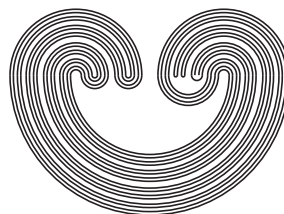


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CHARACTERIZING $C(X)$ AMONG INTERMEDIATE C -RINGS ON X

JOSHUA SACK AND SALEEM WATSON

ABSTRACT. Let X be a completely regular topological space. An intermediate ring is a ring $A(X)$ of continuous functions satisfying $C^*(X) \subseteq A(X) \subseteq C(X)$. We give a characterization of $C(X)$ in terms of extensions of functions in $A(X)$ to real-compactifications of X . We also give equivalences of properties involving the closure in the real-compactifications of X of zero-sets in X ; we use these equivalences to answer an open question about the correspondences of ideals in intermediate rings and z -filters on X .

1. INTRODUCTION

The ring $C(X)$ of all continuous real-valued functions on the completely regular Hausdorff space X has been characterized among its subrings in various ways. Perhaps the best known characterization is the Stone-Weierstrass Theorem: If X is compact, then any subring of $C(X)$ which contains the constants, separates points, and is uniformly closed is $C(X)$ itself [16, p. 291]. Other characterizations can be found in the book *Characterizations of $C(X)$ among its Subalgebras* by R. B. Burkel [2]. In this article we consider intermediate subrings of $C(X)$, that is, subrings $A(X)$ satisfying

$$C^*(X) \subseteq A(X) \subseteq C(X)$$

where $C^*(X)$ is the subring of $C(X)$ consisting of the bounded functions. In [4], to each $A(X)$ there is a realcompact space $v_A X$ to which each $f \in A(X)$ can be continuously extended to a function f^{v_A} defined on

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all of $v_A X$, and f^v can be further extended to all of βX via the Stone extension of f , denoted f^* . A *C-ring* is a ring $A(X)$ that is isomorphic to $C(Y)$ for some completely regular space Y . In this article we give a characterization of $C(X)$ among intermediate C -rings $A(X)$ in terms of the extension of functions to $v_A X$ (Theorem 3.7), and we use this characterization to give a more direct proof of a result by Swardson [14, Corollary 4.3]. We also characterize intermediate C -rings that have the property that the closure of a zero-set in $v_A(X)$ is a zero-set (Theorem 4.2). We furthermore characterize $C^*(X)$ among intermediate rings $A(X)$ in terms of the relationship between the extension f^{v_A} and the Stone extension f^* of f for each f (Theorem 3.10). In the setting of this paper, we give a more direct proof of a result by Rudd (the equivalence of the first two parts of [11, Theorem 1.2]) concerning the relationship between f and f^* (Theorem 3.11 in this paper). Furthermore, we give a class of intermediate rings for which the natural correspondence between ideals and z -filters maps maximal ideals to z -ultrafilters (Theorem 5.2). This answers [13, Question 4] in the negative: it is not the case that $A(X) = C(X)$ if and only if every maximal ideal in $A(X)$ is mapped by this correspondence to a z -ultrafilter on X .

2. PRELIMINARIES

We follow the definitions and notation of [12, 13]. Let X be a completely regular space. A zero-set in X is a set of the form $\mathbf{Z}(f) = \{x \in X : f(x) = 0\}$ for some f in $A(X)$; $\mathbf{Z}[X]$ denotes the family of all zero-sets of X . It is well known (see [7]) that \mathbf{Z} acts as a correspondence between proper ideals I in $C(X)$ and z -filters $\mathbf{Z}[I] = \{\mathbf{Z}(f) : f \in I\}$ on X . There is a different but well known mapping \mathbf{E} mapping ideals in $C^*(X)$ to z -filters on X , given by $\mathbf{E}[I] = \bigcup\{\mathbf{E}_\epsilon(f) : f \in I, \epsilon > 0\}$, where $\mathbf{E}_\epsilon(f) = \{x : |f(x)| \leq \epsilon\}$ (see [7]). However, \mathbf{E} and \mathbf{Z} fail to be correspondences for intermediate rings $A(X)$ strictly between $C^*(X)$ and $C(X)$ and z -filters on X [13, Propositions 3.2 and 4.2]. Thus extensions \mathcal{Z}_A of \mathbf{E} and \mathfrak{Z}_A of \mathbf{Z} to intermediate rings were defined as follows. If $f \in C(X)$ and $E \subseteq X$, we say that f is *E-regular* in $A(X)$ if there exists $g \in A(X)$ such that $f(x) \cdot g(x) = 1$ for all $x \in E$. We may simply write that f is *E-regular* if $A(X)$ is understood from context. For $f \in A(X)$ we have

$$\begin{aligned}\mathcal{Z}_A(f) &= \{E \in \mathbf{Z}[X] : f \text{ is } E^c\text{-regular}\} \\ \mathfrak{Z}_A(f) &= \{E \in \mathbf{Z}[X] : f \text{ is } H\text{-regular for all zero-sets } H \subseteq E^c\}.\end{aligned}$$

Note the difference between $\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A(f)$. For example, if $A(X) = C(X)$ then $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$, the z -filter consisting of all zero-sets containing $\mathbf{Z}(f)$, whereas $\mathcal{Z}_A(f)$ consists of all zero-set neighborhoods of $\mathbf{Z}(f)$.

For each non-invertible f in $A(X)$, both $\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A(f)$ are z -filters on X and $\mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f)$.

For an ideal I in $A(X)$ we define $\mathcal{Z}_A[I] = \bigcup\{\mathcal{Z}_A(f) : f \in I\}$ and $\mathfrak{Z}_A[I] = \bigcup\{\mathfrak{Z}_A(f) : f \in I\}$. Again $\mathcal{Z}_A[I]$ and $\mathfrak{Z}_A[I]$ are z -filters on X . The maps \mathcal{Z}_A and \mathfrak{Z}_A between ideals of $A(X)$ and z -filters on X do extend the correspondences \mathbf{E} and \mathbf{Z} respectively to $A(X)$. In particular, for $A(X) = C^*(X)$ we have $\mathcal{Z}_A[I] = \mathbf{E}[I]$ (from [8, Corollary 1.3]), and for $A(X) = C(X)$ we have $\mathfrak{Z}_A[I] = \mathbf{Z}[I]$ (from [8, Corollary 2.4]).

The *hull* of a z -filter \mathcal{F} is the set

$$h\mathcal{F} = \{\mathcal{U} \mid \mathcal{U} \text{ is a } z\text{-ultrafilter and } \mathcal{F} \subseteq \mathcal{U}\}.$$

Also, the *kernel* of a set \mathfrak{U} of z -ultrafilters is

$$k\mathfrak{U} = \bigcap_{\mathcal{U} \in \mathfrak{U}} \mathcal{U}$$

(as the intersection of z -filters is a z -filter, $k\mathfrak{U}$ is a z -filter). From [8, Theorem 3.1] we have the following relationship between \mathcal{Z}_A and \mathfrak{Z}_A :

$$(2.1) \quad \mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$$

for each $f \in A(X)$. The following is from [9].

Lemma 2.1. *Let $A(X)$ be an intermediate ring, let $f \in A(X)$, and let \mathcal{F} be a z -filter on X . Then $\mathcal{Z}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$.*

For a z -filter \mathcal{F} on X we define the inverse maps $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$ and $\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathcal{F}\}$. It is shown in [3] that for any intermediate ring $A(X)$, $\mathcal{Z}_A^{\leftarrow}$ maps z -filters on X to ideals in $A(X)$, and it is shown in [12, Theorem 14] that the corresponding result for $\mathfrak{Z}_A^{\leftarrow}$ holds for intermediate C -rings $A(X)$. From the definition of $\mathcal{Z}_A^{\leftarrow}$ and Lemma 2.1 we obtain the following.

Lemma 2.2. *Let $A(X)$ be an intermediate ring and let \mathcal{F} be a z -filter on X . Then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \lim_{\mathcal{F}} fh = 0 \text{ for all } h \in A(X)\}$.*

In general, if M is a maximal ideal then $\mathfrak{Z}_A[M]$ and $\mathcal{Z}_A[M]$ need not be z -ultrafilters, but each is contained in a unique z -ultrafilter (see [3] and [8] respectively). However, if \mathcal{U} is a z -ultrafilter on X , then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ and $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ are maximal ideals in $A(X)$, as the following lemma from [8] describes.

Lemma 2.3. *Let $A(X)$ be an intermediate ring, and let \mathcal{U} be a z -ultrafilter on X . Then the following hold:*

- (a) Each of $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ and $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ are maximal ideals in $A(X)$ and $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] = \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$.
- (b) The map $\mathcal{U} \rightarrow \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ defines a one-one correspondence between z -ultrafilters on X and maximal ideals in $A(X)$.

Recall that the Stone-Ćech compactification βX of X can be realized as the set of z -ultrafilters on X topologized with the hull-kernel topology, and X is naturally embedded in βX by the map $p \mapsto \mathcal{U}_p$, where $\mathcal{U}_p = \langle \{p\} \rangle$ is the principal z -ultrafilter consisting of all zero-sets containing $\{p\}$. So a point p in βX corresponds to the z -ultrafilter \mathcal{U}_p on X . Throughout this paper, we identify each $p \in \beta X$ with \mathcal{U}_p . Then the closure of any set $U \subseteq \beta X$ is $hk(U)$, the hull of the kernel of U . By the preceding result $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]$ is a maximal ideal which we denote by M_A^p . Thus the correspondence $p \rightarrow M_A^p$ is one-one between βX and the maximal ideals of $A(X)$.

A z -ultrafilter \mathcal{U} on X is called A -stable if for every f in $A(X)$ there is a member of \mathcal{U} on which f is bounded. For each $A(X)$ we define a subspace $v_A X$ of βX , called the A -compactification of X , by $v_A X = \{p \in \beta X : \mathcal{U}_p \text{ is } A\text{-stable}\}$ [4]. If $A(X) = C^*(X)$ then $v_A X = \beta X$ and if $A(X) = C(X)$ then $v_A X = vX$, the Hewitt realcompactification of X . We have

$$X \subseteq vX \subseteq v_A X \subseteq \beta X.$$

It is shown in [4] that the space $v_A X$ consists of the points of βX to which every f in $A(X)$ can be continuously extended. We denote the extension of f to $v_A X$ by f^{v_A} . It is also shown in [4] that for $f \in A(X)$ and $p \in v_A X$ we have

$$(2.2) \quad f^{v_A}(p) = \lim_{\mathcal{U}_p} f.$$

Theorem 2.4 (from [4]). *An intermediate ring $A(X)$ is a C -ring if and only if $A(X)$ is isomorphic to $C(v_A X)$, via the map $f \mapsto f^{v_A}$.*

Different intermediate rings may induce the same realcompact extension of X . For additional information on the realcompactifications of X , see [1, 4, 6]

3. THE ZERO-SETS OF f , f^{v_A} , AND f^*

If $A(X)$ is an intermediate C -ring, $f \in A(X)$, and f^* is the Stone extension of f (to be defined below), we show the following containments:

$$cl_{v_A X} \mathbf{Z}(f) \subseteq \mathbf{Z}(f^{v_A}) \subseteq \mathbf{Z}(f^*).$$

The first containment, that $cl_{v_A X} \mathbf{Z}(f) \subseteq \mathbf{Z}(f^{v_A})$, is Proposition 3.2, and this containment can be strict by Example 3.3. Furthermore, equality of this containment for every $f \in A(X)$ characterizes $C(X)$ among intermediate C -rings (Theorem 3.7). The second containment, that $\mathbf{Z}(f^{v_A}) \subseteq \mathbf{Z}(f^*)$, is Proposition 3.9, and we show that equality for every $f \in A(X)$ characterizes $C^*(X)$ among intermediate rings (Theorem 3.10).

We make use of the following lemma.

Lemma 3.1. *Let $A(X)$ be an intermediate ring and $f \in A(X)$. Then $\mathbf{Z}(f) = \bigcap \{E : E \in \mathcal{Z}_A(f)\} = \bigcap \{E : E \in \mathfrak{Z}_A(f)\}$.*

Proof. It is proved in [10, Proposition 2.2] that $\mathbf{Z}(f) = \bigcap \{E : E \in \mathcal{Z}_A(f)\}$, so it is sufficient to show that $\bigcap \mathfrak{Z}_A(f) = \bigcap \mathcal{Z}_A(f)$. Since by (2.1), $\mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f)$, it follows that $\bigcap \mathfrak{Z}_A(f) \subseteq \bigcap \mathcal{Z}_A(f)$.

For the other containment, suppose that p is a point of X such that $p \in \bigcap \mathcal{Z}_A(f)$; then $\mathcal{U}_p \supseteq \mathcal{Z}_A(f)$. Since by (2.1), $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$, we have that $\mathfrak{Z}_A(f) \subseteq \mathcal{U}_p$, and it follows that $p \in \bigcap \mathcal{U}_p \subseteq \bigcap \mathfrak{Z}_A(f)$. \square

The A -stable hull of a z -filter \mathcal{F} is defined as in [12]:

$$h^A(\mathcal{F}) = \{\mathcal{U}_p : \mathcal{U}_p \text{ is an } A\text{-stable } z\text{-ultrafilter on } X, \text{ and } \mathcal{F} \subseteq \mathcal{U}_p\}.$$

The closure operator in the topology of $v_A X$ is $h^A k$, because $v_A X$ is a subspace of βX (where the closure operator is hk). If E is a zero-set in X , then $kE = \langle E \rangle$, where $\langle E \rangle$ denotes the z -filter consisting of all zero-sets containing E . So

$$(3.1) \quad cl_{v_A X} E = h^A kE = h^A \langle E \rangle.$$

In other words $cl_{v_A X} E$ consists of the A -stable z -ultrafilters that contain E .

Proposition 3.2. *If $A(X)$ is an intermediate C -ring and $f \in A(X)$, then*

$$cl_{v_A X} \mathbf{Z}(f) \subseteq \mathbf{Z}(f^{v_A}).$$

Proof. Since $\mathbf{Z}(f)$ is a zero-set in X , it immediately follows from (3.1) that $cl_{v_A X} \mathbf{Z}(f) = h^A \langle \mathbf{Z}(f) \rangle$. By Lemma 3.1, $\mathfrak{Z}_A(f) \subseteq \langle \mathbf{Z}(f) \rangle$, so $cl_{v_A X} \mathbf{Z}(f) = h^A \langle \mathbf{Z}(f) \rangle \subseteq h^A \mathfrak{Z}_A(f)$. Also $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$, and by [12, Lemma 5], $kh\mathcal{Z}_A(f) = kh^A \mathcal{Z}_A(f)$. So $h^A \mathfrak{Z}_A(f) = h^A kh\mathcal{Z}_A(f) = h^A kh^A \mathcal{Z}_A(f) = cl_{v_A X} h^A \mathcal{Z}_A(f)$. It is shown in [12, Lemma 4] that $h^A \mathcal{Z}_A(f) = \mathbf{Z}(f^{v_A})$; so we have $cl_{v_A X} h^A \mathcal{Z}_A(f) = cl_{v_A X} \mathbf{Z}(f^{v_A})$. But $\mathbf{Z}(f^{v_A})$ is a closed set in $v_A X$, because it is a zero-set. So $cl_{v_A X} \mathbf{Z}(f^{v_A}) = \mathbf{Z}(f^{v_A})$. This completes the proof. \square

The following example shows that in general the containment in Proposition 3.2 may be strict.

Example 3.3. Let $X = \mathbb{N}$ and $A(X) = C^*(\mathbb{N})$. Suppose E is the set of even numbers in \mathbb{N} , and let g be the functions whose value is zero on the set E and $1/n$ at the odd integer n . Then $cl_{\beta X} \mathbf{Z}(g)$ consists of all z -ultrafilters that contain the set E of even numbers in \mathbb{N} , and hence none of these z -ultrafilters contain the set of odd numbers in \mathbb{N} . Now for any free z -ultrafilter \mathcal{U}_p containing the set of odd integers we have $\lim_{\mathcal{U}_p} g = 0$. So $p \in \mathbf{Z}(g^\beta)$, but clearly $p \notin cl_{\beta X} \mathbf{Z}(g)$. Thus $cl_{\beta X} \mathbf{Z}(g) \subsetneq \mathbf{Z}(g^\beta)$ (strict containment).

Note that this argument does not work for every function with zero-set E . Consider h which maps x to 1 if $x \notin E$ and to 0 otherwise; that is, h is the characteristic function of the complement of E . Here $cl_{\beta X} \mathbf{Z}(h)$ consists of all z -ultrafilters that contain E . As \mathbb{N} is discrete, any z -ultrafilter \mathcal{U}_p not in $cl_{\beta X} \mathbf{Z}(h)$ cannot contain E , and hence must contain the complement of E . But then $\lim_{\mathcal{U}_p} h = 1$ for such a z -ultrafilter \mathcal{U} , and hence by (2.2), $\mathcal{U}_p \notin \mathbf{Z}(h^\beta)$. Thus $cl_{\beta X} \mathbf{Z}(h) = \mathbf{Z}(h^\beta)$.

As we will see in the next lemma, there are a number of characterizations of when $cl_{v_A X} \mathbf{Z}(f)$ is equal to $\mathbf{Z}(f^{v_A})$. One of them is the negation of Condition ii of a theorem of Rudd [11, Theorem 1.2] that is expressed in terms of near-zero-sets. Rudd defines a subset T of X to be a near-zero-set of $f \in C(X)$ if for every $\epsilon > 0$ there exists a point $p \in T$ such that $|f(p)| < \epsilon$. The significance of T being a “near-zero-set” of f is that f is not T -regular with respect to $C^*(X)$. So in the context of intermediate rings we make the following definition.

Definition 3.4. Let $A(X)$ be an intermediate ring of continuous functions and let $f \in C(X)$. A subset T of X is called a *near-zero-set* of f with respect to $A(X)$ if f is not T -regular in $A(X)$.

It is apparent that a set E is a near-zero-set of f in the sense of Rudd [11] if and only if E is a near-zero-set of f with respect to $C^*(X)$ in the sense of Definition 3.4. Also, if $B(X) \subseteq A(X)$ and if E is a near-zero-set of f with respect to $A(X)$, then E is a near-zero-set of f with respect to $B(X)$.

Lemma 3.5. *Let $A(X)$ be an intermediate ring. Then for each $f \in A(X)$, the following are equivalent:*

- (a) $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$.
- (b) *No near-zero-set of f with respect to $A(X)$ is completely separated from $\mathbf{Z}(f)$.*

Proof. (b) \Rightarrow (a). Suppose that $\mathfrak{Z}_A(f) \neq \langle \mathbf{Z}(f) \rangle$. Then by Lemma 3.1 there exists a zero-set E such that $\mathbf{Z}(f) \subseteq E$ and $E \notin \mathfrak{Z}_A(f)$. By the definition of $\mathfrak{Z}_A(f)$ it follows that there exists a zero-set H in the complement of E for which f is not H -regular with respect to $A(X)$.

So H is a near-zero-set of f with respect to $A(X)$. But $\mathbf{Z}(f)$ and H are disjoint zero-sets in X and hence are completely separated ([7, page 17]).

(a) \Rightarrow (b). Suppose that there exists a near-zero-set T of f with respect to $A(X)$ which is completely separated from $\mathbf{Z}(f)$. So there is a zero-set H disjoint from $\mathbf{Z}(f)$ such that $T \subseteq H$. Since f is not T -regular, it follows that f is not H -regular either. So H is a zero-set in the complement of $\mathbf{Z}(f)$ and f is not H -regular in $A(X)$. It follows from the definition of $\mathfrak{Z}_A(f)$ that $\mathbf{Z}(f) \notin \mathfrak{Z}_A(f)$, and so $\mathfrak{Z}_A(f) \neq \langle \mathbf{Z}(f) \rangle$. \square

In the case of C -rings, we have the following lemma.

Lemma 3.6. *Let $A(X)$ be an intermediate C -ring. Then for each $f \in A(X)$, the following are equivalent:*

- (a) $cl_{v_A X} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$.
- (b) $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$.

Proof. (a) \Rightarrow (b). Suppose $cl_{v_A X} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$. By Lemma 3.1, $\mathfrak{Z}_A(f) \subseteq \langle \mathbf{Z}(f) \rangle$. For the other inclusion, suppose E is a zero-set such that $E \supseteq \mathbf{Z}(f)$. Then $cl_{v_A X} E \supseteq cl_{v_A X} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$ by hypothesis. But by [12, Theorem 6], $\mathfrak{Z}_A(f) = \{E \in \mathbf{Z}(X) \mid \mathbf{Z}(f^{v_A}) \subseteq cl_{v_A} E\}$. Hence we have $E \in \mathfrak{Z}_A(f)$.

(b) \Rightarrow (a). Suppose that $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$. Then

$$\begin{aligned} cl_{\beta X} \mathbf{Z}(f^{v_A}) &= cl_{\beta X} h^A \mathfrak{Z}_A(f) && \text{(by [12, Lemma 4])} \\ &= hkh^A \mathfrak{Z}_A(f) && \text{(by definition of closure in } \beta X) \\ &= hkh \mathfrak{Z}_A(f) && \text{(by [12, Lemma 5])} \\ &= h\mathfrak{Z}_A(f) && \text{(by (2.1))} \\ &= h\langle \mathbf{Z}(f) \rangle && \text{(by hypothesis)} \end{aligned}$$

Now, since $\mathbf{Z}(f^{v_A})$ is closed in $v_A X$, we have

$$\begin{aligned} \mathbf{Z}(f^{v_A}) &= cl_{v_A X} \mathbf{Z}(f^{v_A}) = cl_{\beta X} \mathbf{Z}(f^{v_A}) \cap v_A(X) \\ &= h\langle \mathbf{Z}(f) \rangle \cap v_A(X) = h^A \langle \mathbf{Z}(f) \rangle = cl_{v_A X} \mathbf{Z}(f). \quad \square \end{aligned}$$

As a special case of Proposition 3.2, we have that $cl_{\beta X} \mathbf{Z}(f) \subseteq \mathbf{Z}(f^\beta)$ and we know from Example 3.3 that the containment can be strict. But in general $cl_{v_X} \mathbf{Z}(f) = \mathbf{Z}(f^v)$, where vX is the Hewitt realcompactification of X (see [7, §8.8(b)]). The next theorem shows that this latter property characterizes $C(X)$ among intermediate C -rings on X .

Theorem 3.7. *Let $A(X)$ be an intermediate C -ring. Then $A(X) = C(X)$ if and only if every $f \in A(X)$ satisfies $cl_{v_A X} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$.*

Proof. The desired result is immediate from the following equivalences:

$$\begin{aligned} A(X) &= C(X) \\ \Leftrightarrow \mathfrak{Z}_A(f) &= \langle \mathbf{Z}(f) \rangle \text{ for each } f \in A(X) && \text{(by [8, Theorem 2.3])} \\ \Leftrightarrow cl_{v_A X} \mathbf{Z}(f) &= \mathbf{Z}(f^{v_A}) \text{ for each } f \in A(X) && \text{(by Lemma 3.6).} \quad \square \end{aligned}$$

Swardson [14, Corollary 4.3] shows that a completely regular space X satisfies $cl_{\beta X} \mathbf{Z}(f) = \mathbf{Z}(f^\beta)$ for every $f \in C^*(X)$ if and only if X is pseudocompact. We show that this characterization follows from Theorem 3.7 above. First, recall that a space X is pseudocompact if every continuous real-valued function on X is bounded. So X is pseudocompact if and only if $C(X) = C^*(X)$.

Corollary 3.8. [14, Corollary 4.3] *Let X be a completely regular space. Then $cl_{\beta X} \mathbf{Z}(f) = \mathbf{Z}(f^\beta)$ for every $f \in C^*(X)$ if and only if X is pseudocompact.*

Proof. Suppose that $cl_{\beta X} \mathbf{Z}(f) = \mathbf{Z}(f^\beta)$ for every $f \in C^*(X)$. Setting $A(X) = C^*(X)$, we have that $v_A X = \beta X$, and hence $cl_{v_A X} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$, for every $f \in A(X)$. It follows by Theorem 3.5 that $A(X) = C(X)$. But this means that $C^*(X) = C(X)$, and so X is pseudocompact.

Conversely, suppose that X is pseudocompact and set $A(X) = C^*(X)$. As $C^*(X) = C(X)$, we also have $A(X) = C(X)$, and it follows by Theorem 3.7 that $cl_{v_A X} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$. But since $A(X) = C^*(X)$ we have that $v_A X = \beta X$, and it follows that $cl_{\beta X} \mathbf{Z}(f) = \mathbf{Z}(f^\beta)$ for every $f \in C^*(X)$. \square

For $f \in C(X)$, the function f^* denotes the Stone extension of f to the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. This means that $f^* : \beta X \rightarrow \mathbb{R}^*$ extends the function f^{v_A} , and that (2.2) extends as follows: every point $p \in \beta X$ corresponds to a z -ultrafilter \mathcal{U}_p on X , and the value of f^* at p is

$$(3.2) \quad f^*(p) = \lim_{\mathcal{U}_p} f.$$

We observe that for any intermediate ring $A(X)$, the zero-set of f^{v_A} is contained in the zero-set of f^* , as follows.

Proposition 3.9. *If $A(X)$ is an intermediate ring and $f \in A(X)$, then*

$$\mathbf{Z}(f^{v_A}) \subseteq \mathbf{Z}(f^*).$$

Proof. By (3.2), the zero-set of f^* is $\mathbf{Z}(f^*) = \{p \in \beta X : \lim_{\mathcal{U}_p} f = 0\}$. It follows by Lemma 2.1 that a z -ultrafilter \mathcal{U}_p that contains $\mathcal{Z}_A(f)$ must satisfy $\lim_{\mathcal{U}_p} fh = 0$ for every $h \in A(X)$, and in particular for $h = 1$.

Hence \mathcal{U}_p belongs to $\mathbf{Z}(f^*)$, that is $h\mathcal{Z}_A(f) \subseteq \mathbf{Z}(f^*)$. On the other hand, by [12, Lemma 4] we have $\mathbf{Z}(f^{v_A}) = h^A\mathcal{Z}_A(f)$. Thus $\mathbf{Z}(f^{v_A}) = h^A\mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(f) \subseteq \mathbf{Z}(f^*)$. \square

Theorem 3.10. *Let $A(X)$ be an intermediate ring. Then $A(X) = C^*(X)$ if and only if every $f \in A(X)$ satisfies $\mathbf{Z}(f^{v_A}) = \mathbf{Z}(f^*)$.*

Proof. If $A(X) = C^*(X)$ then $v_AX = \beta X$ and $f^{v_A} = f^\beta$. Also, as f is bounded, f^* never attains the value ∞ , and hence $f^{v_A} = f^*$. So $\mathbf{Z}(f^{v_A}) = \mathbf{Z}(f^*)$ for every $f \in C^*(X)$.

Conversely, suppose that $A(X) \neq C^*(X)$. Then there is an unbounded function $g \in A(X)$. Without loss of generality assume that $g > 0$, and let $f = 1/g$. Let \mathcal{U} be any z -ultrafilter containing the zero-sets $\{x \in X : g(x) \geq n\}$. Clearly $\lim_{\mathcal{U}} f = 0$, so $\mathcal{U} \in \mathbf{Z}(f^*)$. But \mathcal{U} is not A -stable because f is unbounded on every set in \mathcal{U} , so $\mathcal{U} \notin \mathbf{Z}(f^{v_A})$. \square

Rudd [11] considers the question of when $\mathbf{Z}(f^*) = cl_{\beta X} \mathbf{Z}(f)$. We give a short proof of the equivalence of the first two parts of [11, Theorem 1.2] by using the results of this article.

Theorem 3.11. *Let $f \in C(X)$. Then $\mathbf{Z}(f^*) \neq cl_{\beta X} \mathbf{Z}(f)$ if and only if there is a near-zero-set of f with respect to $C^*(X)$ which is completely separated from $\mathbf{Z}(f)$.*

Proof. First, note that two sets in X are completely separated if and only if they are contained in disjoint zero-sets [7, §1.15].

(\implies) Suppose that $\mathbf{Z}(f^*) \neq cl_{\beta X} \mathbf{Z}(f)$. Then there is a z -ultrafilter $\mathcal{U}_p \in \mathbf{Z}(f^*)$ but $\mathcal{U}_p \notin cl_{\beta X} \mathbf{Z}(f)$. This means that $\mathbf{Z}(f) \notin \mathcal{U}_p$. But then there is a zero-set E in \mathcal{U}_p which does not meet $\mathbf{Z}(f)$, and hence by [7, §1.15], E is completely separated from $\mathbf{Z}(f)$. But since $E \in \mathcal{U}_p$ and $\lim_{\mathcal{U}_p} f = 0$, it follows that E is a near-zero-set of f with respect to $C^*(X)$.

(\impliedby) Suppose that E is a near-zero-set of f with respect to $C^*(X)$ and suppose that E is completely separated from $\mathbf{Z}(f)$. Then by [7, §1.15], there is a zero-set F containing E such that F and $\mathbf{Z}(f)$ are disjoint. As the property of being a near-zero-set of a function with respect to $A(X)$ is closed under supersets, F is a near-zero-set of f with respect to $C^*(X)$. Since F is a near-zero-set of f with respect to $C^*(X)$, the function f achieves values on F arbitrarily close to zero. Thus the sets $\{x : |f(x)| \leq 1/n\} \cap F$, for $n \in \mathbb{N}$, are nonempty zero-sets with the finite intersection property. So there is a z -ultrafilter \mathcal{U}_p containing these sets. Clearly $\lim_{\mathcal{U}_p} f = 0$, so $p \in \mathbf{Z}(f^*)$. On the other hand, since $\mathbf{Z}(f)$ and F are completely separated, $\mathbf{Z}(f) \notin \mathcal{U}_p$, and so $p \notin cl_{\beta X} \mathbf{Z}(f)$. \square

4. CLOSURES OF ZERO-SETS OF X IN v_AX

We now consider the problem of when the closure of a zero-set in X is a zero-set in v_AX . It was shown by Rudd in [11, Example 1.5] that if $E \in \mathbf{Z}[X]$, then it is possible that $cl_{\beta X} E \notin \mathbf{Z}[\beta X]$. Hence it is not always the case that the closure of a zero-set in X is a zero-set in $v_A(X)$. The following example is an alternative to Rudd's example that fits well with the formalism of this paper.

Example 4.1. Let $A(X) = C^*([0, \infty))$ and let $f \in A(X)$ be any function whose zero-set is $E = \{1, 2, \dots\}$. So $cl_{\beta X} \mathbf{Z}(f)$ consists of those z -ultrafilters containing E . Choose a sequence $F = \{a_n : n = 1, 2, \dots\}$ of positive numbers disjoint from E such that $|a_n - n| < 1$ for each n and $f(a_n) \rightarrow 0$ as $n \rightarrow \infty$. Since the a_n are close to n , the set F is unbounded, and hence F is contained in some free z -ultrafilter \mathcal{U}_p . As \mathcal{U}_p is free, $\lim_{\mathcal{U}_p} f = 0$. Hence by (2.2), $p \in \mathbf{Z}(f^\beta)$. But $p \notin cl_{\beta X} \mathbf{Z}(f)$ because $E \cap F = \emptyset$. So $cl_{\beta X} \mathbf{Z}(f) \subsetneq \mathbf{Z}(f^\beta)$ (strict containment). Since this is true for each f with zero-set E , it follows that $cl_{\beta X} E$ is not a zero-set in βX . (Note that if there were a function $g \in C(\beta X)$ whose zero-set were $cl_{\beta X} E$, then the restriction of g to X would have zero-set E .)

In [14, §5], Swardson asked the following two questions. When is $cl_{\beta X} E$ a zero-set in βX if E is a zero-set of X ? When does $cl_{\beta X} \mathbf{Z}(f) = \mathbf{Z}(f^\beta)$ for $f \in C^*(X)$? Some progress to answering these had already been given. For example in [15, Lemma 5], Terada found a characterization for when $cl_{\beta X} \mathbf{Z}(f)$ is a zero-set of βX . For other results on the closures of zero-sets in βX see [5]. We generalize these questions to $A(X)$, and compare the following two statements.

- (1) For every zero-set E in X , its closure $cl_{v_AX} E$ is a zero-set in v_AX .
- (2) For every $f \in A(X)$ we have $cl_{v_AX} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$.

Let E be a zero-set in X . Statement (2) implies that for every $f \in A(X)$ with $\mathbf{Z}(f) = E$ we have $cl_{v_AX} \mathbf{E} = \mathbf{Z}(f^{v_A})$ whereas Statement (1) follows if this equality holds only for some $f \in A(X)$ (so that $cl_{v_AX} \mathbf{E}$ is a zero-set in v_AX). So Statement (2) clearly implies Statement (1). We have shown (Theorem 3.7) that the second statement characterizes $C(X)$ among intermediate C -rings. We now show that the first statement characterizes those C -rings $A(X)$ for which each z -filter is a \mathfrak{Z}_A -filter. By a \mathfrak{Z}_A -filter we mean a z -filter \mathcal{F} with the property that

$$\mathfrak{Z}_A \mathfrak{Z}_A^\leftarrow[\mathcal{F}] = \mathcal{F}.$$

Note that if $A(X) = C(X)$ then every z -filter is a \mathfrak{Z}_A -filter because in this case $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$ for each f , and it is known that $\mathbf{Z} \mathbf{Z}^\leftarrow[\mathcal{F}] = \mathcal{F}$ for every z -filter \mathcal{F} ([7, p. 26]).

Theorem 4.2. *Let $A(X)$ be an intermediate C -ring. The following are equivalent.*

- (a) *Every z -filter on X is a \mathfrak{Z}_A -filter.*
- (b) *For every zero-set E in X , there exists $f \in A(X)$ such that $E = \mathbf{Z}(f)$ and $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$.*
- (c) *For every zero-set E in X , there exists $f \in A(X)$ such that $E = \mathbf{Z}(f)$ and no near-zero-set of f with respect to $A(X)$ is completely separated from $\mathbf{Z}(f)$.*
- (d) *For every zero-set E in X , its closure $cl_{v_AX} E$ is a zero-set in v_AX .*

Proof. (a) \Rightarrow (b). Let E be a zero-set in X , let $\mathcal{F} = \langle E \rangle$, and let $I = \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$. By hypothesis, $\mathfrak{Z}_A[I] = \mathcal{F}$. So there exists $f \in I$ such that $E \in \mathfrak{Z}_A(f)$. Then $\langle E \rangle \subseteq \mathfrak{Z}_A(f) \subseteq \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] = \mathcal{F} = \langle E \rangle$; hence $\mathfrak{Z}_A(f) = \langle E \rangle$. Thus by Lemma 3.1, we have $\mathbf{Z}(f) = E$, and so $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$.

(b) \Rightarrow (a). Let \mathcal{F} be a z -filter on X . Clearly $\mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] \subseteq \mathcal{F}$. Suppose $E \in \mathcal{F}$. Then by hypothesis there exists $f \in A(X)$ such that $E = \mathbf{Z}(f)$ and $\mathfrak{Z}_A(f) = \langle E \rangle$. Thus $\mathfrak{Z}_A(f) \subseteq \mathcal{F}$ and so $f \in \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$. This means that $E \in \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]]$. Thus $\mathcal{F} \subseteq \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]]$.

(b) \Leftrightarrow (c). This is immediate from Lemma 3.5.

(b) \Rightarrow (d). Let E be a zero-set in X and let $f \in A(X)$ such that $\mathbf{Z}(f) = E$ and $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$. Then, by Lemma 3.6, $cl_{v_AX} \mathbf{Z}(f) = \mathbf{Z}(f^{v_A})$, and so $cl_{v_AX} \mathbf{Z}(f) = cl_{v_AX} E$ is a zero-set in v_AX .

(d) \Rightarrow (b). Let E be a zero-set in X with the property that $cl_{v_AX} E$ is a zero-set in v_AX . Let $\tilde{f} \in C(v_AX)$, such that $\mathbf{Z}(\tilde{f}) = cl_{v_AX} E$. Since $A(X)$ is a C -ring, by Theorem 2.4, $A(X)$ is isomorphic to $C(v_A(X))$ by the isomorphism $f \mapsto f^{v_A}$. Thus there exists f , such that $f^{v_A} = \tilde{f}$. It follows that $\mathbf{Z}(f) = E$ and $\mathbf{Z}(f^{v_A}) = cl_{v_AX} \mathbf{Z}(f)$. It follows by Lemma 3.6 that for this f we have $\mathfrak{Z}_A(f) = \langle \mathbf{Z}(f) \rangle$. \square

5. MAXIMAL IDEALS UNDER \mathfrak{Z}_A

Recall that \mathfrak{Z}_A not only maps ideals in $A(X)$ to z -filters on X , but that \mathfrak{Z}_A extends the map \mathbf{Z} defined for $C(X)$ to all intermediate rings $A(X)$. This means that if $A(X) = C(X)$ then $\mathfrak{Z}_A[I] = \mathbf{Z}[I]$ for every ideal I in $A(X)$. It is known that \mathbf{Z} maps each maximal ideal of $C(X)$ to a z -ultrafilter on X . We investigate the corresponding property for \mathfrak{Z}_A on any intermediate ring $A(X)$. It was shown in [8, Example 4.5] that if M is a maximal ideal in $A(X)$ it does not necessarily follow that $\mathfrak{Z}_A[M]$ is a z -ultrafilter on X . This raises the following question: does the property of \mathfrak{Z}_A mapping maximal ideals to z -ultrafilters characterize $C(X)$ among intermediate rings $A(X)$? We show that the answer to this question is negative by exhibiting intermediate rings $A(X)$, different from

$C(X)$, for which the property does hold (Theorem 5.2). We begin by giving a sufficient condition for an intermediate ring $A(X)$ to have this property.

Theorem 5.1. *Let $A(X)$ be an intermediate C -ring for which every z -filter on X is a \mathfrak{Z}_A -filter. If M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is a z -ultrafilter on X .*

Proof. By Lemma 2.3(b), there is a unique z -ultrafilter \mathcal{U} such that $\mathfrak{Z}_A[M] \subseteq \mathcal{U}$. Now, let $E \in \mathcal{U}$. Then by Theorem 4.2 (the fact that (a) implies (b)), there exists $f \in A(X)$ such that $\mathfrak{Z}_A(f) = \langle E \rangle \subseteq \mathcal{U}$. By definition of \mathfrak{Z}_A , it holds that $f \in M$. Hence $E \in \mathfrak{Z}_A[M]$. \square

A P -space is a completely regular Hausdorff space in which every zero-set is open. We show that if $A(X)$ is an intermediate ring on a P -space space X , then \mathfrak{Z}_A does map maximal ideals to z -ultrafilters. In particular, if X is a P -space, then $C^*(X)$ has this property, and so the property does not characterize $C(X)$ among intermediate rings.

Theorem 5.2. *If $A(X)$ is an intermediate C -ring on a P -space X and if M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is a z -ultrafilter on X .*

Proof. Because X is a P -space, for every zero-set E in X , it holds that E is open, and hence the characteristic function f of the complement of E is continuous (and hence in $A(X)$). By definition of f , we have that $\mathfrak{Z}(f) = E$, and it is easy to see from the definition of \mathfrak{Z}_A that $\mathfrak{Z}_A(f) = \langle E \rangle$. Then by Theorem 4.2 (the fact that (b) implies (a)), every z -filter on X is a \mathfrak{Z}_A -filter. Then by Theorem 5.1, if M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is a z -ultrafilter on X . \square

The following problem remains open.

Problem 5.3. *Characterize those intermediate rings $A(X)$ for which $\mathfrak{Z}_A[M]$ is a z -ultrafilters on X whenever M is a maximal ideal in $A(X)$.*

Any characterization of the intermediate rings satisfying the condition in the problem must involve properties of both the topology of X and the ring structure of $A(X)$. This is because from Theorem 5.2 we know that if X is a P -space, then the condition holds for any intermediate ring $A(X)$, whereas it is well known that if $A(X) = C(X)$ then the condition holds for any completely regular topological space X .

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