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TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 67–72

http://topology.nipissingu.ca/tp/

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Electronically published on April 17, 2014

Topology Proceedings

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ISSN: 0146-4124

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TOPOLOGY PROCEEDINGS
Volume 46 (2015)
Pages 67-72

E-Published on April 17, 2014

ON THE STABILITY OF ORBITS FOR ITERATED FUNCTION SYSTEMS

ALIREZA ZAMANI BAHABADI

Abstract. In this paper, we consider the stability of orbits for iterated function systems. Precisely, we prove that there is a residual set $\mathcal{R} \subset \mathcal{H}(X)\times\mathcal{H}(X)$ such that for every $(f_0, f_1) \in \mathcal{R}$, $IFS(f_0, f_1)$ is weak orbitally stable, 1-inverse weak orbitally stable, and w-orbitally stable, where $w \in \Sigma^2$. As well, we show that for every $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$, $IFS(f_0, f_1)$ is 2-inverse weak orbitally stable.

1. Introduction

Stability is an important topic in the theory of dynamical systems. Nominally, this means that after a small perturbation, the dynamical invariants do not change too much. Another interesting notion of stability, due to Zeeman and Floris Takens, is tolerance stable (see [6], [7]): A diffeomorphism f is tolerance stable if the corresponding orbit's structure varies only a little after small perturbations. Zeeman's tolerance stability conjecture expresses that tolerance stable diffeomorphisms are generic in the space of all diffeomorphisms with C^1-topology. The tolerance stable is well known generically for homeomorphisms and, by it, the genericity of weak shadowing and weak inverse shadowing properties was obtained (see [5], [3]). The tolerance stable became a motivation for us to study the stability of orbits for iterated function systems. The study of iterated function systems plays an important role for understanding certain dynamical systems. It has also a remarkable role in producing fractals and chaos game [1]. In [2], David Broomhead, Demetris Hadjiloucas,

2010 *Mathematics Subject Classification.* 54H20.
Key words and phrases. Iterated function systems, orbitally stable, weak orbitally stable.
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and Matthew Nicol studied the orbit stability and stability of mixing of $T(x,y) = (f(x), g(x,y))$ under deterministic and random perturbation of g and proved that these systems are stable in the sense that for every $\epsilon > 0$ there is a pair of orbits of the perturbed and unperturbed system such that paired orbits stay within a distance of each other except for a fraction of time. Their findings have applications to the stability of iterated function systems. Krzysztof Lesniak in [4] studied the stability and invariance of multivalued iterated function systems. In this paper, we study the stability of orbits for iterated function systems generically. We prove that there exists a residual set $\mathcal{R} \subset \mathcal{H}(X) \times \mathcal{H}(X)$ such that for every $(f_0, f_1) \in \mathcal{R}$, $IFS(f_0, f_1)$ is weak orbitally stable, 1-inverse weak orbitally stable, and w-orbitally stable, where $w \in \Sigma^2$. We also show that for every $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$, $IFS(f_0, f_1)$ is 2-inverse weak orbitally stable. Let us mention some notation.

Let (X, d) denote a compact metric space and let $f : X \to X$ be a homeomorphism. The space of all homeomorphisms from X to itself, which is endowed with C^0-topology, is denoted by $\mathcal{H}(X)$.

Let Σ^2 be the space of one-sided infinite sequences over the alphabet $\{0, 1\}$, and let $\sigma : \Sigma^2 \to \Sigma^2$ be the Bernoulli shift. We equip Σ^2 with the metric $d_{\Sigma^2}(w, w') = 2^{-\min\{|n|;\ w_n \neq w'_n\}}$. For $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$, the iterated function system $IFS(f_0, f_1)$ on X generated by f_0 and f_1 is given by iterates $f_{i_1} \circ \cdots \circ f_{i_k}$ with $i_j \in \{0, 1\}$. For $x \in X$, the orbit of x for $IFS(f_0, f_1)$ is denoted by

$$\mathcal{O}_{(f_0,f_1)}(x) = \bigcup_{w \in \Sigma^2} \{f_w^n(x)\}_{n=0}^{\infty},$$

where, for $w = w_0 w_1 w_2 \cdots \in \Sigma^2$, $f_w^n(x) = f_{w_n} \circ \cdots \circ f_{w_1} \circ f_{w_0}(x)$.

We say that $IFS(f_0, f_1)$ is *orbitally stable* if, for every $\varepsilon > 0$, there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every ${f'}_0 \in \mathcal{U}_0$ and ${f'}_1 \in \mathcal{U}_1$ and for every $y \in X$, there exists a point $x \in X$ such that

$$\mathcal{O}_{({f'}_0,{f'}_1)}(y) \subset \mathcal{N}_\varepsilon\big(\mathcal{O}_{(f_0,f_1)}(x)\big) \tag{1.1}$$

and

$$\mathcal{O}_{(f_0,f_1)}(x) \subset \mathcal{N}_\varepsilon\big(\mathcal{O}_{({f'}_0,{f'}_1)}(y)\big). \tag{1.2}$$

We say that $IFS(f_0, f_1)$ is *weak orbitally stable* if only (1.1) holds.

For $w \in \Sigma^2$, we say that $IFS(f_0, f_1)$ is *w-orbitally stable* if, for every $\varepsilon > 0$, there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every ${f'}_0 \in \mathcal{U}_0$, ${f'}_1 \in \mathcal{U}_1$, and $y \in X$, there exists a point $x \in X$ in which $\{{f'}_w^n(y)\}_{n=0}^{\infty} \subset \mathcal{N}_\varepsilon(\{f_w^n(x)\}_{n=0}^{\infty})$ and $\{f_w^n(x)\}_{n=0}^{\infty} \subset \mathcal{N}_\varepsilon(\{{f'}_w^n(y)\}_{n=0}^{\infty}$.

A property P is said to be generic for elements of a topological space X if the set of points $x \in X$ satisfying P is residual; i.e., it includes a countable intersection of open and dense subsets of X.

2. Main Results

In this section, we prove the genericity of w-orbitally stable and weak orbitally stable. With the proposed methods in literature, one cannot show the genericity of orbitally stable. So the following problem arises.

Question 2.1. Is there a residual set $\mathcal{R} \subset \mathcal{H}(X) \times \mathcal{H}(X)$ such that, for every f_0 and $f_1 \in \mathcal{R}$, $IFS(f_0, f_1)$ is orbitally stable?

Theorem 2.2. *For $w \in \Sigma^2$, there exists a residual set $\mathcal{R}_w \subset \mathcal{H}(X) \times \mathcal{H}(X)$ such that, for every $(f_0, f_1) \in \mathcal{R}_w$, $IFS(f_0, f_1)$ is w-orbitally stable.*

Proof. Let $w \in \Sigma^2$ and $\varepsilon > 0$ be given. Since X is compact, we can find a finite covering $V_\varepsilon = \{V_i\}_{i=1}^k$ of X by open set with diameters less than ε. Put $T = \{1, 2, \cdots, k\}$. For $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$, let $\mathcal{I}_{(f_0,f_1)}$ be a subset of $P(T)$ consisting of subsets L of T for which there exists $x \in X$ such that, for all $i \in L$,

$$\{f_w^n(x)\}_{n=0}^\infty \cap V_i \neq \emptyset.$$

Let $\mathcal{R}_{V_\varepsilon}$ be the set of all $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$ such that there exist open neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every ${f'}_0 \in \mathcal{U}_0$ and ${f'}_1 \in \mathcal{U}_1$, we have $\mathcal{I}_{(f_0,f_1)} = \mathcal{I}_{({f'}_0,{f'}_1)}$. It is clear that $\mathcal{R}_{V_\varepsilon}$ is an open subset of $\mathcal{H}(X) \times \mathcal{H}(X)$. We show that $\mathcal{R}_{V_\varepsilon}$ is dense in $\mathcal{H}(X) \times \mathcal{H}(X)$. Suppose that $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$. If $L \in \mathcal{I}(f_0, f_1)$, then there exists $x \in X$ such that, for all $i \in L$,

$$\{f_w^n(x)\}_{n=0}^\infty \cap V_i \neq \emptyset.$$

So, for every $i \in L$, $f_w^{n_i}(x) \in V_i$ for some $n_i \in \mathbb{N}$.

Consider $N = \max\{n_i | \ i \in L\}$. Therefore, there exist open neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every $({f'}_0, {f'}_1) \in \mathcal{U}_0 \times \mathcal{U}_1$ and for all $i \in L$, we have

$$\{{f'}_w^j(x)\}_{j=0}^N \cap V_i \neq \emptyset.$$

Hence, $L \in \mathcal{I}_{({f'}_0,{f'}_1)}$.

Since $P(T)$ is finite, then we can find an open neighborhood $\mathcal{W}$ of (f_0, f_1) such that, for every $({f'}_0, {f'}_1) \in \mathcal{W}$, $\mathcal{I}_{(f_0,f_1)} \subset \mathcal{I}_{({f'}_0,{f'}_1)}$. We can continue the above process for $(f_0', {f'}_1)$ and find an open neighborhood $\mathcal{W}'$ of (f_0, f_1) such that, for every $(f_0'', f_1'') \in \mathcal{W}'$, $\mathcal{I}_{({f'}_0,{f'}_1)} \subset \mathcal{I}_{(f_0'',f_1'')}$. Since $P(T)$ is finite, by continuity of the above process, we can find $(g_0, g_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$ near (f_0, f_1) such that $(g_0, g_1) \in \mathcal{R}_{V_\varepsilon}$. Hence, $\mathcal{R}_w = \bigcap_{n=1}^\infty \mathcal{R}_{V_{\frac{1}{n}}}$ is the required residual set. Let $(f_0, f_1) \in \mathcal{R}_w$ and $\varepsilon > 0$ be given. So $(f_0, f_1) \in \mathcal{R}_{V_{\frac{1}{n}}}$, where $\frac{1}{n} < \varepsilon$. Hence, there exist open sets $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every $({f'}_0, {f'}_1) \in \mathcal{U}_0 \times \mathcal{U}_1$, we have $\mathcal{I}_{(f_0,f_1)} = \mathcal{I}_{({f'}_0,{f'}_1)}$. For any $y \in X$, suppose that $L \in \mathcal{I}_{({f'}_0,{f'}_1)}$ is such that for all $i \in L$,

$$\{{f'}^n_w(y)\}_{n=0}^{\infty} \cap V_i \neq \emptyset,$$

and

$$\{{f'}^n_w(y)\}_{n=0}^{\infty} \subset \bigcup_{i \in L} V_i.$$

Since $L \in \mathcal{I}_{(f_0,{f'}_1)}$, there exists a point $x \in X$ such that $\{f^n_w(x)\}_{n=0}^{\infty} \cap V_i \neq \emptyset$ for all $i \in L$.

This completes the proof. □

Theorem 2.3. *There exists a residual set* $\mathcal{R}_1 \subset \mathcal{H}(X) \times \mathcal{H}(X)$ *such that, for every* $(f_0, f_1) \in \mathcal{R}_1$, $IFS(f_0, f_1)$ *is weak orbitally stable.*

Proof. Let $w \in \Sigma^2$ be such that for any finite alphabet $(w'_0 w'_1 \cdots w'_n)$, there exists $l \in \mathbb{N}$ such that $\big(\sigma^l(w)\big)_i = w'_i$ for all $0 \leq i \leq n$. Let $\mathcal{R}_w$ be as in Theorem 2.2. We show that $\mathcal{R}_w$ is the required residual set. Suppose that $(f_0, f_1) \in \mathcal{R}_w$ and $\varepsilon > 0$ are given. Therefore, $(f_0, f_1) \in \mathcal{R}_{V_\varepsilon}$. Let $\mathcal{U}_0$ and $\mathcal{U}_1$ be open neighborhoods of f_0 and f_1 as in $\mathcal{R}_{V_\varepsilon}$. Let ${f'}^n_{w'}(y) \in \mathcal{O}_{({f'}_0,{f'}_1)}(y)$ where $y \in X$ and $({f'}_0, {f'}_1) \in \mathcal{U}_0 \times \mathcal{U}_1$. There exists $l \in \mathbb{N}$ such that $\big(\sigma^l(w)\big)_i = w'_i$ for $0 \leq i \leq n$. Choose $y' \in X$ such that $y = {f'}_{w_l} \circ \cdots \circ {f'}_{w_1} \circ {f'}_{w_0}(y')$ so ${f'}^n_{w'}(y) = {f'}^{n+l}_w(y')$. Therefore, there exists $x \in X$ such that ${f'}^{n+l}_w(y') \in \mathcal{N}_\varepsilon\big(\{f^n_w(x)\}_{n=0}^{\infty}\big)$. So we have

$${f'}^n_{w'}(y) \in \mathcal{N}_\varepsilon\big(\{f^n_w(x)\}_{n=0}^{\infty}\big) \subset \mathcal{N}_\varepsilon\big(\mathcal{O}_{(f_0,f_1)}(x)\big).$$

Hence, (f_0, f_1) is weak orbitally stable.

We say that $IFS(f_0, f_1)$ is 1-*inverse weak orbitally stable* if, for every $\varepsilon > 0$, there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0, and f_1, respectively, such that, for every ${f'}_0 \in \mathcal{U}_0$ and ${f'}_1 \in \mathcal{U}_1$ and every $x \in X$, there exists a point $y \in X$ such that

$$\mathcal{O}_{({f'}_0,{f'}_1)}(y) \subset \mathcal{N}_\varepsilon\big(\mathcal{O}_{(f_0,f_1)}(x)\big).$$

We define 2-inverse weak orbitally stable in a similar manner. Specifically, $IFS(f_0, f_1)$ is 2-*inverse weak orbitally stable* if, for every $\varepsilon > 0$, there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every ${f'}_0 \in \mathcal{U}_0$ and ${f'}_1 \in \mathcal{U}_1$ and every $x \in X$, there exists a point $y \in X$ such that

$$\mathcal{O}_{(f_0,f_1)}(x) \subset \mathcal{N}_\varepsilon\big(\mathcal{O}_{({f'}_0,{f'}_1)}(y)\big).$$ □

Theorem 2.4. *There exists a residual set* $\mathcal{R}_2 \subset \mathcal{H}(X) \times \mathcal{H}(X)$ *such that, for every* $(f_0, f_1) \in \mathcal{R}_2$, $IFS(f_0, f_1)$ *is* 1*-inverse weak orbitally stable.*

Proof. Let $w \in \Sigma^2$ and $\mathcal{R}_w$ be as in the proof of Theorem 2.2. We can see that $\mathcal{R}_w$ is the required residual set. □

Theorem 2.5. *For every* $(f_0, f_1) \in \mathcal{H}(X) \times \mathcal{H}(X)$, $IFS(f_0, f_1)$ *is* 2*-inverse weak orbitally stable.*

Proof. Let $\varepsilon > 0$ be given. As in the proof of Theorem 2.2, we can find open neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of f_0 and f_1, respectively, such that, for every $({f'}_0, {f'}_1) \in \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{I}_{(f_0, f_1)} \subset \mathcal{I}_{({f'}_0, {f'}_1)}$. Suppose that $L \in \mathcal{I}_{(f_0, f_1)}$ is such that $\{f_w^n(x')\}_{n=0}^{\infty} \cap V_i \neq \varnothing$ for all $i \in L$ and $\{f_w^n(x')\}_{n=0^\infty} \subset \bigcup_{i \in L} V_i$, where w is as in the proof of Theorem 2.3. Since $L \in \mathcal{I}_{({f'}_0, {f'}_1)}$, there exists $y' \in X$ such that, for all $i \in L$,

$$\{{f'}_w^n(y')\}_{n=0}^{\infty} \cap V_i \neq \emptyset.$$

Now suppose that $f_{w'}^n(x) \in \mathcal{O}_{IFS(f_0, f_1)}(x)$ is arbitrary. Similar to the proof of Theorem 2.3, we can find $x' \in X$ and $l \in \mathbb{N}$ such that $f_{w'}^n(x) = f_w^{n+l}(x')$. So we can see that

$$f_{w'}^n(x) = f_w^{n+l}(x') \in \mathcal{N}_\varepsilon\big(\{{f'}_w^n(y')\}_{n=0}^{\infty}\big) \subset \mathcal{N}_\varepsilon\big(\mathcal{O}_{({f'}_0, {f'}_1)}(y')\big).$$

This completes the proof. □

Theorem 2.6. *There exists* $w \in \Sigma^2$ *and a residual set* $\mathcal{R} \subset \mathcal{H}(X) \times \mathcal{H}(X)$ *such that, for every* $(f_0, f_1) \in \mathcal{R}$, $IFS(f_0, f_1)$ *is* w*-orbitally stable, weak orbitally stable, and* 1*-inverse weak orbitally stable.*

Proof. Let w be as in the proof of Theorem 2.3. Consider $\mathcal{R} = \mathcal{R}_w \cap \mathcal{R}_1 \cap \mathcal{R}_2$, where $\mathcal{R}_w$, $\mathcal{R}_1$, and $\mathcal{R}_2$ are as in the Theorem 2.2, Theorem 2.3, and Theorem 2.4, respectively. □

Acknowledgments. I am grateful to the Editor-in-Chief and to the referee for his/her valuable comments and corrections on the earlier version of this manuscript. I also thank Professor Fatemeh Helen Ghane who helped me to improve the exposition of the introduction of this paper.

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