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by

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CONVERGENCE S-COMPLETIONS

B. LOSERT AND G. RICHARDSON

ABSTRACT. A Cauchy semigroup acting continuously on a Cauchy space is investigated. In particular, the question as to when the action can be continuously extended to a completion of the Cauchy space is studied. Moreover, completions of generalized quotient spaces are considered.

1. Introduction

A topological group acting continuously on a topological space has been the subject of several research articles. Wayne R. Park [13], Nandita Rath [15], and H. Boustique et al. [2], [3] have studied this notion in the larger category of convergence spaces. A completion theory in this context is the main thrust of the present work and a convenient category for this study is the category of Cauchy spaces. Cauchy spaces date back to Hans-Joachim Kowalsky [7]. The formulation employed here was first defined by H. H. Keller [5]. Cauchy spaces have been found to be useful in several areas of research; for example, Kelly McKennon [10] used Cauchy spaces in the study of C^*-algebras and Richard N. Ball [1] found Cauchy space completions of lattice ordered groups.

In this paper, we generalize one of the results in [6] that says that, given a limit space (X, p) there is an isomorphism between the ordered set of precompact Cauchy structures on X that induce p and the ordered set of equivalence classes of strict regular compactifications (X, p). The generalization is done in the context of "S-spaces" and is given in Theorem 3.7. We also investigate the notion of a "generalized quotient space" from the Cauchy-space perspective. These spaces were introduced by Józef Burzyk

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et al. [4] in the topological setting. P. Mikusiński ([12], [11]) used generalized quotient spaces to study generalized functions. A good reference on convergence spaces, Cauchy spaces, and categorical terminology is the book by Gerhard Preuss [14]. An excellent treatment of Cauchy spaces is the monograph of Eva Lowen-Colebunders [9].

2. Preliminaries

Let X be a set, let $\mathbf{P}(X)$ be the power set of X, and let $\mathbf{F}(X)$ be the set of filters on X. Given $x \in X$, we will use $\dot{x}$ to denote the fixed ultrafilter on X generated by $\{\{x\}\}$. Given $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$, we will write $\mathcal{F} \geq \mathcal{G}$ (read "$\mathcal{F}$ is finer than $\mathcal{G}$") if and only if $\mathcal{G} \subseteq \mathcal{F}$. The relation $\geq$ is a partial order on $\mathbf{F}(X)$ and we will write $\mathcal{F} \vee \mathcal{G}$ for the least upper bound of $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ with respect to this partial order, which exists whenever $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

The pair (X, p) is called a *limit space* and p is called a *limit structure* on X whenever X is a set and $p\colon \mathbf{F}(X) \to \mathbf{P}(X)$ satisfies the following conditions:

(LS1) $x \in p(\dot{x})$.
(LS2) $\mathcal{G} \geq \mathcal{F}$ implies $p(\mathcal{F}) \subseteq q(\mathcal{G})$.
(LS3) $x \in p(\mathcal{F})$ and $x \in p(\mathcal{G})$ implies $x \in p(\mathcal{F} \cap \mathcal{G})$.

Conventions. The notation $x \in p(\mathcal{F})$ is read as "$\mathcal{F}$ p-converges to x" or "$\mathcal{F}$ converges to x in X" or "$\mathcal{F}$ converges to x" and is usually written "$\mathcal{F} \xrightarrow{p} x$" or "$\mathcal{F} \to x$ in X" or "$\mathcal{F} \to x$." When we do not need to make reference to the limit structure of a limit space (X, p), we will write the space as X.

A function $f\colon X \to Y$ between limit spaces is *continuous* provided that $f^{\to}\mathcal{F} \to f(x)$ in Y whenever $\mathcal{F} \to x$ in X. Here, $f^{\to}\mathcal{F}$ denotes the filter on Y generated by $\{f(F)\colon F \in \mathcal{F}\}$. Given $\mathcal{G} \in \mathbf{F}(Y)$, we use $f^{\leftarrow}\mathcal{G}$ to denote the filter on X generated by $\{f^{-1}(G)\colon G \in \mathcal{G}\}$ whenever the latter set does not contain $\emptyset$.

Let Lim denote the category of limit spaces and continuous functions and let X be an object in Lim. We say X is *reciprocal* if whenever a filter $\mathcal{F}$ on X converges to two points $x, y \in X$, then x and y have the same convergent filters. We say that X is *Hausdorff* provided that each filter on X converges to at most one point. Note that every Hausdorff space is reciprocal. We say that X is *regular* if whenever $\mathcal{F} \to x$ in X, the filter $\operatorname{cl}_X \mathcal{F}$ generated by $\{\operatorname{cl}_X F\colon F \in \mathcal{F}\}$ converges to x. Here, cl_X denotes the closure operator on X. A point $x \in X$ is an *adherent point* of $\mathcal{F} \in \mathbf{F}(X)$ whenever there exists a $\mathcal{G} \geq \mathcal{F}$ such that $\mathcal{G} \to x$ in X. We use $\operatorname{adh}_X \mathcal{F}$ to denote the set of all adherent points of $\mathcal{F}$. The *neighborhood filter* of a point $x \in X$ is the intersection of all filters converging to x and is denoted $\mathcal{U}_X(x)$.

We say X is *compact* provided that $\operatorname{adh}_X \mathcal{F} \neq \emptyset$ for each $\mathcal{F} \in \mathbf{F}(X)$ or, equivalently, if each ultrafilter on X converges. A *compactification* in Lim of X is a pair (Y, f) where Y is a compact Hausdorff limit space and $f\colon X \to Y$ is a dense embedding in Lim. Observe that *compactifications are required to be Hausdorff.* A compactification (Y, f) is called *regular* whenever Y is regular and *strict* if, whenever $\mathcal{G} \to y$ in Y, there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $\mathcal{G} \geq \operatorname{cl}_Y f^{\rightarrow}\mathcal{F}$.

The pair $(X, \mathcal{C})$ is called a *Cauchy space* and $\mathcal{C}$ is called a *Cauchy structure* whenever X is a set and $\mathcal{C} \subseteq \mathbf{F}(X)$ satisfies the following conditions:

(CS1) $\dot{x} \in \mathcal{C}$ for all $x \in X$.
(CS2) $\mathcal{G} \geq \mathcal{F} \in \mathcal{C}$ implies $\mathcal{G} \in \mathcal{C}$.
(CS3) If $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ and $\mathcal{F} \vee \mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$.

Conventions. When we do not need to make reference to the Cauchy structure of a Cauchy space $(X, \mathcal{C})$, we will write the space as X and call the filters in $\mathcal{C}$ the *Cauchy filters* on X.

Given two Cauchy structures $\mathcal{C}$ and $\mathcal{D}$ on a set X, we write $\mathcal{C} \geq \mathcal{D}$ to mean $\mathcal{C} \subseteq \mathcal{D}$. Any collection of Cauchy structures on a set X is partially ordered by $\geq$ and whenever we think of a collection of Cauchy structures on X as an ordered set, it will be with respect to $\geq$.

A function $f\colon X \to Y$ between Cauchy spaces is called *Cauchy continuous* if it maps Cauchy filters on X to Cauchy filters on Y. The category whose objects are Cauchy spaces and whose morphisms are Cauchy continuous functions is denoted Chy. It is well known that Chy is a topological and cartesian closed category ([9], [14]).

It is shown in Keller [5] that the category Lim_R of reciprocal limit spaces is coreflective in Chy. The Lim_R-coreflection of a Cauchy space $(X, \mathcal{C})$ is the reciprocal limit space $(X, p_{\mathcal{C}})$ where $p_{\mathcal{C}}$ is defined so that $\mathcal{F} \xrightarrow{p_{\mathcal{C}}} x$ if and only if $\mathcal{F} \cap \dot{x} \in \mathcal{C}$. Concepts such as convergence, adherent points, closure and denseness of filters, and subsets of a Cauchy space will be understood to be with respect to its Lim_R-coreflection.

A Cauchy space X is *complete* if every Cauchy filter on X converges, *totally bounded* if every ultrafilter on X is Cauchy, *Hausdorff* if its Lim_R-coreflection is Hausdorff, and *regular* if $\operatorname{cl}_X \mathcal{F}$ is Cauchy whenever $\mathcal{F} \in \mathbf{F}(X)$ is Cauchy. A *completion* of a Cauchy space X is a pair (Y, f) where Y is a complete Hausdorff Cauchy space and $f\colon X \to Y$ is a dense embedding in Chy. Observe that *completions are required to be Hausdorff.* A *strict completion* (Y, f) of a Cauchy space X is a completion that satisfies the following condition: If $\mathcal{G} \in \mathbf{F}(Y)$ is Cauchy, then there exists a Cauchy filter $\mathcal{F}$ on X such that $\mathcal{G} \geq \operatorname{cl}_Y f^{\rightarrow}\mathcal{F}$. A totally bounded Cauchy space having a regular completion is called *precompact.*

The category Lim_R is isomorphic to the full subcategory of Chy whose objects are complete Cauchy spaces. The embedding is defined as follows: Given a reciprocal limit space (X, p), let $(X, \mathcal{C}^p)$ be the complete Cauchy space where $\mathcal{C}^p = \{\mathcal{F} \in \mathbf{F}(X)\colon \mathcal{F}\ p\text{-converges}\ \}$. In light of this, we will consider complete Cauchy spaces as reciprocal limit spaces and vice versa. The embedding of Lim_R in Chy is also concretely coreflective (see [9]), which means that it preserves colimits and, in particular, quotients. For our purposes, this implies the following: If $f\colon X \to Y$ is a quotient map in Lim_R, then it is also a quotient map in Chy.

A monoid equipped with a limit structure making its binary operation continuous is called a *limit monoid*. A monoid equipped with a Cauchy structure making its binary operation Cauchy continuous is called a *Cauchy monoid*.

Conventions. All monoids will be multiplicative by default and their identity element will be denoted e.

Let S be monoid and let X be a set. An *action of S on X* is a function $\lambda\colon X \times S \to X$ such that

(A1) $\lambda(x, e) = x$ for all $x \in X$ and
(A2) $\lambda(\lambda(x, s), t) = \lambda(x, st)$ for all $x \in X$ and $s, t \in S$.

If S is a limit monoid, X is a limit space, and λ is continuous, then λ is called a *continuous action of S on X*. If S is a Cauchy monoid, X is a Cauchy space, and λ is Cauchy continuous, then λ is called a *Cauchy continuous action of S on X*.

Let $\mathsf{CA}_{\mathsf{Lim}}$ denote the category whose objects consist of triples of the form (X, S, λ) where X is a limit space, S is a limit monoid, and λ is a continuous action of S on X, and whose morphisms are of the form

$$(f, g)\colon (X, S, \lambda) \longrightarrow (Y, T, \mu)$$

where

(C1) $f\colon X \to Y$ is a continuous function,
(C2) $g\colon S \to T$ is a continuous homomorphism, and
(C3) $\mu \circ (f \times g) = f \circ \lambda$.

The category $\mathsf{CA}_{\mathsf{Chy}}$ is defined in an analogous manner. A *compactification* of an object (X, S, λ) in $\mathsf{CA}_{\mathsf{Lim}}$ is a pair $((Y, S, \mu), f)$ where

- (Y, S, μ) is an object in $\mathsf{CA}_{\mathsf{Lim}}$,
- (Y, f) is a compactification in Lim of X, and
- $(f, \mathrm{id}_S)\colon (X, S, \lambda) \to (Y, S, \mu)$ is a morphism in $\mathsf{CA}_{\mathsf{Lim}}$.

Completions of objects in $\mathsf{CA}_{\mathsf{Chy}}$ are defined analogously.

Conventions. The objects of $\mathsf{CA}_{\mathsf{Lim}}$ ($\mathsf{CA}_{\mathsf{Chy}}$, respectively) whose objects have S as the acting monoid will be called *limit S-spaces* (*Cauchy S-spaces*, respectively). To simplify notation, limit and Cauchy S-spaces will usually be written as (X,λ) or X. Just as there is a Lim_R-coreflection of Chy, there is a $\mathsf{CA}_{\mathsf{Lim}_R}$-coreflection of $\mathsf{CA}_{\mathsf{Chy}}$. (Here, $\mathsf{CA}_{\mathsf{Lim}_R}$ denotes the full subcategory of $\mathsf{CA}_{\mathsf{Lim}}$ whose objects (X,S,λ) have the additional property that X is reciprocal.) And just like Lim_R is embedded in Chy, $\mathsf{CA}_{\mathsf{Lim}_R}$ is embedded in $\mathsf{CA}_{\mathsf{Chy}}$). Compactifications in $\mathsf{CA}_{\mathsf{Lim}}$ of limit S-spaces will be called *S-compactifications* and completions in $\mathsf{CA}_{\mathsf{Chy}}$ of Cauchy S-spaces will be called *S-completions*.

If $((Y,\mu),f)$ and $((Z,\nu),g)$ are two S-compactifications of a limit S-space (X,λ), we will write $((Y,\mu),f)\geq((Z,\nu),g)$ or, more simply, $Y\geq Z$ whenever there exists a continuous function $h\colon Y\to Z$ such that $h\circ f=g$ and $h\circ\mu=\nu\circ(h\times\mathrm{id}_S)$. If Y and Z are two S-compactifications of a limit S-space X and $Y\geq Z$ and $Z\geq Y$, then Y and Z are isomorphic objects in $\mathsf{CA}_{\mathsf{Lim}}$ and we say that Y and Z are *equivalent S-compactifications* of X. By extending the relation $\geq$ to the set of equivalence classes of equivalent S-compactifications of X, the relation $\geq$ becomes a partial ordering. This is the ordering that we will be referring to when we say "ordered set of equivalence classes of equivalent S-compactifications."

A limit S-space (X,λ) is called *adherence restrictive* if, for each $\mathcal{F}\in\mathbf{F}(X)$ with no adherent points and every convergent filter $\mathcal{G}$ on S, the filter $\lambda^{\to}(\mathcal{F}\times\mathcal{G})$ has no adherent points. Similarly, a Cauchy S-space (X,λ) is called *adherence restrictive* if, for every Cauchy filter $\mathcal{F}$ on X with no adherent points and every convergent filter $\mathcal{G}$ on S, the filter $\lambda^{\to}(\mathcal{F}\times\mathcal{G})$ has no adherence points.

An S-compactification or an S-completion $((Y,\mu),f)$ of an S-space (X,λ) will be called *remainder invariant* if $\mu((Y-f(X))\times S)\subseteq Y-f(X)$.

3. Cauchy S-spaces

In this section, we will establish a relationship between the S-compactifications of a limit S-space X and the precompact Cauchy S-spaces whose $\mathsf{CA}_{\mathsf{Lim}_R}$-coreflections are equal to X. Before we do so, we will need some preliminary results.

A limit space is called *completely regular* if the limit space is regular and Hausdorff and agrees on ultrafilter convergence with a completely regular topological space. The *pretopological modification* πX of a limit space X is defined so that $\mathcal{F}\in\mathbf{F}(X)$ converges to x in πX if and only if $\mathcal{F}\geq\mathcal{U}_X(x)$ in X.

Theorem 3.1 ([17]). *A regular Hausdorff limit space X has a strict regular compactification in* Lim *if and only if*

(1) *X and its pretopological modification πX agree on ultrafilter convergence.*
(2) *πX is a completely regular topological space.*

It follows that a limit space X has a strict regular compactification in Lim *if and only if it is completely regular. Moreover, if X is completely regular, then it has a largest strict regular compactification in* Lim*.*

Lemma 3.2. *Any limit S-space that has a regular S-compactification also has a strict regular S-compactification.*

Proof. Let $((X,q),\lambda)$ be a limit S-space and let $(((Y,r),\mu),f)$ be a regular S-compactification of $((X,q),\lambda)$. Let t be a limit structure on Y defined so that $\mathcal{H} \xrightarrow{t} y$ if and only if $\mathcal{H} \geq \mathrm{cl}_r\, f^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \xrightarrow{r} y$. Note that r and t agree on ultrafilter convergence, which means that (Y,t) is a strict, regular compactification of (X,q). We now show that μ is a continuous action on (Y,t).

Suppose $\mathcal{H} \xrightarrow{t} y$ and $\mathcal{G} \to s$ in S. Then $\mathcal{H} \geq \mathrm{cl}_r\, f^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \xrightarrow{r} y$. Since μ is a continuous action on (Y,r), we have that $\mu^{\rightarrow}(f^{\rightarrow}\mathcal{F}\times\mathcal{G}) \xrightarrow{r} \mu(y,s)$. Since $\mu^{\rightarrow}(f^{\rightarrow}\mathcal{F}\times\mathcal{G}) = f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F}\times\mathcal{G}))$ and

$$\mu^{\rightarrow}(\mathcal{H}\times\mathcal{G}) \geq \mu^{\rightarrow}(\mathrm{cl}_r\, f^{\rightarrow}\mathcal{F}\times\mathcal{G}) \geq \mathrm{cl}_r\, \mu^{\rightarrow}(f^{\rightarrow}\mathcal{F}\times\mathcal{G}) = \mathrm{cl}_r\, f^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F}\times\mathcal{G}))$$

it follows that $\mu^{\rightarrow}(\mathcal{H}\times\mathcal{G}) \xrightarrow{t} \mu(y,s)$. This concludes the proof that μ is a continuous action on (Y,t), and it follows that $(((Y,r),\mu),f)$ is a strict regular S-compactification of $((X,q),\lambda)$. □

This lemma, coupled with [8, Theorem 3.5], yields the following theorem.

Theorem 3.3. *Let X be a completely regular limit space, let $((Y,r),f)$ be a strict regular compactification of X, and let $\mathcal{C}$ be the Cauchy structure on X defined so that $\mathcal{F} \in \mathcal{C}$ if and only if $f^{\rightarrow}\mathcal{F}$ r-converges. If S is a complete Cauchy monoid and if λ is a continuous action of S on X making (X,S,λ) a limit S-space, then there is an action μ of S on Y such that $(((Y,r),\mu),f)$ is a strict regular S-compactification of (X,λ) if and only if λ is a Cauchy continuous action on $(X,\mathcal{C})$. Moreover, when $((Y,r),f)$ is the largest regular compactification of X and the action on X is Cauchy continuous, then $(((Y,r),\mu),f)$ is the largest strict regular S-compactification of X.*

Let X be a Hausdorff Cauchy space. Two Cauchy filters on X are *equivalent* if their intersection is Cauchy. Given a Cauchy filter $\mathcal{F}$ on X, let $[\mathcal{F}]$ denote the equivalence class of Cauchy filters equivalent to $\mathcal{F}$ and let $\tilde{X}$ denote the set of all such equivalence classes. A completion (Y,f) of X is said to be in *standard form* if $Y = \tilde{X}$, $f\colon X \to Y$ is given by $f(x) = [\dot{x}]$ and $f^{\rightarrow}\mathcal{F} \to [\mathcal{F}]$ in Y for all Cauchy filters $\mathcal{F}$ on X.

Theorem 3.4 ([16, Theorem 5]). *Every completion of a Cauchy space is equivalent to one in standard form.*

Given a subset A of X, let $\Sigma A \subseteq \tilde{X}$ be defined so that $[\mathcal{F}] \in \Sigma A$ if and only if $A \in \mathcal{G}$ for some $\mathcal{G} \in [\mathcal{F}]$. Given $\mathcal{F} \in \mathbf{F}(X)$, since $\Sigma(A \cap B) \subseteq \Sigma A \cap \Sigma B$ for all $A, B \subseteq X$, $\{\Sigma F \colon F \in \mathcal{F}\}$ is a basis for a filter on $\tilde{X}$, which we denote $\Sigma\mathcal{F}$. Let $\mathcal{C}$ be the Cauchy structure on X and let $\tilde{\mathcal{C}} = \{\mathcal{H} \in \mathbf{F}(\tilde{X}) \colon \mathcal{H} \geq \Sigma\mathcal{F} \text{ for some } \mathcal{F} \in \mathcal{C}\}$. In general, $\tilde{\mathcal{C}}$ fails to satisfy (CS3), so, in general, $\tilde{\mathcal{C}}$ is not a Cauchy structure on $\tilde{X}$. If $\tilde{\mathcal{C}}$ is a Cauchy structure, then $\mathcal{F} \in \mathcal{C}$ if and only if $f^{\rightarrow}\mathcal{F} \in \tilde{\mathcal{C}}$.

Conventions. Whenever we consider $\tilde{X}$ as a Cauchy space, $\tilde{\mathcal{C}}$ will be its corresponding Cauchy structure, and whenever we consider $\tilde{X}$ as a completion of X, it will be with respect to the embedding $x \mapsto [\dot{x}] \colon X \to \tilde{X}$, which we will call the *canonical embedding of X in $\tilde{X}$*.

Theorem 3.5 ([6, Corollary 1.6 and Theorem 2.2]). *Let X be a Hausdorff Cauchy space. Then $\tilde{X}$ is the only possible candidate for a strict regular completion of X in standard form. Moreover, if X is precompact, then $\tilde{X}$ is a strict regular completion of X.*

Lemma 3.6. *Let X be a Cauchy S-space. If (Y, f) is a strict regular completion of X, then there exists a Cauchy continuous action of S on Y making Y a strict regular S-completion of X.*

Proof. Let $\mathcal{C}$ be the Cauchy structure on X and let λ be the Cauchy continuous action of S on X. According to theorems 3.4 and 3.5, we may assume without loss of generality that $Y = \tilde{X}$.

Define $\tilde{\lambda} \colon \tilde{X} \times S \to \tilde{X}$ by $\tilde{\lambda}([\mathcal{F}], s) = [\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})]$. Note that $\tilde{\lambda}$ is well defined, for if $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ are equivalent, then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$ and $\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \cap \lambda^{\rightarrow}(\mathcal{G} \times \dot{s}) = \lambda^{\rightarrow}((\mathcal{F} \cap \mathcal{G}) \times \dot{s}) \in \mathcal{C}$, which means $\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})$ and $\lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$ are equivalent, which means $\tilde{\lambda}([\mathcal{F}], s) = \tilde{\lambda}([\mathcal{G}], s)$. We now prove that $\tilde{\lambda}$ is an action on $\tilde{X}$: Let $\mathcal{F} \in \mathcal{C}$ and $s, t \in S$ be arbitrary. Then $\tilde{\lambda}([\mathcal{F}], e) = [\lambda^{\rightarrow}(\mathcal{F} \times \dot{e})] = [\mathcal{F}]$ and $\tilde{\lambda}(\tilde{\lambda}([\mathcal{F}], s), t) = \tilde{\lambda}([\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})], t) = [\lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \dot{s}) \times \dot{t})] = [\lambda^{\rightarrow}(\mathcal{F} \times \dot{st})] = \tilde{\lambda}([\mathcal{F}], st)$.

Before we prove that $\tilde{\lambda}$ is Cauchy continuous, we prove that $\tilde{\lambda}(\Sigma A \times B) \subseteq \Sigma\lambda(A \times B)$ for all $A \subseteq X$ and $B \subseteq S$: Suppose that $[\mathcal{F}] \in \Sigma A$ and $s \in B$. Then $A \in \mathcal{G}$ for some $\mathcal{G} \in \mathcal{C}$ equivalent to $\mathcal{F}$, hence $\lambda(A \times \{s\}) \in \lambda^{\rightarrow}(\mathcal{G} \times \dot{s})$, hence $\lambda^{\rightarrow}(\mathcal{G} \times \dot{s}) \in [\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})] = \tilde{\lambda}([\mathcal{F}], s)$, hence $[\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})] \in \Sigma\lambda(A \times B)$, hence $\tilde{\lambda}(\Sigma A \times B) \subseteq \Sigma\lambda(A \times B)$, as claimed.

Now we are ready to prove that $\tilde{\lambda}$ is Cauchy continuous: Let $\mathcal{H} \in \tilde{\mathcal{C}}$ and let $\mathcal{G}$ be a Cauchy filter on S. Then $\mathcal{H} \geq \Sigma\mathcal{F}$ for some $\mathcal{F} \in \mathcal{C}$, hence $\tilde{\lambda}^{\rightarrow}(\mathcal{H} \times \mathcal{G}) \geq \tilde{\lambda}^{\rightarrow}(\Sigma\mathcal{F} \times \mathcal{G}) \geq \Sigma\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \in \tilde{\mathcal{C}}$.

Finally, letting f denote the canonical embedding of X in $\tilde{X}$, since $\tilde{\lambda} \circ (f \times \mathrm{id}_S) = f \circ \lambda$, it follows that $((\tilde{X}, \tilde{\lambda}), f)$ is a strict regular S-completion of (X, λ). □

Conventions. Whenever we regard $\tilde{X}$ as an S-completion of an S-space (X, λ), the action on $\tilde{X}$ will be the action $\tilde{\lambda}$ as defined in Theorem 3.5.

Theorem 3.7. *Let X be a completely regular limit space, let S be a complete Cauchy monoid, and let λ be a continuous action of S on X. Then there is an isomorphism between the ordered set of all precompact Cauchy S-spaces whose $\mathsf{CA}_{\mathsf{Lim}_R}$-coreflections are isomorphic to X and the ordered set of all equivalence classes of strict regular S-compactifications of X.*

Proof. Let λ be the action on X. For notational convenience, let $\mathfrak{X}$ be the set of all precompact Cauchy S-spaces whose $\mathsf{CA}_{\mathsf{Lim}_R}$-coreflections are isomorphic to X. Without loss of generality, we assume that the elements of $\mathfrak{X}$ have the form $((X, \mathcal{C}), \lambda)$. In this way, the ordering on $\mathfrak{X}$ is given by the ordering of the precompact Cauchy structures of the elements of $\mathfrak{X}$; i.e., $((X, \mathcal{C}_1), \lambda) \geq ((X, \mathcal{C}_2), \lambda)$ if and only if $\mathcal{C}_1 \geq \mathcal{C}_2$ if and only if $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Also, we will let $\mathfrak{K}$ denote a fixed but complete set of non-equivalent strict regular S-compactifications of X, ordered in the way as explained in §2. Here, $\mathfrak{K}$ will play the role of the ordered set of all equivalence classes of strict regular S-compactifications of X.

Theorem 3.3 gives the necessary and sufficient conditions for the existence of a strict regular S-compactification of the limit S-space X. Let $X_0 \in \mathfrak{X}$. By Theorem 3.5 and Lemma 3.6, $\tilde{X}_0$ is a strict regular S-completion of X_0. Since X_0 is totally bounded, $\tilde{X}_0$ is a strict regular S-compactification of X. Define $\theta\colon \mathfrak{X} \to \mathfrak{K}$ so that $\theta(X_0) = \tilde{X}_0$.

Let us prove that θ is injective. Let $X_1, X_2 \in \mathfrak{X}$ and $((X, \mathcal{C}_2), \lambda)$ be two elements of $\mathfrak{X}$. For $i = 1, 2$, let $[\mathcal{F}]_i$ denote the equivalence class of Cauchy filters on X_i equivalent to the Cauchy filter $\mathcal{F}$ on X_i, let f_i denote the canonical embedding of X_i in $\tilde{X}_i$, and let $\tilde{\lambda}_i$ be the action on $\tilde{X}_i$. Suppose $\tilde{X}_1$ and $\tilde{X}_2$ are equivalent S-compactifications of X. Then there is a homeomorphism $h\colon \tilde{X}_1 \to \tilde{X}_2$ such that $h \circ f_1 = f_2$ and $\tilde{\lambda}_2 \circ (h \times \mathrm{id}_S) = h \circ \tilde{\lambda}_1$. If $\mathcal{F}$ is a Cauchy filter on X_1, then $f_1^{\rightarrow}\mathcal{F} \to [\mathcal{F}]_1$ in $\tilde{X}_1$, hence $(h \circ f_1)^{\rightarrow}\mathcal{F} = f_2^{\rightarrow}\mathcal{F} \to h([\mathcal{F}]_1)$ in $\tilde{X}_2$, hence $f_2^{\rightarrow}\mathcal{F}$ is Cauchy filter on $\tilde{X}_2$, hence $\mathcal{F}$ is a Cauchy filter on X_2. This proves that $X_1 \geq X_2$ and, since h is a homeomorphism, the same argument with X_1 and X_2 swapped proves that $X_2 \geq X_1$, which means $X_1 = X_2$.

Let us now prove that θ is surjective. Let $((Y, \mu), g)$ be an arbitrary strict regular remainder-invariant S-compactification of X, let $\mathcal{C}$ be the collection of all filters $\mathcal{F} \in \mathbf{F}(X)$ such that $g^{\rightarrow}\mathcal{F}$ converges in Y, and let $\mathcal{D}$ be the collection of all convergent filters on Y. Then $g\colon (X,\mathcal{C}) \to (Y,\mathcal{D})$ is a dense embedding in Chy and, consequently, $((Y,\mathcal{D}),g)$ is a totally bounded regular completion of $(X,\mathcal{C})$ in Chy. Also, λ is a Cauchy continuous action of S on $(X,\mathcal{C})$: Given $\mathcal{F} \in \mathcal{C}$ and a Cauchy filter $\mathcal{G}$ on S, we have that $g^{\to}\mathcal{F} \to y$ in Y and $\mathcal{G} \to s$ for some $y \in Y$ and $s \in S$, and since $g \circ \lambda = \mu \circ (g \times \mathrm{id}_S)$, we have that $g^{\to}(\lambda^{\to}(\mathcal{F} \times \mathcal{G})) = \mu^{\to}(g^{\to}\mathcal{F} \times \mathcal{G}) \to \mu(y,s)$, hence $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \in \mathcal{C}$, hence $((X,\mathcal{C}),\lambda)$ is a precompact Cauchy S-space in $\mathfrak{X}$. Finally, note that θ maps $(X,\mathcal{C})$ to $\tilde{X}$ and that $\tilde{X}$ and Y are equivalent S-compactifications.

Lastly, we prove that θ is order preserving. Let $X_1, X_2 \in \mathfrak{X}$ and suppose $X_1 \geq X_2$. Define $h\colon \tilde{X}_1 \to \tilde{X}_2$ by $h([\mathcal{F}]_1) = [\mathcal{F}]_2$. Let Σ_i be the Σ operator for $\tilde{X}_i$, $i = 1,2$. Since $h^{\to}(\Sigma_1\mathcal{F}) \geq \Sigma_2\mathcal{F}$, it follows that h is continuous. We also have that $h \circ f_1 = f_2$ and that $\tilde{\lambda}_2 \circ (h \times \mathrm{id}_S) = h \circ \tilde{\lambda}_1$, for if $\mathcal{F}$ is Cauchy filter on X_1 and $s \in S$, then $\tilde{\lambda}_2 \circ (h \times \mathrm{id}_S)([\mathcal{F}]_1, s) = \tilde{\lambda}_2([\mathcal{F}]_2, s) = [\lambda^{\to}(\mathcal{F} \times \dot{s})]_2 = h([\lambda^{\to}(\mathcal{F} \times \dot{s})]_1) = (h \circ \tilde{\lambda}_1)([\mathcal{F}]_1, s)$. Therefore, $\tilde{X}_1 \geq \tilde{X}_2$. Conversely, suppose that $\tilde{X}_1 \geq \tilde{X}_2$. Then there is a continuous function $h\colon \tilde{X}_1 \to \tilde{X}_2$ such that $h \circ f_1 = f_2$ and $\tilde{\lambda}_2 \circ (h \times \mathrm{id}_S) = h \circ \tilde{\lambda}_1$. If $\mathcal{F}$ is a Cauchy filter on X_1, then $f_1^{\to}\mathcal{F} \to [\mathcal{F}]_1$ in $\tilde{X}_1$, hence $(h \circ f_1)^{\to}\mathcal{F} = f_2^{\to}\mathcal{F} \to h([\mathcal{F}]_1)$ in $\tilde{X}_2$, hence $f_2^{\to}\mathcal{F}$ is a Cauchy filter on $\tilde{X}_2$, hence $\mathcal{F}$ is a Cauchy filter on X_2. This proves that $X_1 \geq X_2$. □

4. Generalized Quotients

Generalized quotients in the topological setting were introduced by Burzyk et al. [4] for the purpose of studying generalized functions. Extensions to the category of convergence spaces can be found in Boustique et al. [2], [3]. In this section we investigate generalized quotients in the context of S-completions.

Let $\mathsf{GQ}_{\mathsf{Chy}}$ denote the full subcategory of $\mathsf{CA}_{\mathsf{Chy}}$ whose objects (X,S,λ) satisfy the following conditions:

(G1) S is a commutative monoid.
(G2) $\lambda(\cdot, s)$ is injective for each fixed $s \in S$.

The category $\mathsf{GQ}_{\mathsf{Lim}}$ is defined analogously.

Let (X,S,λ) be an object in $\mathsf{GQ}_{\mathsf{Chy}}$. Define a relation $\sim$ on $X \times S$ so that $(x,s) \sim (y,t)$ if and only if $\lambda(x,t) = \lambda(y,s)$. The relation $\sim$ is an equivalence relation. Let $B(X,S) = (X \times S)/\sim$ and define $\theta_X\colon (X \times S) \to B(X,S)$ so that $\theta_X(x,s)$ is the equivalence class containing (x,s). Let $\mathcal{C}_X$ be the structure of the Cauchy quotient on $B(X,S)$ with respect to the canonical surjection θ_X. Note that $\mathcal{C}_X$ is the finest Cauchy structure on $B(X,S)$ making θ_X Cauchy continuous and that θ_X is a quotient map in Chy.

Define $\Lambda_X \colon (X \times S) \times S \to X \times S$ by $\Lambda_X((x,s),t) = (\lambda(x,t),s)$ and $\lambda_B \colon B(X,S) \times S \to B(X,S)$ by $\lambda_B(\theta_X(x,s),t) = \theta_X(\lambda(x,t),s)$. It is not hard to verify that these actions are valid and that the diagram below commutes.

$$\begin{CD} (X \times S) \times S @>{\Lambda_X}>> X \times S \\ @V{\theta_X \times \mathrm{id}_S}VV @VV{\theta_X}V \\ B(X,S) \times S @>{\lambda_B}>> B(X,S) \end{CD}$$

Since λ is Cauchy-continuous, it follows that Λ_X is Cauchy continuous, and since θ_X is a quotient map in Chy and Chy is cartesian closed, $\theta_X \times \mathrm{id}_S$ is also a quotient map in Chy. It follows from the diagram above that λ_B is Cauchy continuous, thus proving that $B(X,S)$ is a Cauchy S-space with Cauchy structure $\mathcal{C}_X$ and action λ_B. We will call the $B(X,S)$ the *generalized quotient* of (X,S,λ).

Theorem 4.1. *Let (X,S,λ) be a object in $\mathsf{GQ}_{\mathsf{Chy}}$ and let $B(X,S)$ be its generalized quotient. Let $((Y,\mu),f)$ be a strict regular remainder-invariant S-completion of (X,λ). If S is complete, then the generalized quotient $(B(Y,S),h)$ is an S-completion of $B(X,S)$, where $h \colon B(X,S) \to B(Y,S)$ is defined by $h \circ \theta_X = \theta_Y \circ (f \times \mathrm{id}_S)$.*

Proof. Observe that h is well defined. Consider the following commutative diagram.

$$\begin{CD} X \times S @>{\theta_X}>> B(X,S) \\ @V{f \times \mathrm{id}_S}VV @VV{h}V \\ Y \times S @>{\theta_Y}>> B(Y,S) \end{CD}$$

We now prove that h is an injection. If $\theta_X(x_1,s_1) \neq \theta_X(x_2,s_2)$, then $\lambda(x_1,s_2) \neq \lambda(x_2,s_1)$, hence $\mu(f(x_1),s_2) = f(\lambda(x_1,s_2)) \neq f(\lambda(x_2,s_1)) = \mu(f(x_2),s_1)$, hence $(h \circ \theta_X)(x_1,s_1)) = \theta_Y(f(x_1),s_1) \neq \theta_Y(f(x_2),s_2) = (h \circ \theta_X)(x_2,s_2)$.

Since θ_X is a quotient map in Chy and $\theta_Y \circ (f \times \mathrm{id}_S) = h \circ \theta_X$ is Cauchy continuous, it follows that h is Cauchy continuous.

Next, we prove that h is an embedding in Chy. Since (Y,S,μ) is an object in $\mathsf{GQ}_{\mathsf{Lim}}$ and since Y is Hausdorff, it follows by [2, Theorem 4.1] that $B(Y,S)$ is a Hausdorff limit space. Since S and Y are complete, $Y \times S$ is complete and can therefore be regarded as a reciprocal limit space. Since $B(Y,S)$ is a Hausdorff (hence reciprocal) limit space and θ_Y is a quotient map in Lim (and hence in Lim_R), it follows that θ_Y is a quotient map in Chy. Moreover, this means that $B(Y,S)$ is complete.

We now prove the following claim: If $(a,b) \in \theta_Y(f(x),s)$, then $a \in f(X)$. Since $\mu(a,s) = \mu(f(x),b) = f(\lambda(x,b)) \in f(X)$ and since Y is remainder invariant, $a \in f(X)$. This proves that $\theta_Y(f(x),s)$ is determined by $f(X)$ and S.

Let $\mathcal{H}$ be a filter on $B(X,S)$ such that $h^{\to}\mathcal{H}$ is a Cauchy filter on $B(Y,S)$. Since $B(Y,S)$ is complete, $h^{\to}\mathcal{H} \to \theta_Y(y,s)$ in $B(Y,S)$, so there exists a $\mathcal{K} \in \mathbf{F}(Y)$ and a $\mathcal{L} \in \mathbf{F}(S)$ and a $y_1 \in Y$ and an $s_1 \in S$ such that $\mathcal{K} \to y_1$ in Y and $\mathcal{L} \to s_1$ in S and $\theta_Y(y_1,s_1) = \theta_Y(y,s)$ and $h^{\to}\mathcal{H} \geq {\theta_Y}^{\to}(\mathcal{K} \times \mathcal{L})$. Since Y is a strict regular completion of X, there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\to}\mathcal{F} \to y_1$ in Y and $\mathcal{K} \geq \mathrm{cl}_Y f^{\to}\mathcal{F}$. Thus, $h^{\to}\mathcal{H} \geq {\theta_Y}^{\to}(\mathcal{K} \times \mathcal{L}) \geq {\theta_Y}^{\to}(\mathrm{cl}_Y f^{\to}\mathcal{F} \times \mathcal{L})$, and since $\theta_Y(f(X) \times S) \in h^{\to}\mathcal{H}$, it follows that $h^{\to}\mathcal{H} \geq {\theta_Y}^{\to}(f^{\to}(\mathrm{cl}_X \mathcal{F}) \times \mathcal{L}) = (\theta_Y \circ (f \times \mathrm{id}_S))^{\to}(\mathrm{cl}_X \mathcal{F} \times \mathcal{L}) = (h \times \theta_X)^{\to}(\mathrm{cl}_X \mathcal{F} \times \mathcal{L})$. However, since h is an injection, it follows that $\mathcal{H} \geq {\theta_X}^{\to}(\mathrm{cl}_X \mathcal{F} \times \mathcal{L}) \in \mathcal{C}_X$ since $\mathrm{cl}_X \mathcal{F}$ and $\mathcal{L}$ are Cauchy. This concludes the proof that h is an embedding. □

5. Conclusion

Let X be an adherence-restrictive limit S-space. Suppose X has a regular compactification in Lim. According to Theorem 3.1, there exists a largest strict regular compactification $((Y,r),f)$ of X. Let $\mathcal{C}$ be the Cauchy structure on X defined so that $\mathcal{F} \in \mathcal{C}$ if and only if $f^{\to}\mathcal{F}$ converges in Y, let S be a complete Cauchy monoid, and let λ be a continuous action of S on X making (X,S,λ) a limit S-space. By Theorem 3.3, there is an action μ of S on Y and a limit structure t on Y such that $(((Y,t),\mu),f)$ is the largest strict regular S-compactification of (X,λ) if and only if λ is a Cauchy continuous action on $(X,\mathcal{C})$. By Theorem 3.7, there exists a largest precompact adherence-restrictive Cauchy S-space whose $\mathsf{CA}_{\mathsf{Lim}_R}$-coreflection is isomorphic to (X,λ).

In the other direction, it is shown in [8, Theorem 3.2] that if X is an completely regular adherence-restrictive limit S-space, then X has a one-point strict regular remainder-invariant S-compactification if and only if X is locally compact. By Theorem 3.7, it follows that if X is a locally compact completely regular adherence-restrictive limit S-space, then there exists a precompact Cauchy S-space with a one-point remainder-invariant completion whose $\mathsf{CA}_{\mathsf{Lim}_R}$-coreflection is isomorphic to X.

Regular S-completions were discussed in §3. Using a similar construction, S-completions can be constructed by relaxing regularity somewhat. Assume that $(X,\mathcal{C})$ is a Hausdorff Cauchy space and let η be the collection of equivalence classes $[\mathcal{F}]$ such that $\mathcal{F}$ fails to converge. Define for each $A \subseteq X$, $\hat{A} = f(A) \cup (\Sigma A \cap \eta)$, where f is the canonical embedding of X in $\tilde{X}$. Note that $\widehat{A \cap B} \subseteq \hat{A} \cap \hat{B}$ and $\widehat{A \cup B} = \hat{A} \cup \hat{B}$ for all $A, B \subseteq X$. Given

$\mathcal{F} \in \mathbf{F}(X)$, let $\hat{\mathcal{F}}$ denote the filter on $\tilde{X}$ generated by $\{\hat{F}: F \in \mathcal{F}\}$. Note that $\hat{\mathcal{F}} \geq \Sigma\mathcal{F}$. Let us say that the Cauchy space X is *separated* if $\hat{\mathcal{K}} \vee \hat{L}$ fails to exist whenever $\mathcal{K} \vee \mathcal{L}$ fails to exist for every $\mathcal{K}, \mathcal{L} \in \mathcal{C}$. Suppose X is separated and let $\hat{\mathcal{C}}$ be the collection of filters $\mathcal{H}$ on $\tilde{X}$ such that $\mathcal{H} \geq \hat{\mathcal{F}}$ for some Cauchy filter $\mathcal{F}$ on X. Then $\hat{X} = (\tilde{X}, \hat{\mathcal{C}})$ is a Hausdorff Cauchy space and we can conclude the following result.

Theorem 5.1. *Let $((X, \mathcal{C}), \lambda)$ be a separated adherence-restrictive Cauchy S-space. Then*

(i) *$((\hat{X}, \tilde{\lambda}), f)$ is an S-completion of (X, λ), where f is the canonical embedding of X in $\tilde{X}$,*
(ii) *$\hat{X}$ is totally bounded whenever X is totally bounded,*
(iii) *$\hat{\mathcal{C}} = \tilde{\mathcal{C}}$ whenever X is regular, and*
(iv) *$(B(\hat{X}, S), S, \mu_B)$ is an S-completion of $(B(X, S), S, \lambda_B)$ whenever (X, S, λ) is an object in $\mathsf{GQ}_{\mathsf{Chy}}$ and S is complete.*

Proof. (i) Since X is separated, $(\hat{X}, f)$ is a completion of X in Chy. We now prove that if $A \subseteq X$ and $G \subseteq S$, then $\tilde{\lambda}(\hat{A} \times G) \subseteq \lambda\widehat{(A \times G)}$: If $[\mathcal{F}] \in \hat{A} \cap \eta$ and $s \in G$, then $A \in \mathcal{K}$ for some Cauchy filter $\mathcal{K}$ equivalent to $\mathcal{F}$, hence $\lambda(A \times G) \in \lambda^{\rightarrow}(\mathcal{K} \times \dot{s})$. Since $\operatorname{adh}_X \mathcal{K} = \emptyset$ and X is adherence restrictive, $\operatorname{adh}_X \lambda^{\rightarrow}(\mathcal{K} \times \dot{s}) = \emptyset$, hence $[\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})] = [\lambda^{\rightarrow}(\mathcal{K} \times \dot{s})] \in \tilde{X}$, hence $[\lambda^{\rightarrow}(\mathcal{F} \times \dot{s})] \in \lambda\widehat{(A \times G)}$, hence $\tilde{\lambda}(\hat{A} \times G) \subseteq \lambda\widehat{(A \times G)}$. Having proven this, it follows that $\tilde{\lambda}^{\rightarrow}(\hat{\mathcal{F}} \times \mathcal{G}) \geq \lambda^{\rightarrow}\widehat{(\mathcal{F} \times \mathcal{G})}$ for all Cauchy filters $\mathcal{F}$ on X and $\mathcal{G}$ on S, which proves that $\tilde{\lambda}$ is a Cauchy continuous action on $\hat{X}$ and that $((\hat{X}, \tilde{\lambda}), f)$ is an S-completion of (X, λ).

(ii) Let $\mathcal{H}$ be an ultrafilter on $\tilde{X}$. For each $[\mathcal{F}] \in \eta$, choose an ultrafilter $\mathcal{U}_{[\mathcal{F}]} \in [\mathcal{F}]$ and define $A^\sigma = f(A) \cup \{[\mathcal{F}]: A \in \mathcal{U}_{[\mathcal{F}]}\}$ for each $A \subseteq X$. Since $A^\sigma \cap B^\sigma = (A \cap B)^\sigma$ and $(A \cup B)^\sigma = A^\sigma \cup B^\sigma$ for every $A, B \subseteq X$, $\mathcal{H}_\sigma := \{A \subseteq X: A^\sigma \in \mathcal{H}\}$ is an ultrafilter on X. Let $(\mathcal{H}_\sigma)^\sigma$ denote the filter on $\tilde{X}$ generated by $\{A^\sigma: A \in \mathcal{H}_\sigma\}$. Then $\mathcal{H} \geq (\mathcal{H}_\sigma)^\sigma \geq \hat{\mathcal{H}}_\sigma$. Since X is totally bounded, $\mathcal{H}_\sigma$ is a Cauchy filter on X and so $\hat{\mathcal{H}}_\sigma$ is a Cauchy filter on $\hat{X}$, which means $\mathcal{H}$ is a Cauchy filter on $\hat{X}$, which means $\hat{X}$ is totally bounded.

(iii) Let $\mathcal{H}$ be a Cauchy filter on $\tilde{X}$. Then $\mathcal{H} \geq \Sigma\mathcal{F}$ for some Cauchy filter $\mathcal{F}$ on X. Observe that $f(\operatorname{cl}_X A) \cup \hat{A} = \Sigma A$ for each $A \subseteq X$, hence $\Sigma\mathcal{F} = f^{\rightarrow}(\operatorname{cl}_X \mathcal{F}) \cap \hat{\mathcal{F}}$; and since X is regular, $\Sigma\mathcal{F}$ is a Cauchy filter on $\hat{X}$, which proves that $\hat{\mathcal{C}} = \tilde{\mathcal{C}}$.

(iv) The argument here follows from the argument in the proof of Theorem 4.1. ∎

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