http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 33–43

http://topology.nipissingu.ca/tp/

Several Remarks on the Combinatorial Hodge Star

by

Masaharu Tanabe

Electronically published on April 4, 2014

Topology Proceedings

Web: http://topology.auburn.edu/tp/

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 46 (2015)
Pages 33-43

E-Published on April 4, 2014

SEVERAL REMARKS ON THE COMBINATORIAL HODGE STAR

MASAHARU TANABE

Abstract. In 2007, Scott O. Wilson defined the combinatorial Hodge star operator $\bigstar$ on cochains of a triangulated manifold. This operator depends on the choice of a cochain inner product, but he showed that for a certain inner product it converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero. He also stated that $\bigstar^2 \neq \pm\mathrm{Id}$ in general and raised a question if $\bigstar^2$ approaches $\pm\mathrm{Id}$. In this paper, we solve this problem affirmatively. We also give a remark about the definition of holomorphic 1-cochains given by Wilson using $\bigstar$.

1. Introduction

For cochains equipped with an inner product, Scott O. Wilson defined the combinatorial Hodge star operator $\bigstar$ in [7]. The definition is formally analogous to that of the smooth Hodge star operator on differential forms of a Riemannian manifold. By using a certain cochain inner product which he named the Whitney inner product, he showed that the combinatorial Hodge star operator, defined on the simplicial cochains of a triangulated Riemannian manifold, converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero. A precise statement for this is Theorem 2.12 and in more detailed form Theorem 2.13. These theorems are given by using a map from cochains into differential forms, defined by Hassler Whitney [6]. Jozef Dodziuk [2] and Dodziuk and V. K. Patodi [3] stated the approximation properties of this map. These are Theorem 2.6 and Theorem 2.7.

2010 *Mathematics Subject Classification.* Primary 57R12; Secondary 58A10, 58A14.

Key words and phrases. cochains, Hodge star.

Wilson [7] also stated that $\bigstar^2 \neq \pm\mathrm{Id}$, in general, is contrary to the case of smooth Hodge star and raised a question if $\bigstar^2$ approaches $\pm\mathrm{Id}$. For the circle S^1 with the standard metric, he computed the operator $\bigstar$ explicitly and suggested that this is the case. In this paper, we solve this problem affirmatively. We will see this in Theorem 3.1 and Theorem 3.2.

In §2 we recall some facts about cochains and differential forms and describe previous results concerning our work.

In §3 we show approximation theorems for $\bigstar^2$ which state that $\bigstar^2$ approaches $\pm\mathrm{Id}$.

In §4 we consider the definition of holomorphic 1-cochains which was defined by Wilson in [8]. We will see that one additional assumption for a hermitian inner product is needed for such cochains to hold.

2. Preliminaries

We mainly follow the notation of [7] and [8]. Let M be a closed oriented C^∞ manifold of dimension n and K a C^∞ triangulation of M. We denote by $C^j(K)$ the simplicial j-cochains of K with values in $\mathbb{R}$, where $j = 0, 1, \cdots, n$. Given an ordering of the vertices, we have a coboundary operator

$$\delta : C^j \to C^{j+1}.$$

If the cochains $C(K) = \bigoplus_j C^j(K)$ are equipped with an inner product $\langle\,,\,\rangle$, then we can define the adjoint of δ denoted by δ^* as

$$\langle \delta^* \sigma,\, \tau\rangle = \langle \sigma,\, \delta\tau\rangle.$$

Definition 2.1. The combinatorial Laplacian is defined to be $\blacktriangle = \delta^*\delta + \delta\delta^*$, and the space of harmonic j-cochains of K is defined to be

$$\mathcal{H}C^j(K) = \{a \in C^j \,|\, \blacktriangle a = \delta a = \delta^* a = 0\}.$$

Beno Eckmann [4] showed the following theorem which states that any inner product in cochains brings about the combinatorial Hodge decomposition.

Theorem 2.2. *Let (C, δ) be a finite dimensional complex with inner product and induced adjoint δ^*. There is an orthogonal direct sum decomposition*

$$C^j(K) = \delta C^{j-1} \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K)$$

and $\mathcal{H}C^j(K) \cong H^j(K)$, the cohomology of (C, δ) in degree j.

Suppose M is a Riemannian manifold. The Riemannian metric provides the space $\Lambda(M) = \bigoplus \Lambda^j(M)$ of smooth differential forms on M with an inner product

$$\langle \omega,\, \psi\rangle = \int_M \omega \wedge \star\psi, \quad \omega,\, \psi \in \Lambda(M),$$

where $\star$ is the Hodge star operator. We denote by $L^2\Lambda^j$ the completion of $\Lambda^j(M)$ with respect to this inner product.

We now define the Whitney map W of $C^j(K)$ into $L^2\Lambda^j$. To do so we identify K with M and fix some ordering of the set of vertices of K. We denote by μ_k the barycentric coordinate corresponding to the kth vertex p_k in K. Since K is a finite complex, we can identify chains and cochains and write every cochain $c \in C^j(K)$ as the sum $c = \sum c_\tau \cdot \tau$ with $c_\tau \in \mathbb{R}$ and τ running through all j-simplexes $[p_0, p_1, \cdots, p_j]$ whose vertices form an increasing sequence with respect to the ordering of K. Now we define $W\tau$ for such simplexes τ.

Definition 2.3. Let $\tau = [p_0, p_1, \cdots, p_j]$, where $p_0, p_1, \cdots, p_j$ is an increasing sequence of vertices of K. Define $W\tau \in L^2\Lambda^j$ by the formula

$$W\tau = j! \sum_{i=0}^{j} (-1)^i \mu_i \, d\mu_0 \wedge d\mu_1 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j,$$

where $\widehat{\ }$ over a symbol means deletion.

We extend W by linearity to all of $C^j(K)$ and call it the *Whitney map*. Several properties of W are found in [6], including the following.

Lemma 2.4. *The following hold:*

(1) $W\tau = 0$ *on* $M \setminus \overline{St(\tau)}$,
(2) $dW = W\delta$,

where St denotes the open star and the bar denotes closure.

Using the Whitney map W, we can define an inner product on $C(K)$ which is induced by the inner product on differential forms, namely, for σ, $\tau \in C(K)$

$$\langle \sigma, \tau \rangle = \int_M W\sigma \wedge \star W\tau = \langle W\sigma, W\tau \rangle,$$

where we use the same notation $\langle\, ,\, \rangle$ for the inner product on $C(K)$ and for the inner product on $L^2\Lambda$. We call this inner product on C the *Whitney inner product*. Dodzuik [2] proved that the Whitney inner product on C is non-degenerate.

There is a map in the converse direction, the de Rham map R from differential forms to $C(K)$ which is given by integration; i.e., for any differential form ω and chain c, we have

$$R\omega(c) = \int_c \omega.$$

One can check that $RW = \mathrm{Id}$; see [6]. In general, $WR \neq \mathrm{Id}$, but Dodziuk and Patodi [3] showed that WR is approximately equal to the identity.

Definition 2.5. Let K be a triangulation of a Riemannian manifold M. The mesh $\eta = \eta(K)$ of a triangulation is

$$\eta = \sup r(p, q),$$

where r means the geodesic distance in M and the supremum is taken over all the pair of vertices p, q of a 1-simplex in K.

Theorem 2.6. *Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that*

$$\|\omega - WR\omega\| \leq C \cdot \|(\mathrm{Id} + \Delta)^m \omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

For the comparison of the Hodge decomposition of a smooth form ω and the combinatorial Hodge decomposition of $R\omega$, we have the following.

Theorem 2.7. *Let $\omega \in \Lambda^j(M)$ and $R\omega \in C^j(K)$ have Hodge decompositions*

$$\begin{aligned} \omega &= d\omega_1 + \omega_2 + d^*\omega_3, \\ R\omega &= \delta a_1 + a_2 + \delta^* a_3. \end{aligned}$$

Then

$$\begin{aligned} \|d\omega_1 - W\delta a_1\| &\leq C \cdot \|(\mathrm{Id} + \Delta)^m \omega\| \cdot \eta, \\ \|\omega_2 - Wa_2\| &\leq C \cdot \|(\mathrm{Id} + \Delta)^m \omega\| \cdot \eta, \\ \|d^*\omega_3 - W\delta^* a_3\| &\leq C \cdot \|(\mathrm{Id} + \Delta)^m \omega\| \cdot \eta, \end{aligned}$$

where C and m are independent of ω and K.

For details, see [2], [3], and [7].

Whitney [6] also defined a product operation on $C(K)$.

Definition 2.8. We define $\cup : C^j(K) \otimes C^k(K) \to C^{j+k}(K)$ by

$$\sigma \cup \tau = R(W\sigma \wedge W\tau).$$

We see easily that $\delta(\sigma \cup \tau) = \delta\sigma \cup \tau + (-1)^j \sigma \cup \delta\tau$ and $\sigma \cup \tau = (-1)^{jk} \tau \cup \sigma$.

Wilson defined the combinatorial Hodge star operator in [7] and showed that, for the Whitney inner product, this operator converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero. For the rest of this section, all of the statements are attributed to Wilson ([7], [8]).

Definition 2.9. Let K be a triangulation of a closed oriented manifold M of dimension n with simplicial cochains $C = \bigoplus_j C^j$. Let $\langle \, , \, \rangle$ be an inner product on C such that C^i is orthogonal to C^j for $i \neq j$. For $\sigma \in C^j$ we define $\bigstar\sigma \in C^{n-j}$ by

$$\langle \bigstar\sigma, \tau \rangle = (\sigma \cup \tau)[M],$$

where $[M]$ denotes the fundamental class of M. We call $\bigstar$ the combinatorial Hodge star operator.

Several properties of the combinatorial star operator are given below.

Lemma 2.10. *The following hold:*

(1) $\bigstar\delta = (-1)^{j+1}\delta^*\bigstar$, *i.e.,* $\bigstar$ *is a chain map.*
(2) *For* $\sigma \in C^j$ *and* $\tau \in C^{n-j}$, $\langle \bigstar\sigma, \tau\rangle = (-1)^{j(n-j)}\langle \sigma, \bigstar\tau\rangle$, *i.e.,* $\bigstar$ *is (graded) skew-adjoint.*
(3) $\bigstar$ *induces isomorphisms* $\mathcal{H}C^j(K) \to \mathcal{H}C^{n-j}(K)$ *on harmonic cochains.*

From now on, we work under the assumption that the inner product on C is the Whitney inner product unless otherwise mentioned. Let π denote the orthogonal projection of $L^2\Lambda^j$ onto the image of $C^j(K)$ under the Whitney map W.

Lemma 2.11. $W\bigstar = \pi \star W$.

This lemma is the key in showing the following theorem which states that $\bigstar$ converges to $\star$ as the mesh η tends to 0.

Theorem 2.12. *Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that*

$$\| \star \omega - W\bigstar R\omega\| \leq C \cdot \|(\mathrm{Id} + \Delta)^m\omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

This approximation respects the Hodge decompositions of $\Lambda(M)$ and $C(K)$.

Theorem 2.13. *Let M be a Riemannian manifold with triangulation K of mesh η. Let $\omega \in \Lambda^j(M)$ and $R\omega \in C^j(K)$ have Hodge decompositions*

$$\omega = d\omega_1 + \omega_2 + d^*\omega_3,$$
$$R\omega = \delta a_1 + a_2 + \delta^* a_3.$$

There exist a positive constant C and a positive integer m, independent of ω and K, such that

$$\| \star d\omega_1 - W\bigstar\delta a_1\| \leq C \cdot (\|(\mathrm{Id} + \Delta)^m\omega\| + \|(\mathrm{Id} + \Delta)^m d\omega_1\|) \cdot \eta,$$
$$\| \star \omega_2 - W\bigstar a_2\| \leq C \cdot (\|(\mathrm{Id} + \Delta)^m\omega\| + \|(\mathrm{Id} + \Delta)^m \omega_2\|) \cdot \eta,$$
$$\| \star d^*\omega_3 - W\bigstar\delta^* a_3\| \leq C \cdot (\|(\mathrm{Id} + \Delta)^m\omega\| + \|(\mathrm{Id} + \Delta)^m d^*\omega_3\|) \cdot \eta.$$

We remark that we can take a common integer m in Theorem 2.6, Theorem 2.7, Theorem 2.12, and Theorem 2.13.

Wilson defined holomorphic 1-cochains on Riemann surfaces by using the combinatorial Hodge star. Holomorphic 1-cochains have several properties analogous to holomorphic 1-forms. For details, see [8].

3. Approximation Theorems

Recall that the smooth Hodge star $\star$ on j-forms satisfies

$$\star^2 = (-1)^{j(n-j)}\mathrm{Id},$$

where n is the dimension of the manifold. Thus, by the next theorem, we see that $\bigstar^2$ converges to $\pm\mathrm{Id}$ as the mesh η of the triangulation tends to 0.

Theorem 3.1. *Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that*

$$\| \star^2\ \omega - W\bigstar^2 R\omega\| \leq C \cdot \|(\mathrm{Id}+\Delta)^m\omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

Proof. Using the triangle inequality and Lemma 2.11, we calculate

$$\begin{aligned} & \| \star^2\ \omega - W\bigstar^2 R\omega\| \\ \leq\ & \| \star^2\ \omega - \star^2 WR\omega\| + \| \star^2\ WR\omega - W\bigstar^2 R\omega\| \\ \leq\ & \| \omega - WR\omega\| + \| \star^2\ WR\omega - \star W\bigstar R\omega\| + \| \star W\bigstar R\omega - W\bigstar^2 R\omega\| \\ =\ & \| \omega - WR\omega\| + \| \star WR\omega - W\bigstar R\omega\| + \| \star W\bigstar R\omega - \pi \star W\bigstar R\omega\|. \end{aligned}$$

The first term is bounded by

$$\| \omega - WR\omega\| \leq C \cdot \|(\mathrm{Id}+\Delta)^m\omega\| \cdot \eta,$$

using the estimate in Theorem 2.6. For the second term, we have

$$\begin{aligned} \| \star WR\omega - W\bigstar R\omega\| &\leq \| \star WR\omega - \star\omega\| + \| \star\omega - W\bigstar R\omega\| \\ &= \|WR\omega - \omega\| + \| \star\omega - W\bigstar R\omega\|, \end{aligned}$$

and these are bounded by

$$\begin{aligned} & \|WR\omega - \omega\| + \| \star\omega - W\bigstar R\omega\| \\ \leq\ & C \cdot \|(\mathrm{Id}+\Delta)^m\omega\| \cdot \eta + C \cdot \|(\mathrm{Id}+\Delta)^m\omega\| \cdot \eta, \end{aligned}$$

using the estimate in Theorem 2.6 and the estimate in Theorem 2.12. For the last term, we estimate

$$\begin{aligned}
& \| \star W \bigstar R\omega - \pi \star W \bigstar R\omega \| \\
\leq \ & \| \star W \bigstar R\omega - W R \star^2 \omega \| \\
\leq \ & \| \star W \bigstar R\omega - \star^2 \omega \| + \| \star^2 \omega - W R \star^2 \omega \| \\
= \ & \| W \bigstar R\omega - \star \omega \| + \| \omega - W R \omega \| \\
\leq \ & C \cdot \| (\mathrm{Id} + \Delta)^m \omega \| \cdot \eta + C \cdot \| (\mathrm{Id} + \Delta)^m \omega \| \cdot \eta. \qquad \square
\end{aligned}$$

The approximation also respects the Hodge decompositions of $\Lambda(M)$ and $C(K)$.

Theorem 3.2. *Let M be a Riemannian manifold with triangulation K of mesh η. Let $\omega \in \Lambda^j(M)$ and $R\omega \in C^j(K)$ have Hodge decompositions*

$$\begin{aligned}
\omega &= d\omega_1 + \omega_2 + d^*\omega_3, \\
R\omega &= \delta a_1 + a_2 + \delta^* a_3.
\end{aligned}$$

There exist a positive constant C and a positive integer m, independent of ω and K, such that

$$\begin{aligned}
\| \star^2 d\omega_1 - W \bigstar^2 \delta a_1 \| &\leq C \cdot (\| (\mathrm{Id} + \Delta)^m \omega \| + \| (\mathrm{Id} + \Delta)^m d\omega_1 \|) \cdot \eta, \\
\| \star^2 \omega_2 - W \bigstar^2 a_2 \| &\leq C \cdot (\| (\mathrm{Id} + \Delta)^m \omega \| + \| (\mathrm{Id} + \Delta)^m \omega_2 \|) \cdot \eta, \\
\| \star^2 d^*\omega_3 - W \bigstar^2 \delta^* a_3 \| &\leq C \cdot (\| (\mathrm{Id} + \Delta)^m \omega \| + \| (\mathrm{Id} + \Delta)^m d^*\omega_3 \|) \cdot \eta.
\end{aligned}$$

Proof. For the first statement, we calculate

$$\begin{aligned}
& \| \star^2 d\omega_1 - W \bigstar^2 \delta a_1 \| \\
\leq \ & \| \star^2 d\omega_1 - \star^2 W \delta a_1 \| + \| \star^2 W \delta a_1 - W \bigstar^2 \delta a_1 \| \\
\leq \ & \| d\omega_1 - W \delta a_1 \| + \| \star^2 W \delta a_1 - \star W \bigstar \delta a_1 \| \\
& \quad + \| \star W \bigstar \delta a_1 - W \bigstar^2 \delta a_1 \| \\
= \ & \| d\omega_1 - W \delta a_1 \| + \| \star W \delta a_1 - W \bigstar \delta a_1 \| \\
& \quad + \| \star W \bigstar \delta a_1 - \pi \star W \bigstar \delta a_1 \|.
\end{aligned}$$

The first term is bounded by using the first estimate in Theorem 2.7. For the second term, we have

$$\| \star W \delta a_1 - W \bigstar \delta a_1 \| \leq \| \star W \delta a_1 - \star d\omega_1 \| + \| \star d\omega_1 - W \bigstar \delta a_1 \|,$$

and these are bounded by using the first estimate in Theorem 2.7 and the first estimate in Theorem 2.13. For the last term, we estimate

$$\begin{aligned} \| \star W \bigstar \delta a_1 &- \pi \star W \bigstar \delta a_1 \| \leq \| \star W \bigstar \delta a_1 - W R \star^2 d\omega_1 \| \\ &\leq \| \star W \bigstar \delta a_1 - \star^2 d\omega_1 \| + \| \star^2 \ d\omega_1 - W R \star^2 d\omega_1 \| \\ &= \| W \bigstar \delta a_1 - \star d\omega_1 \| + \| d\omega_1 - W R\, d\omega_1 \|. \end{aligned}$$

These two terms are bounded by using the first estimate in Theorem 2.13 and the estimate in Theorem 2.6.

The same computations as above lead us to the last two inequalities by using the latter two inequalities in Theorem 2.7 and Theorem 2.13 and Theorem 2.6 applied to ω_2 and $d^*\omega_3$, respectively. □

Lieven Smits [5] showed that, on surfaces, $\|W\delta^* R\omega - d^*\omega\|$ converges to 0 for all C^∞ differential 1-form ω under a certain restriction on the triangulations. Recently, Smits's result was extended to arbitrary dimensions, showing that the above convergence is valid for arbitrary dimension under a certain mesh condition and showing that this mesh condition is necessary [1].

Wilson [7] observed that $\|W\delta R\omega - d\omega\|$ converges to 0, and he also observed that, in short,

$$\pm\delta^* \bigstar = \bigstar \delta \to \pm d^* \star = \star d.$$

He also raised a question if either of $\delta \bigstar$ or $\bigstar \delta^*$ provide a good approximation to $d\star$ or $\star d^*$, respectively. This question seems to be still open.

4. On the Definition of Holomorphic Cochains

In this section, we fix M to be a topological surface. Wilson [8] defined holomorphic cochains for surfaces using the combinatorial star operator. Then he introduced combinatorial period matrices which are the period matrices of holomorphic cochains and gave some (Riemann) bi-linear relations that the periods satisfy, and he proved that for a triangulated Riemannian 2-manifold (or a Riemann surface) and a particularly nice choice of inner product, the combinatorial period matrix converges to the (conformal) Riemann period matrix as the mesh of the triangulation tends to zero.

To define holomorphic 1-cochains, we need to extend some of our definitions to the case of complex valued cochains. Let $\langle\, ,\, \rangle$ be any hermitian inner product on the complex valued simplicial 1-cochains of a triangulation K for a topological surface M. We define the associated combinatorial star operator $\bigstar$ by

$$\langle \bigstar \sigma, \, \tau \rangle = (\sigma \cup \overline{\tau})[M],$$

where the bar denotes complex conjugation and $\cup$ is as in §2, extended over $\mathbb{C}$ linearly. Just as with real coefficients, the Hodge decomposition with complex coefficients holds

$$C^1(K) = \delta C^0(K) \oplus H^1(K) \oplus \delta^* C^2(K),$$

where H^1 is the space of complex valued harmonic 1-cochains.

By Lemma 2.10, $\bigstar$ induces an isomorphism of H^1 and is skew-adjoint. Since $\bigstar$ induces an isomorphism of H^1, this map admits a unique polar decomposition $\bigstar = HU$ where H is positive definite hermitian and U is unitary. Since $\bigstar$ is skew-adjoint, so is U, and therefore the eigenvalues of U are $\pm i$.

Wilson defined holomorphic 1-cochains as follows.

Definition 4.1 ([8, Definition 6.1])**.** Let K and $\langle\, , \rangle$ be as above. Let $\bigstar$ denote the map on complex valued harmonic cochains, as in Lemma 2.10, with polar decomposition $\bigstar = HU$. The subspace of holomorphic 1-cochains $\mathcal{H}^{1,0}(K)$ is defined to be

$$\mathcal{H}^{1,0}(K) = \{\omega \in H^1(K) |\, U\omega = -i\omega\}.$$

The subspace of anti-holomorphic 1-cochains $\mathcal{H}^{0,1}(K)$ is defined to be

$$\mathcal{H}^{0,1}(K) = \{\omega \in H^1(K) |\, U\omega = i\omega\}.$$

Subsequently, he remarked the following.

Remark 4.2 ([8, Remark 6.2])**.** An equivalent definition is to let $\mathcal{H}^{1,0}(K)$ be the span of the eigenvectors for non-positive imaginary eigenvalues of $\bigstar$ and let $\mathcal{H}^{0,1}(K)$ be the span of the eigenvectors for non-negative imaginary eigenvalues of $\bigstar$.

Then he stated the following lemma.

Lemma 4.3 ([8, Lemma 6.3])**.** *Let K be a triangulation of a surface M of genus g. A hermitian inner product on the simplicial* 1*-cochains of K gives an orthogonal direct sum decomposition*

$$H^1(K) = \mathcal{H}^{1,0}(K) \oplus \mathcal{H}^{0,1}(K).$$

Each summand on the right has complex dimension g and complex conjugation maps $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$ and vice versa.

As Wilson indicated, the decomposition follows from the property of skew-adjoint operators; that is, eigenspaces of distinct eigenvalues are orthogonal. However, complex conjugation does not map $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$ in general.

For example, let K be a triangulation of a torus M and let a and b be $\mathbb{R}$-valued harmonic 1-cochains which satisfy

$$(a \cup b)[M] = 1.$$

Then a and b form a basis for $H^1(K)$. We put

$$\sigma_1 = \sqrt{2}a + \frac{i}{\sqrt{2}}b, \quad \sigma_2 = \frac{1}{\sqrt{2}}b\,.$$

Let $\langle\, ,\, \rangle$ be a hermitian inner product for which σ_1, σ_2 is an orthonormal basis of $H^1(K)$. Then the matrix representation of $\bigstar$ with respect to the basis σ_1, σ_2 is

$$\begin{pmatrix} 2i & -1 \\ 1 & 0 \end{pmatrix},$$

the eigenvalues are $(1+\sqrt{2})i$ and $(1-\sqrt{2})i$, and the corresponding eigenvectors are constant multiples of $\begin{pmatrix} (1+\sqrt{2})i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} (1-\sqrt{2})i \\ 1 \end{pmatrix}$, respectively. Remark 4.2 defines

$$\mathcal{H}^{1,0}(K) = \left\{ c \begin{pmatrix} (1-\sqrt{2})i \\ 1 \end{pmatrix} ; \, c \in \mathbb{C} \right\}$$

and

$$\mathcal{H}^{0,1}(K) = \left\{ c \begin{pmatrix} (1+\sqrt{2})i \\ 1 \end{pmatrix} ; \, c \in \mathbb{C} \right\}.$$

We see that complex conjugation does *not* map $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$.

The main result of [8] is that the combinatorial period matrix converges to the (conformal) Riemann period matrix as the mesh of the triangulation tends to zero. To show this, Wilson used Riemann's bi-linear relations below (see [8, Theorem 6.5]). Let $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ be a canonical homology basis for M. (Here "canonical" means that the intersection of any two basis elements is non-zero only for a_j and b_j, in which case it equals one.)

Definition 4.4. For $h \in \mathcal{H}^{1,0}(K)$, the A-periods and B-periods of h are the following complex numbers:

$$A_j = h(a_j) \quad \text{and} \quad B_j = h(b_j) \quad for\ 1 \leq j \leq g.$$

Theorem 4.5 (Riemann's bi-linear relations)**.** *If σ, $\sigma' \in \mathcal{H}^{1,0}(K)$ have A-periods A_j, A'_j and B-periods B_j, B'_j, respectively, then*

$$\sum_{j=1}^{g} (A_j B'_j - B_j A'_j) = 0.$$

In the proof of Riemann's bi-linear relations, Wilson used the statement that complex conjugation maps $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$. Thus, to hold Wilson's results, the operator $\bigstar$ must be $\mathbb{R}$-valued on $\mathbb{R}$-cochains. Then, if τ is a $\mathbb{C}$-cochain, it is straightforward to check that $\bigstar\tau = i\lambda\tau$ $(\lambda \in \mathbb{R})$ implies $\bigstar\overline{\tau} = -i\lambda\overline{\tau}$, which means that complex conjugation maps $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$.

By the definition of the combinatorial star operator $\bigstar$

$$\langle \bigstar\sigma,\, \tau\rangle = (\sigma \cup \overline{\tau})[M];$$

$\bigstar$ is $\mathbb{R}$-valued on $\mathbb{R}$-cochains if and only if the hermitian inner product is $\mathbb{R}$-valued on $\mathbb{R}$-cochains. Thus, we need one additional assumption, that a hermitian inner product on the cochains to be $\mathbb{R}$-valued on $\mathbb{R}$-cochains. This assumption is natural and, of course, the Whitney inner product satisfies it.

Consequently, we offer the following definition.

Definition 4.6. Let $\langle\, ,\, \rangle$ be a hermitian inner product on the complex valued simplicial 1-cochains which is $\mathbb{R}$-valued on $\mathbb{R}$-cochains. We define $\mathcal{H}^{1,0}(K)$ to be the span of the eigenvectors for non-positive imaginary eigenvalues of $\bigstar$ and $\mathcal{H}^{0,1}(K)$ to be the span of the eigenvectors for non-negative imaginary eigenvalues of $\bigstar$.

Then complex conjugation maps $\mathcal{H}^{1,0}(K)$ to $\mathcal{H}^{0,1}(K)$, and all of the statements of Wilson's paper [8] hold.

References

[1] Douglas Arnold, Richard Falk, Johnny Guzmán, and Gantumur Tsogtgerel, *On the consistency of the combinatorial codifferential.* To appear in Transactions of the American Mathematical Society. Available at arXiv:1212.4472v2 [math.DG].

[2] Jozef Dodziuk, *Finite-difference approach to the Hodge theory of harmonic forms*, Amer. J. Math. **98** (1976), no. 1, 79–104.

[3] J. Dodziuk and V. K. Patodi, *Riemannian structures and triangulations of manifolds*, J. Indian Math. Soc. (N.S.) **40** (1976), no. 1-4, 1–52 (1977).

[4] Beno Eckmann, *Harmonische funktionen und randwertaufgaben in einem komplex*, Comment. Math. Helv. **17**, (1945), 240–255.

[5] Lieven Smits, *Combinatorial approximation to the divergence of one-forms on surfaces*, Israel J. Math. **75** (1991), no. 2-3, 257–271.

[6] Hassler Whitney, *Geometric Integration Theory.* Princeton, N. J.: Princeton University Press, 1957.

[7] Scott O. Wilson, *Cochain algebra on manifolds and convergence under refinement*, Topology Appl. **154** (2007), no. 9, 1898–1920.

[8] ———, *Conformal cochains*, Trans. Amer. Math. Soc. **360** (2008), no. 10, 5247–5264.

Department of Mathematics; Tokyo Institute of Technology; Ohokayama, Meguro, Tokyo, 152-8551, Japan

E-mail address: tanabe.m.ac@m.titech.ac.jp