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by

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A NOTE ON THE SET FUNCTIONS $\mathcal{T}$ AND $\mathcal{K}$

SERGIO MACÍAS

In Memoriam: Professor Julien Doucet

Abstract. We present results about the continuity of the set functions $\mathcal{T}$ and $\mathcal{K}$. We also give a sufficient condition for the idempotency of $\mathcal{T}$ on closed set and present an example showing the condition is not necessary.

1. Introduction

F. Burton Jones [5] defined the set functions $\mathcal{T}$ and $\mathcal{K}$ in 1948. Since then many properties related with this functions have been studied.

In 1970, David P. Bellamy [1] gave properties of continua for which the set function $\mathcal{T}$ is continuous. In [8] and [12], classes of decomposable nonlocally connected one-dimensional continua for which $\mathcal{T}$ is continuous are given. Further study of the continuity of the set function $\mathcal{T}$ may be found in [9], [10], [11], [13], and [3]. A study of the continuity of the set function $\mathcal{K}$ may be found in [15], [4], [11], and [3].

This paper is divided in six sections. After the section on definitions, in §3, we give a class of continua for which the set function $\mathcal{T}$ is continuous (Theorem 3.4) and give a partial positive answer to [2, Conjecture 23] (Theorem 3.5), which is known to have a negative answer [9]. We also answer a question posed by Isabel Puga in a private conversation with the author (Corollary 3.8). In §4, we present partial answers to [7, Question 7.2.11]: We prove that if X is a continuum for which $\mathcal{T}_X$ is continuous for X and Z is an open monotone image of X, then $\mathcal{T}_Z$ is continuous on continua for Z (Theorem 4.1). Also, we present a couple of consequences of

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this. In §5, we answer [11, Question 3.5]; i.e., we are able to remove from [11, Theorem 3.4] the hypothesis that for each element x of the continuum X, $\mathcal{T}(\{x\})$ is a terminal subcontinuum of X (Theorem 5.4). We also remove the same hypothesis from [11, Corollary 3.6] (Corollary 5.5). In §6, we present a miscellaneous collection of results, starting with results about the continuity of the set functions $\mathcal{T}$ and $\mathcal{K}$ (Theorem 6.2 and Corollary 6.5); we give a characterization of local connectedness of aposyndetic continua using the set function $\mathcal{T}$ (Theorem 6.8); we present a sufficient condition for the idempotency of $\mathcal{T}$ on closed sets (Theorem 6.9) and give an example showing that the condition is not necessary (Example 6.10).

2. Definitions

If (Z, d) is a metric space, then given a subset A of Z, the interior of A is denoted by $Int(A)$, the closure of A is denoted by $Cl(A)$, and, given a positive number ε, $\mathcal{V}^d_\varepsilon(A)$ denotes the open ball about A of radius ε.

Given a metric space Z, a *decomposition* of Z is a family $\mathcal{G}$ of nonempty and mutually disjoint subsets of Z such that $\bigcup \mathcal{G} = Z$. A decomposition $\mathcal{G}$ of a metric space Z is said to be *upper semicontinuous* if the quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{G}$ is closed. The decomposition $\mathcal{G}$ is *continuous* provided that q is open and closed. A decomposition $\mathcal{G}$ is *monotone* provided that each element of $\mathcal{G}$ is connected.

A *continuum* is a compact connected metric space. A *subcontinuum* of a space Z is a continuum contained in Z. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum X is *hereditarily decomposable* provided that each nondegenerate subcontinuum of X is decomposable. A continuum is *indecomposable* if it is not decomposable.

A λ-*dendroid* is a hereditarily decomposable continuum X such that the intersection of every pair of its subcontinua is connected. A *dendroid* is an arcwise connected λ-dendroid. A *dendrite* is a locally connected dendroid.

A subcontinuum Y of a continuum X is *terminal* provided that for each subcontinuum K of X such that $K \cap Y \neq \emptyset$, we have that either $K \subset Y$ or $Y \subset K$.

Let X be a continuum. A subcontinuum W of X is a *continuum domain of* X if $W = Cl(Int(W))$. A continuum domain W of X is a *strong continuum domain of* X if $Int(W)$ is connected.

Given a continuum X, we consider the following *hyperspaces* of X:

$$2^X = \{A \subset X \mid A \text{ is nonempty and closed}\},$$

$$\mathcal{C}(X) = \{A \in 2^X \mid A \text{ is connected}\},$$

and

$$\mathcal{F}_1(X) = \{\{x\} \mid x \in X\}.$$

These spaces are topologized with the Hausdorff metric defined as follows:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\};$$

$\mathcal{H}$ always denotes the Hausdorff metric on 2^X. Since 2^X is a continuum [16, Theorem (1.13)], we may consider the hyperspace 2^{2^X} with the Hausdorff metric $\mathcal{H}_2$ induced by the Hausdorff metric $\mathcal{H}$.

If X and Y are continua and $f\colon X \to Y$ is a map, then we define the *induced maps* $2^f\colon 2^X \to 2^Y$ and $\mathcal{C}(f)\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ by $2^f(A) = f(A)$ and $\mathcal{C}(f)(A) = f(A)$. Note that 2^f and $\mathcal{C}(f)$ are both continuous [7, Theorem 1.8.22 and Corollary 1.8.23].

Given a continuum X, we define the set function $\mathcal{T}$ as follows: If A is a subset of X, then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in Int(W) \subset W \subset X \setminus A\}.$$

A continuum X is *aposyndetic* provided that $\mathcal{T}(\{p\}) = \{p\}$ for every $p \in X$.

Given a continuum X, we define the set function $\mathcal{K}$ as follows: If A is a subset of X, then

$$\mathcal{K}(A) = \bigcap\{W \mid W \in \mathcal{C}(X) \text{ and } A \subset Int_X(W)\}.$$

Let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$. We write $\mathcal{L}_X$ if there is a possibility of confusion. Let us observe that for any subset A of X, $\mathcal{L}(A)$ is a closed subset of X and $A \subset \mathcal{L}(A)$.

Let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$. We say that $\mathcal{L}$ is *continuous for a continuum* X if $\mathcal{L}\colon 2^X \to 2^X$ is continuous. We say that $\mathcal{L}$ is *continuous on continua for* X if $\mathcal{L}|_{\mathcal{C}(X)}\colon \mathcal{C}(X) \to 2^X$ is continuous. We say that $\mathcal{L}$ is *continuous on singletons for* X provided that $\mathcal{L}|_{\mathcal{F}_1(X)}\colon \mathcal{F}_1(X) \to 2^X$ is continuous.

Let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$. The continuum X is *point $\mathcal{L}$-symmetric* provided that for every pair of points x_1 and x_2 of X, $x_1 \in \mathcal{L}(\{x_2\})$ if and only if $x_2 \in \mathcal{L}(\{x_1\})$. X is *$\mathcal{L}$-symmetric* if, for each pair of elements A and B of 2^X, $A \cap \mathcal{L}(B) \neq \emptyset$ if and only if $\mathcal{L}(A) \cap B \neq \emptyset$. X is *$\mathcal{L}$-additive* provided that, for every two elements A and B of 2^X, $\mathcal{L}(A \cup B) = \mathcal{L}(A) \cup \mathcal{L}(B)$.

Let X be a continuum and let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$. We say that $\mathcal{L}$ is *idempotent* provided that $\mathcal{L}(\mathcal{L}(A)) = \mathcal{L}^2(A) = \mathcal{L}(A)$ for every subset A of X. We say that $\mathcal{L}$ is *idempotent on closed sets* if $\mathcal{L}^2(A) = \mathcal{L}(A)$ for every element $A \in 2^X$. We say that $\mathcal{L}$ is *idempotent on singletons* provided that $\mathcal{L}^2(\{x\}) = \mathcal{L}(\{x\})$ for each $x \in X$.

3. Continuity of $\mathcal{T}$

We give a class of continua for which the set function $\mathcal{T}$ is continuous (Theorem 3.4) and give a partial positive answer to [2, Conjecture 23] (Theorem 3.5), which is known to have a negative answer [9]. We also answer a question posed by Puga in a private conversation with the author (Corollary 3.8).

The following result is [14, Theorem 3.10].

Theorem 3.1. *Let X be an aposyndetic continuum. If $\dim(\mathcal{C}(X)) < \infty$, then X is locally connected.*

Theorem 3.2. *Let X be a continuum and let $\mathcal{G}$ be a monotone continuous decomposition of X. Then there exists an embedding of $\mathcal{C}(X/\mathcal{G})$ into $\mathcal{C}(X)$.*

Proof. Let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Note that q is open and monotone. Let $\xi\colon \mathcal{C}(X/\mathcal{G}) \to \mathcal{C}(X)$ be given by $\xi(\Gamma) = q^{-1}(\Gamma)$. Since q is monotone, ξ is well defined. Since q is open, by [7, Theorem 1.8.24], ξ is continuous. We show that ξ is one-to-one. Let Γ_1 and Γ_2 be two distinct elements of $\mathcal{C}(X/\mathcal{G})$. Without loss of generality, we assume that there exists $\chi \in \Gamma_1 \setminus \Gamma_2$. This implies that $q^{-1}(\chi) \subset q^{-1}(\Gamma_1) \setminus q^{-1}(\Gamma_2)$. Hence, $q^{-1}(\Gamma_1) \neq q^{-1}(\Gamma_2)$. Thus, $\xi(\Gamma_1) \neq \xi(\Gamma_2)$. Therefore, ξ is an embedding. □

Corollary 3.3. *Let X be a continuum and let $\mathcal{G}$ be a monotone continuous decomposition of X. Then $\dim(\mathcal{C}(X/\mathcal{G})) \leq \dim(\mathcal{C}(X))$.*

A decomposition $\mathcal{G}$ of a continuum X is *terminal* if each element of $\mathcal{G}$ is a terminal subcontinuum of X.

Theorem 3.4. *Let X be a continuum and let $\mathcal{G}$ be a terminal continuous decomposition of X. If $X/\mathcal{G}$ is an aposyndetic continuum and $\dim(\mathcal{C}(X)) < \infty$, then $\mathcal{T}$ is continuous for X.*

Proof. Since $\dim(\mathcal{C}(X)) < \infty$, by Corollary 3.3, $\dim(\mathcal{C}(X/\mathcal{G})) < \infty$. Hence, since $X/\mathcal{G}$ is aposyndetic, by Theorem 3.1, $X/\mathcal{G}$ is locally connected. Let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Since $\mathcal{G}$ is a continuous terminal decomposition of X, q is monotone, open, and, for every proper subcontinuum W of X, $q(W) \neq X/\mathcal{G}$. Therefore, by [7, Theorem 3.2.2], $\mathcal{T}$ is continuous for X. □

A continuum X is said to be *homogeneous* provided that, for each pair of points x_1 and x_2 of X, there exists a homeomorphism $h\colon X \twoheadrightarrow X$ such that $h(x_1) = x_2$.

Theorem 3.5. *If X is a homogeneous continuum such that $\dim(\mathcal{C}(X)) < \infty$, then $\mathcal{T}$ is continuous for X.*

Proof. If X is indecomposable, then $\mathcal{T}$ is a constant map [7, Theorem 3.1.34]. Hence, $\mathcal{T}$ is continuous.

If X is an aposyndetic continuum, then the theorem follows from Theorem 3.1 and the fact that the set function $\mathcal{T}$ is the identity map on locally connected continua [7, Theorem 3.1.31].

Suppose X is not aposyndetic. Then, by Jones's Aposyndetic Decomposition Theorem (see [7, Theorem 5.1.19]), $\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is a terminal continuous decomposition of X such that $X/\mathcal{G}$ is an aposyndetic continuum. Therefore, by Theorem 3.4, $\mathcal{T}$ is continuous for X. □

Remark 3.6. Note that if X is a continuum such that $\dim(\mathcal{C}(X)) < \infty$, then $\dim(X) = 1$ [6, Theorem 2.1]. Hence, Theorem 3.5 gives a partial positive answer to [2, Conjecture 23 (Nadler-Bellamy)]. In fact, Bellamy wrote,

> Let X be a homogeneous one-dimensional continuum. Then $\mathcal{T}$ is continuous for X.[1]

We give in [9] a negative answer to this conjecture and a characterization of the class of homogeneous continua for which $\mathcal{T}$ is continuous.

In a private conversation, Puga asked the author if the set function $\mathcal{T}$ can be continuous on a dendroid. We answer the question for a wider class of continua in Corollary 3.8. To this end, we need the following theorem.

Theorem 3.7. *Let X be a point $\mathcal{T}$-symmetric, hereditarily decomposable continuum. Then $\mathcal{T}$ is continuous for X if and only if X is a locally connected continuum.*

Proof. If X is a locally connected continuum, then the set function $\mathcal{T}$ is the identity map on 2^X, by [7, Theorem 3.1.31]. Hence, $\mathcal{T}$ is continuous.

Suppose X is a point $\mathcal{T}$-symmetric hereditarily decomposable continuum such that $\mathcal{T}$ is continuous for X. If we prove that X is aposyndetic, by [7, Corollary 3.2.16], we have that X is locally connected.

Assume X is not aposyndetic. Since X is point $\mathcal{T}$-symmetric, by [10, Theorem 3.8], $\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is a continuous decomposition of X. Since X is not aposyndetic, there exists point x_0 in X such that $\mathcal{T}(\{x_0\})$ is a nondegenerate continuum [7, Theorem 3.1.21]. Thus, by the continuity of $\mathcal{T}$, there exists an open subset U of X such that $x_0 \in U$ and $\mathcal{T}(\{x\})$ is nondegenerate for all $x \in U$. Hence, by [10, Theorem 3.8], there exists $x_1 \in U$ such that $\mathcal{T}(\{x_1\})$ is an indecomposable continuum, a contradiction. Therefore, X is aposyndetic and, then, locally connected. □

[1] Bellamy proposed this conjecture at the VI Joint Meeting AMS–SMM, celebrated May 13-15, 2004, in Houston, Texas, during his talk *Problems, in and out of context.*

Corollary 3.8. *Let X be a λ-dendroid. Then $\mathcal{T}$ is continuous for X if and only if X is a dendrite.*

Proof. Suppose $\mathcal{T}$ is continuous for X. Since X is hereditarily unicoherent, by [7, Theorem 3.1.45], X is $\mathcal{T}$-additive. Hence, by [7, Corollary 3.2.15], X is point $\mathcal{T}$-symmetric. The corollary now follows from Theorem 3.7. □

4. Continuity of $\mathcal{T}$ on Continua

In this section, we present partial answers to [7, Question 7.2.11].

Theorem 4.1. *Let X be a continuum for which $\mathcal{T}_X$ is continuous on continua and let Z be a continuum. If $f\colon X \twoheadrightarrow Z$ is a surjective, monotone, and open map, then $\mathcal{T}_Z$ is continuous on continua.*

Proof. Since f is surjective and monotone, $\mathcal{T}_Z(B) = f\mathcal{T}_X f^{-1}(B)$ for all subsets B of Z [7, Theorem 3.1.64(c)]. In particular, the equality is true for all subcontinua B of Z. Let $\zeta\colon \mathcal{C}(Z) \to \mathcal{C}(X)$ be given by $\zeta(B) = f^{-1}(B)$. Since f is monotone, ζ is well defined. Since f is open, ζ is continuous [7, Theorem 1.8.24]. Hence, $\mathcal{T}_Z|_{\mathcal{C}(Z)} = \mathcal{C}(f) \circ \mathcal{T}_X|_{\mathcal{C}(X)} \circ \zeta$ and $\mathcal{T}_Z|_{\mathcal{C}(Z)}$ is the composition of three maps. Therefore, $\mathcal{T}_Z$ is continuous on continua for Z. □

Corollary 4.2. *Let X and Y be continua. If $\mathcal{T}_{X\times Y}$ is continuous on continua for $X \times Y$, then $\mathcal{T}_X$ and $\mathcal{T}_Y$ are continuous on continua for X and Y, respectively.*

Proof. The projection maps $\pi_X\colon X \times Y \twoheadrightarrow X$ and $\pi_Y\colon X \times Y \twoheadrightarrow Y$ are surjective, monotone, and open. The corollary now follows from Theorem 4.1. □

The following result may be found in [13, Theorem 6.5].

Theorem 4.3. *Let X be a point $\mathcal{T}_X$-symmetric decomposable continuum for which $\mathcal{T}_X$ is idempotent on singletons and $\mathcal{T}_X$ is continuous on singletons. Then $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a continuous decomposition of X such that the quotient space $X/\mathcal{G}$ is an aposyndetic continuum. Moreover, all the elements of $\mathcal{G}$ are nowhere dense in X and there exists a dense G_δ subset $\mathcal{W}$ of $X/\mathcal{G}$ such that if $q(z) \in \mathcal{W}$, then $\mathcal{T}_X(\{z\})$ is an indecomposable continuum, where $q\colon X \twoheadrightarrow X/\mathcal{G}$ is the quotient map.*

Corollary 4.4. *Let X be a continuum satisfying the hypothesis of Theorem 4.3 for which $\mathcal{T}_X$ is continuous on continua for X. Then $\mathcal{T}_{X/\mathcal{G}}$ is continuous on continua for $X/\mathcal{G}$.*

Proof. Let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Then q is monotone and open. The corollary now follows from Theorem 4.1. □

5. Continuity of $\mathcal{K}$

In Theorem 5.4, we answer [11, Question 3.5] in the positive; i.e., we are able to remove from [11, Theorem 3.4] the hypothesis that, for each element x of the continuum X, $\mathcal{T}(\{x\})$ is a terminal subcontinuum of X. We also remove the same hypothesis from [11, Corollary 3.6] (Corollary 5.5).

Theorem 5.1. *Let X be a continuum and let A be a subset of X. If $\mathcal{T}^2(A) = \mathcal{T}(A)$, then the components of $X \setminus \mathcal{T}(A)$ are open.*

Proof. If $A = \emptyset$ or $\mathcal{T}(A) = X$, then the result is clear. Let A be a nonempty subset of X such that $\mathcal{T}(A) \neq X$. Let L be a component of $X \setminus \mathcal{T}(A)$ and let $x \in L$. Since $\mathcal{T}^2(A) = \mathcal{T}(A)$, $x \in X \setminus \mathcal{T}^2(A)$. Hence, there exists a subcontinuum W of X such that $x \in Int(W) \subset W \subset X \setminus \mathcal{T}(A)$. Since L is a component of $X \setminus \mathcal{T}(A)$ and $L \cap W \neq \emptyset$, $W \subset L$. Thus, x is an interior point of L. Therefore, L is open. □

Theorem 5.2. *Let X be a continuum such that $\mathcal{T}$ is idempotent on closed sets. If $A \in 2^X$ and $x \in X \setminus \mathcal{T}(A)$, then there exists a strong continuum domain W of X such that $x \in Int(W) \subset W \subset X \setminus \mathcal{T}(A) \subset X \setminus A$.*

Proof. Let $A \in 2^X$ and let $x \in X \setminus \mathcal{T}(A)$. Since $\mathcal{T}$ is idempotent on closed sets, we have that $x \in X \setminus \mathcal{T}^2(A)$. By [7, Corollary 3.1.20], there exists an open subset U of X such that $\mathcal{T}(A) \subset U$ and $x \in X \setminus \mathcal{T}(Cl(U))$. Let L be a component of $X \setminus \mathcal{T}(Cl(U))$ containing x. Since $\mathcal{T}$ is idempotent on closed sets, by Theorem 5.1, L is open in X. Let $W = Cl(L)$. Then W is a strong continuum domain of X such that $x \in Int(W) \subset W \subset X \setminus U \subset X \setminus \mathcal{T}(A)$. □

Theorem 5.3. *Let X be continuum for which $\mathcal{T}$ is continuous and let $A \in \mathcal{C}(X)$. If $x \in X \setminus \mathcal{K}(A)$, then there exists a strong continuum domain W of X such that $x \in X \setminus W$ and $A \subset Int_X(W)$.*

Proof. Let $x \in X \setminus \mathcal{K}(A)$. Then there exists a subcontinuum W' of X such that $x \in X \setminus W'$ and $A \subset Int_X(W')$. This implies that $A \cap \mathcal{T}(\{x\}) = \emptyset$. Note that since $\mathcal{T}$ is continuous, then $\mathcal{T}$ is idempotent [7, Theorem 3.2.8]. Hence, by Theorem 5.2, for each $a \in A$, there exists a strong continuum domain W_a of X such that $a \in Int_X(W_a) \subset W_a \subset X \setminus \{x\}$. Note that $\{Int_X(W_a) \mid a \in A\}$ is an open cover of A. Since A is compact, there exist $a_1, \ldots, a_n \in A$ such that $A \subset \cup_{j=1}^{n} Int_X(W_{a_j})$. Observe that $U = \cup_{j=1}^{n} Int_X(W_{a_j})$ is an open connected subset of X. Let $W = Cl_X(U)$. Then W is a strong continuum domain of X, $x \in X \setminus W$, and $A \subset Int_X(W)$. □

Let us recall that if X is a point $\mathcal{T}$-symmetric continuum for which $\mathcal{T}$ is continuous, then $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a continuous decomposition of X [10, Theorem 3.8].

Theorem 5.4. *Let X be a point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous. If $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$, then $\mathcal{K}_X(A) = q^{-1}\mathcal{K}_{X/\mathcal{G}}q(A)$ for every $A \in 2^X$, where $q\colon X \twoheadrightarrow X/\mathcal{G}$ is the quotient map.*

Proof. Let $A \in 2^X$. Let $x \in X \setminus q^{-1}\mathcal{K}_{X/\mathcal{G}}q(A)$. Then $q(x) \in X/\mathcal{G} \setminus \mathcal{K}_{X/\mathcal{G}}q(A)$. Thus, there exists a subcontinuum $\mathcal{W}$ of $X/\mathcal{G}$ such that $q(x) \in X/\mathcal{G} \setminus \mathcal{W}$ and $q(A) \subset Int_{X/\mathcal{G}}(\mathcal{W})$. Hence, since q is monotone, $q^{-1}(\mathcal{W})$ is a subcontinuum of X such that $x \in q^{-1}(q(x)) \subset X \setminus q^{-1}(\mathcal{W})$ and $A \subset q^{-1}(q(A)) \subset Int_X\left(q^{-1}(\mathcal{W})\right)$. Therefore, $x \in X \setminus \mathcal{K}(A)$.

Next, let $x \in X \setminus \mathcal{K}(A)$. Then, by Theorem 5.3, there exists a strong continuum domain W of X such that $x \in X \setminus W$ and $A \subset Int_X(W)$. Note that, by [7, Corollary 3.2.15 and Theorem 3.2.13], $\mathcal{T}(W) = W$. As a consequence of this, since X is $\mathcal{T}$-additive [7, Theorem 3.2.13], we have that $W = \cup\{\mathcal{T}(\{w\}) \mid w \in W\}$. Thus, $\mathcal{T}_X(\{x\}) \cap W = \emptyset$. Hence, since q is continuous and open, $q(W)$ is a subcontinuum of $X/\mathcal{G}$ such that $q(x) \in X/\mathcal{G} \setminus q(W)$ and $q(A) \subset Int_{X/\mathcal{G}}(q(W))$. This implies that $q(x) \in X/\mathcal{G} \setminus \mathcal{K}_{X/\mathcal{G}}(q(A))$. Thus, $x \in q^{-1}(q(x)) \subset X \setminus q^{-1}\mathcal{K}_{X/\mathcal{G}}q(A)$. Therefore, $\mathcal{K}_X(A) = q^{-1}\mathcal{K}_{X/\mathcal{G}}q(A)$. □

Now, we are able to remove the hypothesis in [11, Corollary 3.6] that for each element x of the continuum X, $\mathcal{T}(\{x\})$ is terminal.

Corollary 5.5. *Let X be a point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous. Then $\mathcal{K}_X$ is continuous.*

Proof. Since $X/\mathcal{G}$ is a locally connected continuum [10, Theorem 3.8], $\mathcal{K}_{X/\mathcal{G}}$ is continuous [4, Theorem 36]. Let $\Im\colon 2^{X/\mathcal{G}} \to 2^X$ be given by $\Im(\Gamma) = q^{-1}(\Gamma)$. Since q is open, by [7, Theorem 1.8.24], $\Im$ is continuous. By [7, Theorem 1.8.22], 2^q is continuous. Since $\mathcal{K}_X(A) = q^{-1}\mathcal{K}_{X/\mathcal{G}}q(A)$ for every $A \in 2^X$ (Theorem 5.4), we have that $\mathcal{K}_X = \Im \circ \mathcal{K}_{X/\mathcal{G}} \circ 2^q$. Therefore, $\mathcal{K}_X$ is continuous. □

6. Miscellaneous Results

The first three results in this section are already known for the set function $\mathcal{T}$ [3, Lemma 6.3, Theorem 6.4, Corollary 6.5]. We include the details of the proofs for the convenience of the reader.

Lemma 6.1. *Let X be a continuum, let $A \in 2^X$, let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$, and let $\mathcal{A} = \{\mathcal{L}(\{a\}) \mid a \in A\}$. If $\mathcal{L}$ is continuous on singletons for X, then $\mathcal{A}$ is closed in 2^X.*

Proof. Let $B \in Cl_{2^X}(\mathcal{A})$. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points of A such that $\lim_{n\to\infty} \mathcal{L}(\{a_n\}) = B$. Since A is compact, without loss of generality, we assume that $\{a_n\}_{n=1}^{\infty}$ converges to a point $a \in A$. Since

$\mathcal{L}|_{\mathcal{F}_1(X)}$ is continuous, we have that $B = \lim_{n\to\infty} \mathcal{L}(\{a_n\}) = \mathcal{L}(\{a\})$. Therefore, $B \in \mathcal{A}$ and $\mathcal{A}$ is closed in 2^X. □

Theorem 6.2. *Let X be a continuum and let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$ be such that $\mathcal{L}(A) = \cup\{\mathcal{L}(\{a\}) \mid a \in A\}$ for each $A \in 2^X$. If $\mathcal{L}$ is continuous on singletons for X, then $\mathcal{L}$ is continuous for X.*

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ given by the uniform continuity of $\mathcal{L}|_{\mathcal{F}_1(X)}$.

Let $A, B \in 2^X$ be such that $\mathcal{H}(A, B) < \delta$. Let $\mathcal{A} = \{\mathcal{L}(\{a\}) \mid a \in A\}$ and $\mathcal{B} = \{\mathcal{L}(\{b\}) \mid b \in B\}$. By Lemma 6.1, $\mathcal{A}, \mathcal{B} \in 2^{2^X}$. Let $\mathcal{L}(\{a\}) \in \mathcal{A}$. Since $\mathcal{H}(A, B) < \delta$, there exists $b \in B$ such that $\mathcal{H}(\{a\}, \{b\}) < \delta$. Hence, by the choice of δ, $\mathcal{H}(\mathcal{L}(\{a\}), \mathcal{L}(\{b\})) < \varepsilon$. Thus, $\mathcal{A} \subset \mathcal{V}^{\mathcal{H}}_{\varepsilon}(\mathcal{B})$. Similarly, $\mathcal{B} \subset \mathcal{V}^{\mathcal{H}}_{\varepsilon}(\mathcal{A})$. Therefore, $\mathcal{H}_2(\mathcal{A}, \mathcal{B}) < \varepsilon$. By [16, Lemma (1.48)], $\mathcal{H}(\cup\mathcal{A}, \cup\mathcal{B}) \leq \mathcal{H}_2(\mathcal{A}, \mathcal{B}) < \varepsilon$.

By hypothesis, $\mathcal{L}(A) = \cup\mathcal{A}$ and $\mathcal{L}(B) = \cup\mathcal{B}$. Hence, we have proved that if $\mathcal{H}(A, B) < \delta$, then $\mathcal{H}(\mathcal{L}(A), \mathcal{L}(B)) < \varepsilon$. Therefore, $\mathcal{L}$ is continuous. □

As a consequence of [7, Theorem 3.1.43], [15, Theorem 3.11], and Theorem 6.2, we have the following.

Corollary 6.3. *Let X be an $\mathcal{L}$-additive continuum. If $\mathcal{L}$ is continuous on singletons for X, then $\mathcal{L}$ is continuous for X.*

Corollary 6.4. *Let X be an $\mathcal{L}$-additive continuum. If $\mathcal{L}$ is continuous on continua for X, then $\mathcal{L}$ is continuous.*

As a consequence of [7, Theorem 3.1.44], [15, Lemma 3.23], and Corollary 6.3, we obtain the following.

Corollary 6.5. *Let X be an $\mathcal{L}$-symmetric continuum. If $\mathcal{L}$ is continuous on singletons for X, then $\mathcal{L}$ is continuous for X.*

Remark 6.6. Let us observe that Corollary 6.5 for $\mathcal{T}$ is not true if we replace $\mathcal{T}$-symmetry with point $\mathcal{T}$-symmetry. The continuum M constructed by Janusz R. Prajs [17] is a point $\mathcal{T}$-symmetric continuum for which $\mathcal{T}$ is continuous on singletons for X, but $\mathcal{T}$ is not continuous [9, p. 3398]. Also note the following: If X is a homogeneous continuum, then X is colocally connected. Hence, $\mathcal{K}$ is the identity map on 2^X [4, Theorem 26]. In particular, $\mathcal{K}$ is continuous for X.

Lemma 6.7. *Let X be an aposyndetic continuum. If $A \in \mathcal{C}(X)$ and $x \in X \setminus A$, then there exists $W \in \mathcal{C}(X)$ such that $A \subset Int(W) \subset W \subset X \setminus \{x\}$.*

Proof. Let $A \in \mathcal{C}(X)$ and let $x \in X \setminus A$. Since X is aposyndetic, for each $a \in A$, there exists $W_a \in \mathcal{C}(X)$ such that $a \in Int(W_a) \subset W_a \subset X \setminus \{x\}$. Since A is compact, there exists $a_1, \ldots, a_n \in A$ such that $A \subset$

$\cup_{j=1}^{n} Int(W_{a_j})$. Let $W = \cup_{j=1}^{n} W_{a_j}$. Then W is a subcontinuum of X, $A \subset Int(W)$, and $x \in X \setminus W$. □

Theorem 6.8. *Let X be an aposyndetic continuum. If $\mathcal{T}(W) = W$ for each $W \in \mathcal{C}(X)$ such that $Int(W) \neq \emptyset$, then X is locally connected.*

Proof. Let $A \in \mathcal{C}(X)$ and let $x \in X \setminus A$. By Lemma 6.7, there exists $W \in \mathcal{C}(X)$ such that $A \subset Int(W) \subset W \subset X \setminus \{x\}$. By our assumption, $\mathcal{T}(W) = W$. Hence, by [7, Proposition 3.1.7], $\mathcal{T}(A) \subset \mathcal{T}(W) \subset X \setminus \{x\}$. Thus, $\mathcal{T}(A) = A$. Therefore, by [7, Theorem 3.1.32], X is locally connected. □

In a private conversation, Bellamy and the author agreed that the following result is true and Bellamy asked the author if the converse is true. We give an example showing that the converse is not true (Example 6.10).

Theorem 6.9. *Let X be a continuum. If for each $A \in 2^X$ and each $K \in \mathcal{C}(X)$ such that $Int(K) \neq \emptyset$ and $A \cap K = \emptyset$, there exists $W \in \mathcal{C}(X)$ such that $K \subset Int(W) \subset W \subset X \setminus A$, then $\mathcal{T}$ is idempotent on closed sets.*

Proof. Suppose that for each $A \in 2^X$ and each $K \in \mathcal{C}(X)$ such that $Int(K) \neq \emptyset$ and $K \cap A = \emptyset$, there exists $W \in \mathcal{C}(X)$ such that $K \subset Int(W) \subset W \subset X \setminus A$. Let $A \in 2^X$ and let $x \in X \setminus \mathcal{T}(A)$. Then there exists $K \in \mathcal{C}(X)$ such that $x \in Int(K) \subset K \subset X \setminus A$. By hypothesis, there exists $W \in \mathcal{C}(X)$ such that $K \subset Int(W) \subset W \subset X \setminus A$. This implies that $x \in X \setminus \mathcal{T}^2(A)$. Hence, $\mathcal{T}^2(A) \subset \mathcal{T}(A)$. By [7, Remark 3.1.5 and Proposition 3.1.7], $\mathcal{T}(A) \subset \mathcal{T}^2(A)$. Therefore, $\mathcal{T}$ is idempotent on closed sets. □

The next example shows that the converse of Theorem 6.9 is not true.

Example 6.10. Let $X = (\{0\} \times [-1, 2]) \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \mid x \in \left(0, \frac{\pi}{2}\right] \right\}$, let $K = \{0\} \times \left[0, \frac{3}{2}\right]$, and let $A = \{0\} \times \left[-\frac{1}{3}, -\frac{1}{2}\right]$. Then $\mathcal{T}$ is idempotent on closed sets; also, $Int(K) = \{0\} \times \left(1, \frac{3}{2}\right)$, $K \cap A = \emptyset$, and if W is a subcontinuum of X such that $K \subset Int(W)$, then $A \subset W$.

Recall that there is a characterization of the idempotency of $\mathcal{T}$ [7, Theorem 3.1.54]. Hence, we ask the following question.

Question 6.11. Does there exist a characterization of the idempotency of $\mathcal{T}$ on closed sets for continua?

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