assume that (X, λ) is an S-space, $\mathcal{F}_i \in \mathbf{F}(X)$, $\mathcal{G} \in \mathbf{F}(S)$, and $\mathcal{F}_1 \lesssim \mathcal{F}_2$, $i = 1, 2$. Then:*

(i) $\mathrm{i}(\mathrm{c}\,\mathcal{F}_1) = \mathrm{c}(\mathrm{i}\,\mathcal{F}_1) = \mathrm{i}\,\mathcal{F}_1$, $\mathrm{d}(\mathrm{c}\,\mathcal{F}_1) = \mathrm{c}(\mathrm{d}\,\mathcal{F}_1) = \mathrm{d}\,\mathcal{F}_1$ *and consequently* $\mathrm{i}\,\mathcal{F}_1$ *and* $\mathrm{d}\,\mathcal{F}_1$ *are convex filters on* X*;*
(ii) $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}) = \mathrm{c}\,\lambda^{\rightarrow}(\mathrm{c}\,\mathcal{F}_1 \times \mathcal{G})$*;*
(iii) $\mathrm{c}\,\mathcal{F}_1 \lesssim \mathrm{c}\,\mathcal{F}_2$
(iv) $\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}) \lesssim \lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G})$
(v) $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}) \lesssim \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G})$.

Proof. Verification of (iv) is given here. First, it is shown that if $A \subseteq X$ and $B \subseteq S$, then $\lambda(\mathrm{i}(A) \times B) \subseteq \mathrm{i}(\lambda(A \times B))$. Let $(x, s) \in \mathrm{i}(A) \times B$. Then there exists $a \in A$ such that $a \leq x$. Since λ is increasing in its first component, $\lambda(a, s) \leq \lambda(x, s)$. Thus $\lambda(x, s) \in \mathrm{i}(\lambda(A \times B))$ and consequently $\lambda(\mathrm{i}(A) \times B) \subseteq \mathrm{i}(\lambda(A \times B))$. It follows that $\mathrm{i}\,\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}) \subseteq \lambda^{\rightarrow}(\mathrm{i}\,\mathcal{F}_1 \times \mathcal{G}) \subseteq \lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G})$. Likewise, $\mathrm{d}\,\lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G}) \subseteq \lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G})$ and thus $\lambda^{\rightarrow}(\mathcal{F}_1 \times \mathcal{G}) \lesssim \lambda^{\rightarrow}(\mathcal{F}_2 \times \mathcal{G})$. □

Definition 1.10. An object $(X, \lambda) \in |\mathsf{CA}|$ is said to be ***adherence restrictive*** (resp. ***c-adherence restrictive***) if for each filter $\mathcal{F} \in \mathbf{F}(X)$ (resp. $\mathcal{F} \in \hat{\mathbf{F}}(X)$) and each convergent filter $\mathcal{G}$ on S, $\mathrm{adh}\,\mathcal{F} = \emptyset$ implies that $\mathrm{adh}\,\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = \emptyset$ (resp. $\mathrm{adh}\,\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = \emptyset$). Let (Y, μ) be an S-compactification of (X, λ) with dense embedding $f : X \to Y$. Then (Y, μ) is called ***remainder invariant*** provided that $\mu((Y - f(X)) \times S) \subseteq Y - f(X)$.

Verification of the following result involves only a minor modification of that given in [8] for spaces which are not necessarily locally convex.

Theorem 1.11. *Suppose that X is not compact, Y is Hausdorff, and (Y, μ) is an S-compactification of (X, λ) with dense embedding $f : X \to Y$. Then (X, λ) is adherence restrictive if and only if (Y, μ) is remainder invariant.*

Proof. Suppose Y is not remainder invariant. Then there exist $y \in Y - f(X)$ and $s \in S$ such that $\mu(y, s) = f(x)$. Since $f(X)$ is dense in Y, there exists $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}(\mathcal{F}) \to y$. Since f is increasing,

$f^{\to}(\mathrm{c}\,\mathcal{F}) \geq \mathrm{c}\,f^{\to}(\mathcal{F}) \to y$. Hence $\operatorname{adh}\mathrm{c}\,\mathcal{F} = \emptyset$ and

$$\begin{aligned} f^{\to}(\lambda^{\to}(\mathrm{c}\,\mathcal{F} \times \dot{s})) &= (f \circ \lambda)^{\to}(\mathrm{c}\,\mathcal{F} \times \dot{s}) \\ &= (\mu \circ (f \times \mathrm{id}_S))^{\to}(\mathrm{c}\,\mathcal{F} \times \dot{s}) \\ &= \mu^{\to}(f^{\to}(\mathrm{c}\,\mathcal{F}) \times \dot{s}) \\ &\to \mu(y, s) = f(x). \end{aligned}$$

Therefore, $\lambda^{\to}(\mathrm{c}\,\mathcal{F} \times \dot{s}) \to x$, which means $\operatorname{adh}\lambda^{\to}(\mathrm{c}\,\mathcal{F} \times \dot{s})$ is nonempty, and so (X, λ) is not adherence restrictive.

Conversely, suppose that (X, λ) is not adherence restrictive. Then there exists a convex filter $\mathcal{F}$ on X and $\mathcal{G} \to s$ in S such that $\operatorname{adh}\mathcal{F} = \emptyset$, yet $x \in \operatorname{adh}\lambda^{\to}(\mathcal{F} \times \mathcal{G})$ for some $x \in X$. Choose an ultrafilter $\mathcal{H}$ such that $\mathcal{H} \to x$ in X and $\mathcal{H} \geq \lambda^{\to}(\mathcal{F} \times \mathcal{G})$. Then there exists an ultrafilter $\mathcal{K} \geq \mathcal{F} \times \mathcal{G}$ such that $\mathcal{H} = \lambda^{\to}(\mathcal{K})$. Since $\pi_1^{\to}(\mathcal{K}) \geq \mathcal{F}$, we have that $\operatorname{adh}\pi_1^{\to}(\mathcal{K}) = \emptyset$, and the compactness of Y implies that $(f \circ \pi_1)^{\to}(\mathcal{K}) \to y \in Y - f(X)$ for some y. Thus:

$$\begin{aligned} f^{\to}(\mathcal{H}) &= (f \circ \lambda)^{\to}(\mathcal{K}) \\ &\geq (f \circ \lambda)^{\to}(\pi_1^{\to}(\mathcal{K}) \times \mathcal{G}) \\ &= (\mu \circ (f \times \mathrm{id}_S))^{\to}(\pi_1^{\to}(\mathcal{K}) \times \mathcal{G}) \\ &= \mu^{\to}((f \circ \pi_1)^{\to}(\mathcal{K}) \times \mathcal{G}) \\ &\to \mu(y, s). \end{aligned}$$

However, since Y is Hausdorff and $f^{\to}(\mathcal{H}) \to f(x)$, we have that $\mu(y, s) = f(x)$. Hence (Y, μ) is not remainder invariant. □

2. One-Point S-Compactifications

Let (X, λ) be an S-space, where $(X, \leq, q) \in |\mathsf{LCS}|$ is not compact. Choose $\omega \notin X$, let $\omega X = X \cup \{\,\omega\,\}$ and let $j : X \to \omega X$ be the natural injection $j(x) = x$. Define $a \lesssim b$ in ωX if and only if either $a \leq b$ in X or $a = b = \omega$. Then for each $x \in X$, $j(x)$ and ω are not comparable. Note that $(\omega X, \lesssim)$ is a poset and $j : X \to \omega X$ is increasing. Also note that for each $A \subseteq X$, $\mathrm{c}(j(A)) = j(\mathrm{c}(A))$ and that if $\mathcal{F}$ is a convex filter on X, then both $j^{\to}(\mathcal{F})$ and $j^{\to}(\mathcal{F} \cap \dot{\omega})$ are convex filters on ωX.

Define the convergence structure ωq on ωX as follows:

(i) $\mathcal{H} \xrightarrow{\omega q} j(x)$ if and only if there exists a convex filter $\mathcal{F}$ on X such that $\mathcal{F} \to x$ and $\mathcal{H} \geq j^{\to}(\mathcal{F})$.

(ii) $\mathcal{H} \xrightarrow{\omega q} \omega$ if and only if there exists a convex filter $\mathcal{F}$ on X such that $\operatorname{adh}\mathcal{F} = \emptyset$ and $\mathcal{H} \geq j^{\to}(\mathcal{F}) \cap \dot{\omega}$.

One can readily show that $(\omega X, \lesssim, \omega q) \in |\mathsf{LCS}|$. Define $\delta : \omega X \times S \to \omega X$ by $\delta(j(x), s) = j(\lambda(x, s))$ and $\delta(\omega, s) = \omega$.

Theorem 2.1. *Assume that* $(X, \lambda) \in |\mathsf{CA}|$ *is c-adherence restrictive and* X *is a noncompact (Hausdorff) space. Then* $(\omega X, \delta)$ *defined above is an (Hausdorff)* S*-compactification of* (X, λ) *with dense embedding* j*.*

Proof. It is not difficult to show that δ is a action of S on ωX. We will prove that δ is continuous. Suppose that $\mathcal{H} \to j(x)$ in ωX and $\mathcal{G} \to s$ in S. Then $\mathcal{H} \geq j^{\to}(\mathcal{F})$ for some convex filter $\mathcal{F} \to x$ in X. Hence

$$\begin{aligned} \delta^{\to}(\mathcal{H} \times \mathcal{G}) &\geq \delta^{\to}(j^{\to}(\mathcal{F}) \times \mathcal{G}) \\ &= j^{\to}(\lambda^{\to}(\mathcal{F} \times \mathcal{G})) \\ &\to j(\lambda(x, s)) = \delta(j(x), s). \end{aligned}$$

Next, assume that $\mathcal{H} \to \omega$ in ωX and $\mathcal{G} \to s$ in S. Then $\mathcal{H} \geq j^{\to}(\mathcal{F}) \cap \dot{\omega}$ for some convex filter $\mathcal{F}$ in X such that $\operatorname{adh} \mathcal{F} = \emptyset$. Since (X, λ) is c-adherence restrictive, $\operatorname{adh} \mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \mathcal{G}) = \emptyset$, and thus it follows that

$$\begin{aligned} \delta^{\to}(\mathcal{H} \times \mathcal{G}) &\geq \delta^{\to}((j^{\to}(\mathcal{F}) \cap \dot{\omega}) \times \mathcal{G}) \\ &= j^{\to}(\lambda^{\to}(\mathcal{F} \times \mathcal{G})) \cap \delta^{\to}(\dot{\omega} \times \mathcal{G}) \\ &\geq j^{\to}(\mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \mathcal{G})) \cap \dot{\omega} \to \omega. \end{aligned}$$

This proves that δ is a continuous action.

We now show that ωX is compact. Assume that $\mathcal{H}$ is an ultrafilter on ωX. Then either $\mathcal{H} = \dot{\omega}$ or $\mathcal{H} = j^{\to}(\mathcal{F})$ for some ultrafilter $\mathcal{F}$ on X. Suppose that the latter case holds. If $x \in \operatorname{adh} \mathrm{c}\, \mathcal{F}$, then there exists an ultrafilter $\mathcal{G} \geq \mathrm{c}\, \mathcal{F}$ such that $\mathcal{G} \to x$ and $\mathrm{c}\, \mathcal{G} \to x$ in X. Since $\mathrm{c}\, \mathcal{G} \geq \mathrm{c}\, \mathcal{F}$ and $\mathrm{c}\, \mathcal{F}$ is a maximal convex filter on X, $\mathrm{c}\, \mathcal{G} = \mathrm{c}\, \mathcal{F} \to x$. Hence $\mathcal{H} \geq j^{\to}(\mathcal{F}) \geq j^{\to}(\mathrm{c}\, \mathcal{F}) \to j(x)$ in ωX. If on the other hand $\operatorname{adh} \mathrm{c}\, \mathcal{F} = \emptyset$, then $\mathcal{H} \geq j^{\to}(\mathrm{c}\, \mathcal{F}) \cap \dot{\omega} \to \omega$. Hence ωX is compact.

Suppose that X is Hausdorff and $\mathcal{H} \to j(x), \omega$ in ωX. Then there exist convex filters $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F} \to x$ in X, $\operatorname{adh} \mathcal{G} = \emptyset$, $\mathcal{H} \geq j^{\to}(\mathcal{F})$ and $\mathcal{H} \geq j^{\to}(\mathcal{G}) \cap \dot{\omega}$. Since $\mathcal{H} \geq j^{\to}(\mathcal{F})$, it follows that $\mathcal{H} \geq j^{\to}(\mathcal{G})$. Thus $\mathcal{F} \vee \mathcal{G}$ exists and consequently $\mathcal{F} \vee \mathcal{G} \to x$ in X. This means $x \in \operatorname{adh} \mathcal{G}$, contradicting that $\operatorname{adh} \mathcal{G} = \emptyset$. Ergo, ωX is Hausdorff whenever X is Hausdorff. Clearly $j : X \to \omega X$ is a dense embedding and j is an increasing function. Furthermore, $j(x) \lesssim j(z)$ and $s \in S$ imply that $\delta(j(x), s) = j(\lambda(x, s)) \lesssim j(\lambda(z, s)) = \delta(j(z), s)$. This shows that $\delta(\cdot, s)$ is an increasing function on ωX. Finally, $(\delta \circ (j \times \mathrm{id}_S))(x, s) = \delta(j(x), s) = j(\lambda(x, s)) = (j \circ \lambda)(x, s)$, and thus $\delta \circ (j \times \mathrm{id}_S) = j \circ \lambda$. This shows that $j \times \mathrm{id}_S : X \times S \to \omega X \times S$ is a morphism in CA, and thus $(\omega X, \delta)$ is an S-compactification of (X, λ). □

Example 2.2. Suppose that $X = \mathbb{R}$ and $S = (0, \infty)$ are equipped with the usual order. Let the action λ be multiplication. Then S is a topological group, λ is a continuous action, $\lambda(\cdot, s)$ is increasing for each $s \in S$, and consequently (X, λ) is an S-space. Define $\mathcal{F}_1$ to be the convex filter on X whose base is $\{ (r, \infty) \mid r > 0 \}$, and let $\mathcal{F}_2 = -\mathcal{F}_1$. If $\mathcal{F}$ is any convex filter on X such that $\operatorname{adh} \mathcal{F} = \emptyset$, then either $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$. Observe that if $\mathcal{G} \to s$ in S, then $\lambda^{\to}(\mathcal{F}_i \times \mathcal{G}) \geq \mathcal{F}_i$, $i = 1, 2$. Hence $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \geq \mathcal{F}_1$ or $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \geq \mathcal{F}_2$, and thus $\operatorname{adh} \mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \mathcal{G}) = \emptyset$. This means (X, λ) is c-adherence restrictive, and thus by Theorem 2.1, $(\omega X, \delta)$ is a one-point S-compactification of (X, λ). Note that ωX agrees on ultrafilter convergence with the Alexandorff one-point compactification of X. However, $j^{\to}(F_i) \to \omega$ in ωX, $i = 1, 2$, but $j^{\to}(F_1) \cap j^{\to}(F_2) \nrightarrow \omega$ in ωX since $\mathrm{c}(\mathcal{F}_1 \cap \mathcal{F}_2) = \dot{\mathbb{R}}$. Hence ωX is not even a limit space.

Definition 2.3. Let GQ denote the full subcategory of CA whose objects (Y, T, μ) have the following two properties:

(i) T is a commutative semigroup.
(ii) $\mu(\cdot, t)$ is an into order isomorphism on Y for each $t \in T$.

Fix $(X, S, \lambda) \in |\mathsf{GQ}|$. Its "generalized quotient" space is defined below. These spaces were introduced in [4] and are used in the study of generalized functions [9].

Define $(x, s) \sim (y, t)$ in $X \times S$ by $\lambda(x, t) = \lambda(y, s)$. It can be shown that $\sim$ is an equivalence relation. Let $B(X, S) = (X \times S)/\sim$ denote the quotient set and let $\theta : X \times S \to B(X, S)$ be the quotient map defined by $\theta(x, s) = \langle (x, s) \rangle$. Define the convergence structure q_B on $B(X, S)$ as follows:

$\mathcal{H} \xrightarrow{q_B} \langle (x, s) \rangle$ if and only if for some $\mathcal{F} \to x_1$ in X and $\mathcal{G} \to s_1$ in S, we have that $\mathcal{H} \geq \mathrm{c}\, \theta^{\to}(\mathcal{F} \times \mathcal{G})$ and $(x_1, s_1) \sim (x, s)$.

The convergence structure q_B is the quotient structure in the category LCS determined by the map $\theta : X \times S \to B(X, S)$. Let $\leq$ be the order on $B(X, S)$ defined as follows:

$\langle (x, s) \rangle \leq \langle (y, t) \rangle$ if and only if $\lambda(x, t) \leq \lambda(y, s)$.

Since $(X, S, \lambda) \in |\mathsf{GQ}|$, $\lambda(\cdot, s)$ is an into order isomorphism for each fixed $s \in S$. Under this assumption on λ, it is shown in [5] that the relation $\leq$ on $B(X, S)$ is a partial order.

Lemma 2.4. *If $(X, \lambda) \in |\mathsf{GQ}|$, then $(B(X, S), \lambda_B)$ is an S-space, where $\lambda_B : B(X, S) \times S \to B(X, S)$ is defined by*

$$\lambda_B(\langle (x, s) \rangle, t) = \langle (\lambda(x, t), s) \rangle .$$

Proof. First, it is shown that λ_B is increasing in its first component. Suppose that $\langle (x_1, s_1) \rangle \leq \langle (x_2, s_2) \rangle$. Then $\lambda(x_1, s_2) \leq \lambda(x_2, s_1)$. Fix $t \in S$. It must be shown that $\lambda_B(\langle (x_1, s_1) \rangle, t) \leq \lambda_B(\langle (x_2, s_2) \rangle, t)$, or $\langle (\lambda(x_1, t), s_1) \rangle \leq \langle (\lambda(x_2, t), s_2) \rangle$. Indeed, $\lambda(\lambda(x_1, t), s_2) = \lambda(\lambda(x_1, s_2), t) \leq \lambda(\lambda(x_2, s_1), t) = \lambda(\lambda(x_2, t), s_1)$ since S is commutative. Hence $\lambda_B(\cdot, t)$ is increasing on $B(X, S)$.

It is shown in Theorem 2.4 [3] that λ_B is an action of S on $B(X,S)$. We now show that λ_B is continuous. Suppose that $\mathcal{H} \to \langle(x,s)\rangle$ in $B(X,S)$ and $\mathcal{K} \to t$ in S. Then there exist $\mathcal{F} \to x_1$ in X, $\mathcal{G} \to s_1$ in S, where $(x_1,s_1) \sim (x,s)$, and $\mathcal{H} \geq \mathrm{c}\,\theta^{\to}(\mathcal{F}\times\mathcal{G})$. Hence

$$\begin{aligned}\lambda_B^{\to}(\mathcal{H}\times\mathcal{K}) &\geq \lambda_B^{\to}(\mathrm{c}\,\lambda^{\to}(\mathcal{F}\times\mathcal{G})\times\mathcal{K})\\ &\geq \mathrm{c}\,\lambda_B^{\to}(\theta^{\to}(\mathcal{F}\times\mathcal{G})\times\mathcal{K})\\ &= \mathrm{c}\,\theta^{\to}(\lambda^{\to}(\mathcal{F}\times\mathcal{K})\times\mathcal{G})\\ &\to \theta(\lambda(x_1,t),s_1)\\ &= \langle(\lambda(x_1,t),s_1)\rangle\\ &= \lambda_B(\langle(x_1,s_1)\rangle,t)\\ &= \lambda_B(\langle(x,s)\rangle,t),\end{aligned}$$

and thus λ_B is a continuous action. Therefore $(B(X,S),\lambda_B)$ is an S-space. □

Corollary 2.5. *Suppose that $(X,\lambda) \in |\mathsf{GQ}|$ is c-adherence restrictive, and let $(\omega X,\delta)$ be the one-point compactification defined above. Then $(B(\omega X,S),\delta_B)$ is an S-space.*

Proof. Verification follows from Lemma 2.1 and the fact that $\delta(\cdot,s)$ is an into order isomorphism for each fixed $s \in S$. □

Let $(X,\lambda) \in |\mathsf{GQ}|$ be c-adherence restrictive. For sake of brevity, let $\leq$ denote the partial order on both $B(X,S)$ and $B(\omega X,S)$. Note that $\langle(\omega,t)\rangle \notin \langle(j(x),s)\rangle$ for each $x \in X$ and $s,t \in S$ since $\omega = \delta(\omega,s) \neq \delta(j(x),s) = j(\lambda(x,t))$. Further, $\langle(\omega,t)\rangle$ and $\langle(j(x),s)\rangle$ are not comparable.

Lemma 2.6. *Assume that the hypotheses of Corollary 2.5 are satisfied and define $h : B(X,S) \to B(\omega X,S)$ by $h(\langle(x,s)\rangle) = \langle(j(x),s)\rangle$. Then h is an increasing function and $h(\mathrm{c}(A)) = \mathrm{c}(h(A))$ for each subset A of $B(X,S)$.*

Proof. Suppose that $\langle(x_1,s_1)\rangle \leq \langle(x_2,s_2)\rangle$. Then $\lambda(x_1,s_2) \leq \lambda(x_2,s_1)$ and so $\delta(j(x_1),s_1) \lesssim \delta(j(x_2),s_2)$. This means $h(\langle(x_1,s_1)\rangle) \leq h(\langle(x_2,s_2)\rangle)$, which proves that h is increasing. Next, assume that $\langle(j(x),s)\rangle \in \mathrm{c}(h(A))$. Then there exist $\langle(x_1,s_1)\rangle, \langle(x_2,s_2)\rangle \in A$ such that

$$\langle(j(x_1),s_1)\rangle \leq \langle(j(x),s)\rangle \leq \langle(j(x_2),s_2)\rangle.$$

It follows that $\delta(j(x_1),s) = j(\lambda(x_1,s)) \lesssim \delta(j(x),s_1) = j(\lambda(x,s_1))$, and thus $\lambda(x_1,s) \leq \lambda(x,s_1)$, or equivalently $\langle(x_1,s_1)\rangle \leq \langle(x,s)\rangle$. Likewise, $\langle(x,s)\rangle \leq \langle(x_2,s_2)\rangle$, and so $\langle(x,s)\rangle \in \mathrm{c}(A)$. Since $\langle(\omega,s)\rangle \notin \mathrm{c}(h(A))$, it follows that $\mathrm{c}(h(A)) \subseteq h(\mathrm{c}(A))$. The reverse holds since h is increasing, and thus $h(\mathrm{c}(A)) = \mathrm{c}(h(A))$. □

Theorem 2.7. *Let $(X,\lambda) \in |\mathsf{GQ}|$ be c-adherence restrictive, with X not compact and S compact. Let $(\omega X, \delta)$ denote the one-point S-compactification of (X,λ) defined above. Then $(B(\omega X, S), \delta_B)$ is a one-point S-compactification of $(B(X,S), \lambda_B)$.*

Proof. It has been shown in Lemma 2.4 and Corollary 2.5 that $(B(X,S),\lambda_B)$ and $(B(\omega X,S),\delta_B)$ are S-spaces. One can readily show that the diagram below commutes, where $h(\langle(x,s)\rangle) = \langle(j(x),s)\rangle$ is an injection and φ is the quotient map in LCS:

$$\begin{array}{ccc} X \times S & \xrightarrow{\theta} & B(X,S) \\ \big\downarrow{\scriptstyle j\times \mathrm{id}_S} & & \big\downarrow{\scriptstyle h} \\ \omega X \times S & \xrightarrow{\varphi} & B(\omega X, S) \end{array}$$

Since θ is a quotient map in LCS and $h \circ \theta = \varphi \circ (j \times \mathrm{id}_S)$ is continuous, it follows that h is continuous. It is shown in Lemma 2.6 that h is an increasing function. We now show that h is an embedding. Suppose that $\mathcal{H}$ is a filter on $B(X,S)$ such that $h^{\to}(\mathcal{H}) \to h(\langle(x,s)\rangle) = \langle(j(x),s)\rangle$ in $B(\omega X,S)$. Then there exist $(x_1,s_1) \sim (x,s)$ and $\mathcal{F} \to x_1$ in X, $\mathcal{G} \to s_1$ in S such that

$$\begin{aligned} h^{\to}(\mathcal{H}) &\geq \mathrm{c}\,\varphi^{\to}(j^{\to}(\mathcal{F}) \times \mathcal{G}) \\ &= \mathrm{c}(\varphi \circ (j \times \mathrm{id}_S))^{\to}(\mathcal{F} \times \mathcal{G}) \\ &= \mathrm{c}(h \circ \theta)^{\to}(\mathcal{F} \times \mathcal{G}). \end{aligned}$$

Employing Lemma 2.6, $h^{\to}(\mathcal{H}) \geq h^{\to}(\mathrm{c}\,\theta^{\to}(\mathcal{F} \times \mathcal{G}))$, and since h is injective, $\mathcal{H} \geq \mathrm{c}\,\theta^{\to}(\mathcal{F} \times \mathcal{G}) \to \langle(x_1,s_1)\rangle = \langle(x,s)\rangle$ in $B(X,S)$. Hence h is an embedding. Letting cl denote the closure operator, observe that

$$\begin{aligned} \mathrm{cl}\, h(B(X,S)) &= \mathrm{cl}(h \circ \theta)(X \times S) \\ &= \mathrm{cl}((\varphi \circ (j \times \mathrm{id}_S))(X \times S)) \\ &\supseteq \varphi(\mathrm{cl}(j(X) \times S)) \\ &= \varphi(\omega X \times S) = B(\omega X, S), \end{aligned}$$

and thus h is a dense embedding.

Finally, $h \times \mathrm{id}_S : (B(X,S),\lambda_B) \to (B(\omega X,S),\delta_B)$ is a morphism in CA. Indeed,

$$\begin{aligned} (\delta_B \circ (h \times \mathrm{id}_S))(\langle(x,s)\rangle, t) &= \delta_B(\langle(j(x),s)\rangle, t) \\ &= \langle(j(\lambda(x,t),s))\rangle \\ &= h(\langle(\lambda(x,t),s)\rangle) \\ &= (h \circ \lambda_B)(\langle(x,s)\rangle, t), \end{aligned}$$

hence $\delta_B \circ (h \times \mathrm{id}_S) = h \circ \lambda_B$. Moreover, $B(\omega X, S)$ is compact since it is the continuous image of the compact set $\omega X \times S$. Therefore $B(\omega X, S)$ is a one-point S-compactification of $B(X, S)$. $\square$

3. More General S-Compactifications

A one-point S-compactification was constructed in Section 2. In this section, a more general S-compactification is considered. It is shown in [7] that each $X \in |\mathsf{LCS}|$ has a compactification in LCS. A modification of this compactification is needed to obtain an S-compactification in CA. Before we start, we will need some definitions.

Definition 3.1. Let $(X, \leq, q) \in |\mathsf{LCS}|$, where (X, q) is noncompact. Define the following notation:

(i) X' denotes the set of all maximal convex filters on X which fail to q-converge.
(ii) $X^* = X \cup \{\lceil \mathcal{F} \rfloor : \mathcal{F} \in X'\}$; here we use the notation $\lceil \mathcal{F} \rfloor$ to distinguish between $\mathcal{F}$ as a filter and $\mathcal{F}$ as an element of X^*.
(iii) $j : X \to X^*$ is the natural injection $j(x) = x$.
(iv) $A^* = j(A) \cup \{ \lceil \mathcal{F} \rfloor \mid A \in \mathcal{F} \}$, $A \subseteq X$.
(v) $\mathcal{F}^*$ denotes the filter on X^* whose base is $\{ F^* \mid F \in \mathcal{F} \}$.

It is shown in [7] that $(X^*, \leq^*)$ is a poset, where $\leq^*$ is defined as follows:

$$\begin{aligned} j(x) \leq^* j(y) &\text{ if and only if } x \leq y \text{ (if and only if } \dot{x} \lesssim \dot{y}),\\ j(x) \leq^* \lceil \mathcal{F} \rfloor &\text{ if and only if } \dot{x} \lesssim \mathcal{F},\\ \lceil \mathcal{F} \rfloor \leq^* \lceil \mathcal{G} \rfloor &\text{ if and only if } \mathcal{F} \lesssim \mathcal{G}. \end{aligned}$$

Supppose (X, λ) is a c-adherence restrictive noncompact S-space. Observe that if $\mathcal{F} \in X'$, then $\operatorname{adh} \mathcal{F} = \emptyset$ and thus $\operatorname{adh} \mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \dot{s}) = \emptyset$ for each $s \in S$. Since $\lambda(\cdot, s)$ is increasing on X for each $s \in S$, it follows from Lemma 1.3 (iii–iv) that $\mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \dot{s}) \in X'$ and thus $\lceil \mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \dot{s}) \rfloor \in X^*$.

Define $\lambda^* : X^* \times S \to X^*$ so that:

$$\begin{aligned} \lambda^*(j(x), s) &= j(\lambda(x, s)),\\ \lambda^*(\lceil \mathcal{F} \rfloor, s) &= \lceil \mathrm{c}\, \lambda^{\to}(\mathcal{F} \times \dot{s}) \rfloor. \end{aligned}$$

Define the convergence structure q^* on X^* as follows:

$\mathcal{H} \xrightarrow{q^*} j(x)$ if and only if $\mathcal{H} \geq \mathcal{F}^*$, for some $\mathcal{F} \to x$ in X.

$\mathcal{H} \xrightarrow{q^*} \lceil \mathcal{F} \rfloor$ if and only if $\mathcal{H} \geq (\mathrm{c}\, \lambda^{\to}(\mathcal{K} \times \mathcal{G}))^*$ for some $\mathrm{c}\, \lambda^{\to}(\mathcal{K} \times \dot{s}) = \mathcal{F}$, $\mathcal{K} \in X'$, $\mathcal{G} \to s$ in S.

It is shown in Proposition 2.3 [7] that if $A \subseteq X$ is convex, then A^* is a convex subset of X^*. This implies that $\mathcal{F}^*$ is a convex filter on X^* whenever $\mathcal{F}$ is a convex filter on X. Hence $(X^*, \leq^*, q^*) \in |\mathsf{LCS}|$.

It should be mentioned that the convergence structure q^* is coarser than the convergence structure on X^* defined in [7]. This is needed to ensure that the action λ^* is continuous as shown below.

Theorem 3.2. *Suppose that* (X, λ) *is a c-adherence restrictive noncompact* S*-space. Then* (X^*, λ^*) *defined above is an* S*-compactification of* (X, λ).

Proof. As mentioned above, $(X^*, \leq^*, q^*) \in |\mathsf{LCS}|$. First, it is shown that λ^* is an action of S on X^*. Indeed, $\lambda^*(j(x), e) = j(\lambda(x, e)) = j(x)$ and $\lambda^*(\lceil \mathcal{F} \rfloor, e) = \lceil \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{e}) \rfloor = \lceil \mathrm{c}\, \mathcal{F} \rfloor = \lceil \mathcal{F} \rfloor$. Also,

$$\begin{aligned} \lambda^*(\lambda^*(j(x), s), t) &= \lambda^*(j(\lambda(x, s)), t) \\ &= j(\lambda(\lambda(x, s), t)) \\ &= j(\lambda(x, s \cdot t)) \\ &= \lambda^*(j(x), s \cdot t), \end{aligned}$$

and

$$\begin{aligned} \lambda^*(\lambda^*(\lceil \mathcal{F} \rfloor, s), t) &= \lambda^*(\lceil \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \rfloor, t) \\ &= \left\lceil \mathrm{c}\, \lambda^{\rightarrow}(\mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \times \dot{t}) \right\rfloor \\ &= \left\lceil \mathrm{c}\, \lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \times \dot{t}) \right\rfloor \end{aligned}$$

by Lemma 1.9 (ii). Hence

$$\lambda^*(\lambda^*(\lceil \mathcal{F} \rfloor, s), t) = \left\lceil \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{\overbrace{s \cdot t}}) \right\rfloor = \lambda^*(\lceil \mathcal{F} \rfloor, s \cdot t),$$

thus proving that λ^* is an action. Next, it is shown that $\lambda^*(\cdot, s)$ is increasing on X^*. Assume that $j(x) \leq^* j(z)$. Then $\lambda^*(j(x), s) = j(\lambda(x, s)) \leq^* j(\lambda(z, s)) = \lambda^*(j(z), s)$ since $\lambda(\cdot, s)$ is increasing on X. Suppose that $j(x) \leq^* \lceil \mathcal{F} \rfloor$. Then $\dot{x} \lesssim \mathcal{F}$ and thus by Lemma 1.9 (v),

$$\mathrm{c}\, \lambda^{\rightarrow}(\dot{x} \times \dot{s}) = \dot{\overbrace{\lambda(x, s)}} \lesssim \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s}).$$

This shows that $\lambda^*(j(x), s) \leq^* \lambda^*(\lceil \mathcal{F} \rfloor, s)$. Furthermore, if $\lceil \mathcal{F} \rfloor \leq^* \lceil \mathcal{G} \rfloor$, then $\mathcal{F} \lesssim \mathcal{G}$ and thus $\mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \lesssim \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$. Therefore, $\lambda^*(\cdot, s)$ is increasing on X^* for each fixed $s \in S$.

Let us show that λ^* is continuous. First, it is shown that if $A \subseteq X$ and $B \subseteq S$, then $\lambda^*(A^* \times B) \subseteq (\mathrm{c}(\lambda(A \times B)))^*$. Assume that $\lceil \mathcal{F} \rfloor \in A^*$ and $s \in B$. Then $\operatorname{adh} \mathcal{F} = \emptyset$, and since (X, λ) is c-adherence restrictive, $\mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \in X'$. Since $A \in \mathcal{F}$ and $s \in B$, $\mathrm{c}(\lambda(A \times B)) \in \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s})$ and thus $\lambda^*(\lceil \mathcal{F} \rfloor, s) = \lceil \mathrm{c}\, \lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \rfloor \in (\mathrm{c}(\lambda(A \times B)))^*$. It follows that $\lambda^*(A^* \times B) \subseteq (\mathrm{c}(\lambda(A \times B)))^*$. Suppose that $\mathcal{H} \to j(x)$ in X^* and $\mathcal{G} \to s$ in S. Then $\mathcal{H} \geq \mathcal{F}^*$ for some $\mathcal{F} \to x$ in X, and so

$$\begin{aligned}\lambda^{*\to}(\mathcal{H}\times\mathcal{G}) &\geq \lambda^{*\to}(\mathcal{F}^*\times\mathcal{G})\\ &\geq (\mathrm{c}\,\lambda^{\to}(\mathcal{F}\times\mathcal{G}))^*\\ &\to j(\lambda(x,s))\\ &= \lambda^*(j(x),s).\end{aligned}$$

Next, assume that $\mathcal{H}\to\lceil\mathcal{F}\rfloor$ in X^* and $\mathcal{G}_1\to s_1$ in S. Then $\mathcal{H}\geq(\mathrm{c}\,\lambda^{\to}(\mathcal{K}\times\mathcal{G}_2))^*$ for some $\mathrm{c}\,\lambda^{\to}(\mathcal{K}\times\dot{s_2})=\mathcal{F}$, where $\mathcal{K}\in X'$ and $\mathcal{G}_2\to s_2$ in S. Hence

$$\begin{aligned}\lambda^{*\to}(\mathcal{H}\times\mathcal{G}_1) &\geq \lambda^{*\to}((\mathrm{c}\,\lambda^{\to}(\mathcal{K}\times\mathcal{G}_2))^*\times\mathcal{G}_1)\\ &\geq (\mathrm{c}\,\lambda^{\to}(\mathrm{c}\,\lambda^{\to}(\mathcal{K}\times\mathcal{G}_2)\times\mathcal{G}_1))^*\\ &= (\mathrm{c}\,\lambda^{\to}(\lambda^{\to}(\mathcal{K}\times\mathcal{G}_2)\times\mathcal{G}_1))^*\end{aligned}$$

by Lemma 3.2 (ii). It follows that

$$\begin{aligned}\lambda^{*\to}(\mathcal{H}\times\mathcal{G}_1) &\geq (\mathrm{c}\,\lambda^{\to}(\mathcal{K}\times\mathcal{G}_2\cdot\mathcal{G}_1))^*\\ &\to \left\lceil \mathrm{c}\,\lambda^{\to}(\mathcal{K}\times(\dot{s_2\cdot s_1}))\right\rfloor\\ &= \lceil \mathrm{c}\,\lambda^{\to}(\lambda^{\to}(\mathcal{K}\times\dot{s_2})\times\dot{s_1})\rfloor\\ &= \lceil \mathrm{c}\,\lambda^{\to}(\mathcal{F}\times\dot{s_1})\rfloor\\ &= \lambda^*(\lceil\mathcal{F}\rfloor,s_1)\end{aligned}$$

and thus λ^* is a continuous action.

It easily follows that $j:X\to X^*$ is a dense embedding and $j\circ\lambda=\lambda^*\circ(j\times\mathrm{id}_S)$. This proves that $j\times\mathrm{id}_S:X\times S\to X^*\times S$ is a morphism in CA. Since q^* is coarser than the structure on X^* given in [7], it follows that (X^*,q^*) is compact. Hence (X^*,λ^*) is an S-compactification of (X,λ). $\square$

Proposition 3.3. *Suppose that (X,λ) is an S-space that is c-adherence restrictive where X is Hausdorff and noncompact. The S-compactification (X^*,λ^*) defined above is Hausdorff if and only if for each $\mathcal{G}_i\to s_i$ in S and $\mathcal{F}_i\in X'$, $\mathrm{c}\,\lambda^{\to}(\mathcal{F}_1\times\mathcal{G}_1)\vee\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\mathcal{G}_2)$ fails to exist whenever $\mathrm{c}\,\lambda^{\to}(\mathcal{F}_1\times\dot{s_1})\vee\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\dot{s_2})$ fails to exist.*

Proof. Observe that if $A\subseteq X$ and $B\subseteq S$, then $A^*\cap B^*\neq\emptyset$ if and only if $A\cap B\neq\emptyset$. Assume that $\mathcal{H}\to j(x_1),j(x_2)$ in X^*. Then $\mathcal{H}\geq\mathcal{F}_i^*$ for some $\mathcal{F}_i\to x_i$ in X, $i=1,2$. Therefore, $\mathcal{F}_1\vee\mathcal{F}_2$ exists and converges to x_1 and x_2. Since X is Hausdorff, $x_1=x_2$. Next, suppose that $\mathcal{H}\to j(x_1),\lceil\mathcal{F}\rfloor$ in X^*. Then $\mathcal{H}\geq\mathcal{F}_1^*$ and $\mathcal{H}\geq(\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\mathcal{G}))^*$ for some $\mathcal{F}_1\to x_1$ in X, $\mathcal{F}_2\in X'$, $\mathcal{G}\to s$ in S, and $\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\dot{s})=\mathcal{F}$. It follows that $\mathcal{F}_1\vee\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\mathcal{G})$ exists and thus $x\in\operatorname{adh}\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\mathcal{G})$. However, this is contrary to the fact that $\operatorname{adh}\mathcal{F}_2=\emptyset$ implies that $\operatorname{adh}\mathrm{c}\,\lambda^{\to}(\mathcal{F}_2\times\mathcal{G})=\emptyset$ since (X,λ) is c-adherence restrictive. Finally, assume that $\mathcal{H}\to\lceil\mathcal{F}_i\rfloor$ in X^*,

$i = 1, 2$. Then $\mathcal{H} \geq (\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_i \times \mathcal{G}_i))^*$ for some $\mathcal{K}_i \in X'$, $\mathcal{G}_i \to s_i$ in S, and $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_i \times \dot{s_i}) = \mathcal{F}_i$, $i = 1, 2$. Hence $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_1 \times \mathcal{G}_1) \vee \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_2 \times \mathcal{G}_2)$ exists and the hypothesis implies that $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_1 \times \dot{s_1}) \vee \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_2 \times \dot{s_2})$ exists. It follows that $\mathcal{F}_1 = \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_1 \times \dot{s_1}) = \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{K}_2 \times \dot{s_2}) = \mathcal{F}_2$ since each is a maximal convex filter. This proves that X^* is Hausdorff. Clearly the condition is necessary for X^* to be Hausdorff. □

Lemma 3.4. *Let (X, λ) be a c-adherence restrictive noncompact S-space. Suppose that (X^*, λ^*) is the S-compactification of (X, λ) as defined above. Assume that $\mathcal{F}, \mathcal{G}$ are convex filters on X and $s \in S$. Then:*

(i) $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \lesssim \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$ *if and only if* $\mathcal{F} \lesssim \mathcal{G}$ *provided that* $\lambda(\cdot, s)$ *is an into order isomorphism on* X.

(ii) $\lambda^*(\cdot, s)$ *is an into order isomorphism on* X^* *if and only if* $\lambda(\cdot, s)$ *is an into order isomorphism on* X.

Proof. (i) According to Lemma 1.9 (v), $\mathcal{F} \lesssim \mathcal{G}$ implies that $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \lesssim \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$. Conversely, assume that $\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \lesssim \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$. Then $\mathrm{i}\,\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \subseteq \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$, and by Lemma 1.9 (i), $\mathrm{i}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) = \mathrm{i}\,\mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \subseteq \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{G} \times \dot{s}) \subseteq \lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$. Fix $F \in \mathcal{F}$. Then the above implies that there exists $G \in \mathcal{G}$ such that $\lambda(G \times \{ x \}) \subseteq \mathrm{i}(\lambda(F \times \{ s \}))$. It is shown that $G \subseteq \mathrm{i}(F)$. Indeed, if $z \in G$, then $\lambda(z, s) \in \mathrm{i}(\lambda(F \times \{ x \}))$ and thus $\lambda(x, s) \leq \lambda(z, s)$ for some $x \in F$. Since $\lambda(\cdot, s)$ is an into order isomorphism, $x \leq z$ and thus $z \in \mathrm{i}(F)$. Therefore $G \subseteq \mathrm{i}(F)$ and $\mathrm{i}\,\mathcal{F} \subseteq \mathcal{G}$. The dual argument shows that $\mathrm{d}\,\mathcal{G} \subseteq \mathcal{F}$ and hence $\mathcal{F} \lesssim \mathcal{G}$.

(ii) Assume that $\lambda(\cdot, s)$ is an into order isomorphism on X. Then, as shown in the proof of Theorem 3.2, $\lambda^*(\cdot, s)$ is increasing on X^*. Next, suppose that $\lambda^*(a, s) \leq \lambda^*(b, s)$. It must be shown that $a \leq^* b$. If $a = j(x)$ and $b = \lceil \mathcal{F} \rfloor$, then $\lambda^*(a, s) = j(\lambda(x, s)) \leq^* \lambda^*(\lceil \mathcal{F} \rfloor, s) = \lceil \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \rfloor$ implies that $\dot{\lambda(x, s)} = \mathrm{c}\,\lambda^{\rightarrow}(\dot{x} \times \dot{s}) \lesssim \mathrm{c}\,\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})$. It follows from (i) above that $\dot{x} \lesssim \mathcal{F}$ and thus $j(x) \leq^* \lceil \mathcal{F} \rfloor$. The other cases follow similarly. □

Assume that $(X, \lambda) \in |\mathsf{GQ}|$ is c-adherence restrictive and X is not compact. Then for each $s \in S$, $\lambda(\cdot, s)$ is an into order isomorphism on X, and by Lemma 1.3 (ii), $\lambda^*(\cdot, s)$ is an into order isomorphism on X^*. This means that the generalized quotient spaces $B(X, S)$ and $B(X^*, S)$ are well-defined. Since $(X^*, \lambda^*) \in |\mathsf{GQ}|$, it follows from Lemma 2.4 that $(B(X, S), \lambda_B)$ and $(B(X^*, S), \lambda_B^*)$ are S-spaces. The next result is just like Theorem 2.7 with (X^*, λ^*) in place of $(\omega X, \delta)$.

Theorem 3.5. *Suppose $(X, \lambda) \in |\mathsf{GQ}|$ is c-adherence restrictive, X is not compact, and S is compact. Let (X^*, λ^*) denote the S-compactification of (X, λ) defined above. Then $(B(X^*, S), \lambda_B^*)$ is an S-compactification of $(B(X, S), \lambda_B)$.*

Proof. Define $h : B(X,S) \to B(X^*,S)$ by $h(\langle (x,s)\rangle) = \langle (j(x),s)\rangle$ and consider the diagram below:

$$\begin{array}{ccc} X \times S & \xrightarrow{\ \theta\ } & B(X,S) \\ \Big\downarrow{\scriptstyle j\times \mathrm{id}_S} & & \Big\downarrow{\scriptstyle h} \\ X^* \times S & \xrightarrow{\ \theta^*\ } & B(X^*,S) \end{array}$$

Observe that $h \circ \theta = \theta^* \circ (j \times \mathrm{id}_S)$, and since θ is a quotient map in LCS, it follows that h is continuous. Moreover, h is an into order isomorphism. Indeed, $h(\langle (x_1,s_1)\rangle) < h(\langle (x_2,s_2)\rangle)$ in $B(X^*,S)$ if and only if $\langle (j(x_1),s_1)\rangle < \langle (j(x_2),s_2)\rangle$ if and only if $\lambda^*(j(x_1),s_2) \leq^* \lambda^*(j(x_2),s_1)$ if and only if $j(\lambda(x_1,s_2)) \leq^* j(\lambda(x_2,s_1))$ if and only if $\langle (x_1,s_1)\rangle < \langle (x_2,s_2)\rangle$ in $B(X,S)$. It follows that if $A \subseteq B(X,S)$, then $\mathrm{c}(h(A)) \cap h(B(X,S)) = h(\mathrm{c}(A))$.

It is shown that h is an embedding. Assume that $h^{\to}(\mathcal{H}) \to h(\langle (x,s)\rangle) = \langle (j(x),s)\rangle$ in $B(X^*,S)$. Then there exist $\mathcal{F} \to x_1$, $\mathcal{G} \to s_1$ such that $h^{\to}(\mathcal{H}) \geq \mathrm{c}\,\theta^{*\to}(j^{\to}(\mathcal{F}) \times \mathcal{G}) = \mathrm{c}(\theta^* \circ (j \times \mathrm{id}_S))^{\to}(\mathcal{F} \times \mathcal{G}) = \mathrm{c}(h \circ \theta)^{\to}(\mathcal{F} \times \mathcal{G})$, where $(x_1,s_1) \sim (x,s)$. As shown above, if $A \in \mathcal{H}$, then $\mathrm{c}(h(A)) \cap h(B(X,S)) = h(\mathrm{c}(A))$, and thus it follows that the trace of $\mathrm{c}\,h^{\to}(\theta^{\to}(\mathcal{F} \times \mathcal{G}))$ on $h(B(X,S))$ coincides with the trace of $h^{\to}(\mathrm{c}\,\theta^{\to}(\mathcal{F} \times \mathcal{G}))$ on $h(B(X,S))$. Hence $h^{\to}(\mathcal{H}) \geq \mathrm{c}\,h^{\to}(\theta^{\to}(\mathcal{F} \times \mathcal{G}))$ implies that $h^{\to}(\mathcal{H}) \geq h^{\to}(\mathrm{c}\,\theta^{\to}(\mathcal{F} \times \mathcal{G}))$. Since h is an injection, $\mathcal{H} \geq \mathrm{c}\,\theta^{\to}(\mathcal{F} \times \mathcal{G}) \to \langle (x_1,s_1)\rangle = \langle (x,s)\rangle$ in $B(X,S)$, and hence h is an embedding.

Note that

$$\begin{aligned} \mathrm{cl}\, h(B(X,S)) &= \mathrm{cl}\,(h \circ \theta)(X \times S) \\ &= \mathrm{cl}\,(\theta^* \circ (j \times \mathrm{id}_S))(X \times S) \\ &= \mathrm{cl}\,\theta^*(j(X) \times S) \\ &\supseteq \theta^*(\mathrm{cl}\, j(X) \times S) \\ &= \theta^*(X^* \times S) = B(X^*,S), \end{aligned}$$

proving that h is a dense embedding. And since S is compact, it follows that $B(X^*,S)$ is a compactification of $B(X,S)$. Furthermore,

$$\begin{aligned} (\lambda_B^* \circ (h \times \mathrm{id}_S))(\langle (x,s)\rangle, t) &= \lambda_B^*(\langle (j(x),s)\rangle, t) \\ &= \langle (\lambda^*(j(x),t),s)\rangle \\ &= \langle (j(\lambda(x,t)),s)\rangle \\ &= h(\lambda_B(\langle (x,s)\rangle, t), \end{aligned}$$

hence $\lambda_B^* \circ (h \times \mathrm{id}_S) = h \circ \lambda_B$. Ergo, $(B(X^*,S), \lambda_B^*)$ is an S-compactification of $(B(X,S), \lambda_B)$. □

4. Conclusion

The notion of a convergence monoid S acting continuously on a locally convex convergence space X is studied. Sufficient conditions are given to ensure that the action can be continuously extended to a one-point locally convex compactification of X. Given any locally convex convergence space X (no action), a locally convex compactification X^* is constructed in [7]. The authors show how to modify the convergence structure on X^* in order to continuously extend the action on X onto X^* . Further, the authors define and study a "locally convex generalized quotient space." An interesting open problem is to determine which convergence space properties remain invariant when passing to the locally convex generalized quotient space. The authors were unable to determine if adherence restrictive implies c-adherence restrictive. Perhaps the following special case is worthy of mention. If (X, λ) is an S-space, where S is a discrete convergence group, then for each fixed s in S, $\lambda(\cdot, s) : X \to X$ is an order isomorphism. It follows that if $\mathcal{F}$ is a convex filter on X, then $c\,\lambda^{\to}(\mathcal{F} \times \dot{s}) = \lambda(\mathcal{F} \times \dot{s})$. Since S is a convergence group, Theorem 2.4 in [8] implies that (X, λ) is adherence restrictive and, under the assumption that S is discrete, (X, λ) is c-adherence restrictive.

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