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by

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NON-NORMALITY POINTS AND BIG PRODUCTS OF METRIZABLE SPACES

SERGEI LOGUNOV

Abstract. By the theorem of Jun Terasawa and Sergei Logunov, every point of the Čech-Stone remainder of a metrizable crowded space is a non-normality point of compactification. Now we obtain the following generalizations:

Theorem 1.1. Let τ be an arbitrary cardinal number and, for every $k < \tau$, let $\mathcal{F}_k$ be a family of metrizable spaces with the following properties: $\mathcal{F}_k$ contains a crowded space and $\mathcal{F}_k$ contains at most one non-compact space. Let a space S be a free union $\bigcup_{k<\tau} S_k$ of Tychonoff products $S_k = \prod\{X : X \in \mathcal{F}_k\}$. Then every point p of S^* is a butterfly-point in βS. Hence, $\beta S \setminus \{p\}$ is not normal.

Corollary 1.2. Let a space S be a free union of arbitrary powers of closed segment $\bigcup_{k<\tau} I^{\tau_k}$. Then every point p of S^* is a butterfly-point in βS. Hence, $\beta S \setminus \{p\}$ is not normal.

Notice that the cardinal numbers τ and τ_k are unrestricted in these results.

Corollary 1.3. Let $S = \omega \times I^{\mathfrak{c}}$. Then every point p of S^* is a butterfly-point in βS. Hence, $\beta S \setminus \{p\}$ is not normal.

1. Introduction

By B. È Šapirovskiǐ [9], a point p of a space X is called a *butterfly-point* (*b-point*) in X if there are sets F and G in $X \setminus \{p\}$ with the following property: $\{p\} = [F] \cap [G]$. We modify this notion as follows: A point p of remainder $X^* = \beta X \setminus X$ of Čech-Stone compactification βX of a

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completely regular space X is called a *butterfly-point* (*b-point*) *in* βX if there are sets F and G in $X^* \setminus \{p\}$ with $[F \cup G]_{\beta X} \subset X^*$ and such that $\{p\} = [F]_{\beta X} \cap [G]_{\beta X}$. Obviously, if p is a butterfly-point in βX, then p is a *non-normality point in* βX, i.e., $\beta X \setminus \{p\}$ is not normal.

It is a classical problem in the theory of the space $\beta\omega$ if every point of ω^* is a non-normality point. The positive answer is not so hard under CH, but still it is only known for very restricted types of points in ZFC. For instance, if p is an accumulation point of some countable discrete subset of ω^*, or p is a strong $R-$point, or p is a Kunen's point, then $\omega^* \setminus \{p\}$ is not normal (A. Błaszczyk and A. Szymański [1], A. Gryzlov [2] and van Douwen (see [8]), respectively). Probably, we should not expect a fast solution of this problem now.

But things are quite different if we have a metrizable crowded space. The whole positive solution of the problem above was first obtained for the real line [3] and then it was generalized for some other spaces (see, for instance, [4] and [5]). The following result was obtained independently by Jun Terasawa and Sergei Logunov.

Theorem ([10], [6])**.** *Let X be a non-compact metrizable crowded space. Then any point $p \in X^*$ is a butterfly-point in βX. Hence, $\beta X \setminus \{p\}$ is not normal.*

In this paper we first investigate Tychonoff products of metrizable spaces S. Only one simple fact makes it really possible: If we have a closed set in every factor, which intersects every member of some family, then the product of these sets is again a closed set in S, which intersects every member of the "box" product of these families. This is a very useful way to split many families in S, which allows us, figuratively speaking, to multiply constructions in factors in order to obtain as a result the desirable construction in S.

Now we obtain the following generalizations of the theorem above.

Theorem 1.1. *Let τ be an arbitrary cardinal number and for every $k < \tau$ let $\mathcal{F}_k$ be a family of metrizable spaces with the following properties: $\mathcal{F}_k$ contains a crowded space and $\mathcal{F}_k$ contains at most one non-compact space. Let a space S be a free union $\bigcup_{k<\tau} S_k$ of Tychonoff products $S_k = \prod\{X : X \in \mathcal{F}_k\}$. Then every point p of S^* is a butterfly-point in βS. Hence, $\beta S \setminus \{p\}$ is not normal.*

Corollary 1.2. *Let a space S be a free union of arbitrary powers of closed segment $\bigcup_{k<\tau} I^{\tau_k}$. Then every point p of S^* is a butterfly-point in βS. Hence, $\beta S \setminus \{p\}$ is not normal.*

Notice that the cardinal numbers τ and τ_k are unrestricted in these results.

Corollary 1.3. *Let $S = \omega \times I^{\mathfrak{c}}$. Then every point p of S^* is a butterfly-point in βS. Hence, $\beta S \setminus \{p\}$ is not normal.*

2. Preliminaries

In this section and at the beginning of the next section, we remind the reader of some facts, which were introduced in [6], for instance. All spaces are metrizable. By $[\,]$ or $[\,]_{\beta X}$, we denote the closure operators in X and βX, respectively, $3 = \{0, 1, 2\}$, $\omega = \{0, 1, 2...\}$, $\mathfrak{c}$ is the real line cardinality. By $\prod_{\alpha<\tau} X_\alpha$, we denote the product of spaces X_α equipped with the Tychonoff topology. By $X = \bigcup_{\alpha<\tau} X_\alpha$, we denote a *free union* of spaces X_α, assuming that all X_α are pairwise disjoint and every X_α is open in X.

Let π and σ be arbitrary families. If members of π are mutually disjoint, then π is called *cellular*. A set U is a *proper subset* of a set V if both $U \subset V$ (i.e., $\forall x(x \in U \Rightarrow x \in V)$) and $U \neq V$. We call a set $U \in \pi$ a *maximal member* of π if U is not a proper subset of any other member of π. We say that π *(strongly) refines* σ, $(\pi \succ \sigma)$ $\pi \succeq \sigma$, if any $U \in \pi$ and $V \in \sigma$ are either disjoint sets or U is a (proper) subset of V. By $\exp(\pi)$, we denote a collection of all subfamilies of π. We define a *projection* $f^\pi_\sigma : \exp(\pi) \to \exp(\sigma)$ in every point $F \in \exp(\pi)$ by the following rule: $f^\pi_\sigma(F) = \{V \in \sigma : \cup F \cap V \neq \emptyset\}$.

A maximal locally finite cellular family of open in X sets is called *nice*. Let π and σ be nice families and $p \in X^*$. A collection $\mathcal{F}$ of subfamilies $F \subset \pi$ is called a *p-filter* on π [6] if the intersection of any of its finite subcollections $\bigcap\{F_0, ..., F_n\}$ contains p in its closure. We write $\pi \succ_{\mathcal{F}} \sigma$ if there is $F \in \mathcal{F}$ with $F \succ \sigma$. Obviously, the union of any increasing family of p-filters is also a p-filter. So by Zorn's lemma, there are *maximal p-filters* or *p-ultrafilters* $\mathcal{F}$ on π, that is, $\mathcal{F} = \mathcal{G}$ whenever a p-filter $\mathcal{G}$ contains $\mathcal{F}$. Enriching any p-filter with new subfamilies of π, while possible, we can embed it into some p-ultrafilter, which is not unique in general. But any p-ultrafilter contains $\pi(O) = \{U \in \pi : U \cap O \neq \emptyset\}$ for any neighborhood O of p.

The notion of cellular refinement of a family π was introduced in [7]:

$$Cell(\pi) = \{\bigcap \phi \setminus [\bigcup(\pi \setminus \phi)] : \phi \subset \pi\}.$$

Lemma 2.1. *If π is a locally finite open cover of X, then Cell (π) is nice.*

Proof. If $\varphi \subset \pi$ has non-empty intersection, then φ is finite. So $\cap\varphi - [\bigcup(\pi \setminus \varphi)]$ is open.

If $U \in \varphi \setminus \varphi'$, then $\bigcap \varphi \subset U$ and $\bigcap \varphi' \cap U = \emptyset$.

If $\psi = \{U \in \pi : U \cap Ox \neq \emptyset\}$ is finite for some $Ox \subset X$, then $\{\varphi \subset \pi : \bigcap \varphi \cap Ox \neq \emptyset\} \subset \exp \psi$ is finite as well.

Let $x \notin [U] \setminus U$ for any $U \in \pi$ and $\varphi = \{U \in \pi : x \in U\}$. Then $x \in \bigcap \varphi \setminus [\bigcup(\pi \setminus \varphi)]$ and $Cell(\pi)$ is maximal. □

3. Special Bases in Metrizable Spaces

Let X be an arbitrary metrizable space. For $\mathcal{P}_0 = \{X\}$ and $\mathcal{B}_0 = \{X\}$, we put $\mathcal{P}_0^* = \{(X, 0)\}$ and $\mathcal{B}_0^* = \{(X, 0)\}$.

If a nice family $\mathcal{B}_i$ has been constructed for some $i \in \omega$, then we define the family of non-empty open sets $\mathcal{W}_i = \{V(\nu) : V \in \mathcal{B}_i \text{ and } \nu \in 3\}$ by the following rules: If $V \in \mathcal{B}_i$ is a one-or-two points set, then we put $V(\nu) = V$ for each $\nu \in 3$. Otherwise, $\{V(\nu) : \nu \in 3\}$ is strongly cellular and $[V(\nu)] \subset V$ for every $\nu \in 3$. We denote $\mathcal{W}_i(\nu) = \{V(\nu) : V \in \mathcal{B}_i\}$. Let $\mathcal{P}_{i+1}$ be an open, locally finite cover of X such that, for every $U \in \mathcal{P}_{i+1}$, the following are true: diam $U \leq \frac{1}{i+1}$ and $V(\nu) \setminus U \neq \emptyset$ whenever $V(\nu) \in \mathcal{W}_i$ is not a singleton. We put $\mathcal{B}_{i+1} = \mathcal{C}ell(\mathcal{B}_i \cup \mathcal{W}_i \cup \mathcal{P}_{i+1})$. We denote $\mathcal{P}_{i+1} \times \{i+1\} = \{(U, i+1) : U \in \mathcal{P}_{i+1}\}$ by $\mathcal{P}_{i+1}^*$ and we denote $\mathcal{B}_{i+1} \times \{i+1\} = \{(V, i+1) : V \in \mathcal{B}_{i+1}\}$ by $\mathcal{B}_{i+1}^*$.

It is folklore and easy to see that $\mathcal{P} = \bigcup_{i\in\omega} \mathcal{P}_i$ is a *regular base of Arhangel'skii*; i.e., $\mathcal{P}$ is a base and, for any point $x \in X$ and for any its neighborhood O, there is another neighborhood O' of x with the following properties: $O' \subset O$ and at most finitely many members of $\mathcal{P}$ meet both O' and $X \setminus O$ simultaneously. Moreover, $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ is a π-base and $\mathcal{B}_{i+1} \succeq \mathcal{B}_i$ for each $i \in \omega$. Any two members of $\mathcal{B}$ are either disjoint or one can be embedded into the other.

Lemma 3.1. *The family of maximal members π' of any cover $\pi \subset \mathcal{P}$ is a locally finite open cover of X.*

Proof. Let a point x belong to some $U \in \pi$. Since the family $\{V \in \pi : U \subset V\}$ is finite by our construction, it contains a maximal member O. Then $x \in O$ and $O \in \pi'$. Since $\mathcal{P}$ is a regular base of Arhangel'skii, there is a neighborhood O' of x as above. Since every member of $\pi' \setminus \{O\}$ intersects $X \setminus O$, at most finitely many members of π' intersect O'. □

Lemma 3.2. *Let $V \in \mathcal{B}_i$ intersect $W \in \mathcal{B}_j$, where $i < j < \omega$. Then either W is a proper subset of V if and only if V is not a singleton or vice versa.*

Proof. If V is a singleton, then the condition $\mathcal{B}_j \succ \mathcal{B}_i$ implies the first claim of our lemma. Vice versa, $W \subset W'$ for some $W' \in \mathcal{B}_{i+1}$, $W' \subset U$ for some $U \in \mathcal{P}_{i+1}$, and $V \setminus U \neq \emptyset$ by the definition of P_{i+1}. □

We define the following (partial) order on $\mathcal{B}^* = \bigcup_{i\in\omega} \mathcal{B}_i^*$ $(\mathcal{P}^* = \bigcup_{i\in\omega} \mathcal{P}_i^*)$:

- $(U, i) =_S (V, j) \Leftrightarrow U = V$ and $i = j$;
- $(U, i) \succ_S (V, j) \Leftrightarrow U \subset V$ and $i > j$;
- $(U, i) \succeq_S (V, j) \Leftrightarrow ((U, i) =_S (V, j)) \vee ((U, i) \succ_S (V, j))$.

Actually, this new order is important only for singletons and we introduce it in order to avoid considering two different cases very often.

If $(U, i) \in \mathcal{P}_i^*$, then

$$\mathcal{B}^*(U, i) = \{(V, i) \in \mathcal{B}_i^* : V \cap U \neq \emptyset \text{ (or, equivalently, } V \subset U)\},$$

where index i is one and the same for $\mathcal{P}_i$ and $\mathcal{B}_i$.

Let $\sigma^*, \delta^* \subset \mathcal{B}^*$. We say that

- $\mathcal{B}(\sigma^*) = \{U : (U, i) \in \sigma^* \text{ for some } i \in \omega\}$ is a projection of σ^* into $\mathcal{B}$;
- σ^* is minimal $\Leftrightarrow \forall (U, i) \in \sigma^* \forall (V, j) \in \sigma^* ((U, i) \neq_S (V, j) \Leftrightarrow U \neq V)$;
- σ^* is locally finite $\Leftrightarrow \sigma^*$ is minimal and $\mathcal{B}(\sigma^*)$ is locally finite;
- σ^* is nice $\Leftrightarrow \sigma^*$ is minimal and $\mathcal{B}(\sigma^*)$ is nice;
- $(U, i) \in \sigma^*$ is minimal in $\sigma^* \Leftrightarrow \forall (V, j) \in \sigma^* \neg((V, j) \prec_S (U, i))$;
- $(U, i) \in \sigma^*$ is s-min $\sigma^* \Leftrightarrow \forall (V, j) \in \sigma^* ((U, i) \preceq_S (V, j))$;
- $\sigma^* \succ_S \delta^* \Leftrightarrow \forall (U, i) \in \sigma^* \forall (V, j) \in \delta^* (U \cap V = \emptyset \vee (U, i) \succ_S (V, j))$;
- σ^* is s-embedded $\Leftrightarrow \forall (U, i) \in \sigma^* \forall (V, j) \in \sigma^* ((U, i) \succeq_S (V, j) \vee (U, i) \preceq_S (U, j))$.

We make analogous definitions for collections $\sigma^*, \delta^* \subset \mathcal{P}^*$.

For any $\pi^* \subset \mathcal{P}^*$, we denote by $\pi^*_{\min}$ all minimal members of π^*. We denote by $\mathcal{B}^*(\pi^*)$ all minimal members of $\bigcup_{(U,i) \in \pi^*_{\min}} \mathcal{B}^*(U, i)$. Since $\pi^*_{\min}$ is locally finite by Lemma 3.1, $\mathcal{B}^*(\pi^*)$ is nice by Lemma 2.1. Define

$$\Sigma^*(X) = \{\sigma^* \subset \mathcal{B}^*(X) : \sigma^* \text{ is nice }\}.$$

4. Special Bases in Products

It is easy to see by the standard arguments that our space $S = \bigcup_{k<\tau} \prod_{\gamma<\tau_k} X_{\gamma k}$, where every τ_k is the cardinality of the family of spaces $\mathcal{F}_k$, is paracompact, and, so, normal. We may assume every space $X_{\gamma k}$ to be compact whenever $\gamma > 0$. Then $Y_k = \prod_{\gamma \in \tau_k \setminus \{0\}} X_{\gamma k}$ is also compact and X_{0k} may or may not be compact. For any $x \in X_{0k}$, we put $Y_x = \{x\} \times Y_k$. We denote also $X = \bigcup_{k<\tau} X_{0k}$ and $Y = \bigcup_{k<\tau} Y_k$.

We can repeat our constructions above in every $X_{\gamma k}$, i.e., construct the families $\mathcal{P}_i(X_{\gamma k})$, $\mathcal{B}_i(X_{\gamma k})$, $\mathcal{P}_i^*(X_{\gamma k})$, $\mathcal{B}_i^*(X_{\gamma k})$, and $\mathcal{W}_i(X_{\gamma k})$ for each $i \in \omega$. We put $\mathcal{P}(X_{\gamma k}) = \bigcup_{i \in \omega} \mathcal{P}_i(X_{\gamma k})$, $\mathcal{B}(X_{\gamma k}) = \bigcup_{i \in \omega} \mathcal{B}_i(X_{\gamma k})$, $\mathcal{P}^*(X_{\gamma k}) = \bigcup_{i \in \omega} \mathcal{P}_i^*(X_{\gamma k})$, and $\mathcal{B}^*(X_{\gamma k}) = \bigcup_{i \in \omega} \mathcal{B}_i^*(X_{\gamma k})$. We denote $\mathcal{P}(X) = \bigcup_{k<\tau} \mathcal{P}(X_{0k})$, $\mathcal{B}(X) = \bigcup_{k<\tau} \mathcal{B}(X_{0k})$, $\mathcal{P}^*(X) = \bigcup_{k<\tau} \mathcal{P}^*(X_{0k})$, and $\mathcal{B}^*(X) = \bigcup_{k<\tau} \mathcal{B}^*(X_{0k})$.

As above, we define $\Sigma^*(X_{0k}) = \{\sigma^* \subset \mathcal{B}^*(X_{0k}) : \sigma^* \text{ is nice in } X_{0k}\}$ and $\Sigma^*(X) = \{\sigma^* \subset \mathcal{B}^*(X) : \sigma^* \text{ is nice in } X\}$. Obviously, $\Sigma^*(X) = \{\bigcup_{k<\tau} \sigma_k^* : \sigma_k^* \in \Sigma^*(X_{0k})\}$.

For every Y_k and $i \in \omega$, we denote the "box" product

$$\prod_{\gamma\in\tau_k\setminus\{0\}} \mathcal{B}_i(X_{\gamma k}) = \left\{ \prod_{\gamma\in\tau_k\setminus\{0\}} V_{\gamma k} : V_{\gamma k} \in \mathcal{B}_i(X_{\gamma k}) \right\}$$

by $\mathcal{B}_i(Y_k)$ and put $\mathcal{B}(Y_k) = \bigcup_{i\in\omega} \mathcal{B}_i(Y_k)$. We denote also $\mathcal{B}_i(Y) = \bigcup_{k<\tau} \mathcal{B}_i(Y_k)$ and $\mathcal{B}(Y) = \bigcup_{i\in\omega} \mathcal{B}_i(Y)$. Then any two sets from $\mathcal{B}(Y)$ are either disjoint or embedded one into another. So any two cellular subfamilies of $\mathcal{B}(Y)$ may be naturally compared by a p-ultrafilter defined on any of them.

For any $\nu \in 3$, we denote the "box" product

$$\prod_{\gamma\in\tau_k\setminus\{0\}} \mathcal{W}_i(X_{\gamma k})(\nu) = \left\{ \prod_{\gamma\in\tau_k\setminus\{0\}} V_{\gamma k}(\nu) : V_{\gamma k}(\nu) \in \mathcal{W}_i(X_{\gamma k})(\nu) \right\}$$

by $\mathcal{W}_i(Y_k)(\nu)$ and put $\mathcal{W}_i(Y_k) = \bigcup_{\nu\in 3} \mathcal{W}_i(Y_k)(\nu)$. Then, if $V = \prod_{\gamma\in\tau_k\setminus\{0\}} V_{\gamma k}$ belongs to $\mathcal{B}_i(Y_k)$ for some $i \in \omega$, then $V(\nu) = \prod_{\gamma\in\tau_k\setminus\{0\}} V_{\gamma k}(\nu)$ belongs to $\mathcal{W}_i(Y_k)(\nu)$. Moreover, $\mathcal{B}_{i+1}(X_{\gamma k}) \succ \mathcal{W}_i(X_{\gamma k}) \succ \mathcal{B}_i(X_{\gamma k})$ for every $\gamma \in \tau_k \setminus \{0\}$ implies

$$\mathcal{B}_{i+1}(Y_k) \succ \mathcal{W}_i(Y_k) \succ \mathcal{B}_i(Y_k).$$

Let $k < \tau$ and let (U, n) be a member of either $\mathcal{P}^*(X_{0k})$ or $\mathcal{B}^*(X_{0k})$. Then we denote the product

$$\{U\} \times \mathcal{B}_n(Y_k) = \{U \times V : V \in \mathcal{B}_n(Y_k)\} \text{ by } (U, n)^S.$$

If $(U, n) \in \mathcal{B}^*(X_{0k})$ and $\nu \in 3$, then we denote the product

$$\{U(\nu)\} \times \mathcal{W}_n(Y_k)(\nu) = \{U(\nu) \times V(\nu) : V \in \mathcal{B}_n(Y_k)\} \text{ by } (U, n)^S(\nu).$$

For any $F^* \subset \mathcal{B}^*(X_{0k})$, we denote the family

$$\bigcup_{(U,n)\in F^*} (U, n)^S = \bigcup_{(U,n)\in F^*} \{U \times V : V \in \mathcal{B}_n(Y_k)\} \text{ by } (F^*)^S.$$

For any $\nu \in 3$, we denote the family

$$\bigcup_{(U,n)\in F^*} (U, n)^S(\nu) = \bigcup_{(U,n)\in F^*} \{U(\nu) \times V(\nu) : V \in \mathcal{B}_n(Y_k)\} \text{ by } (F^*)^S(\nu).$$

For any $F^* \subset \mathcal{B}^*(X)$ and $k < \tau$, we denote the intersection

$$F^* \cap \mathcal{B}^*(X_{0k}) = \{(U, n) \in F^* : (U, n) \in \mathcal{B}^*(X_{0k})\} \text{ by } F_k^*.$$

Then we put

$$(F^*)^S = \bigcup_{k<\tau} (F_k^*)^S \text{ and } (F^*)^S(\nu) = \bigcup_{k<\tau} (F_k^*)^S(\nu).$$

Lemma 4.1. *For any $k < \tau$ and $\sigma^* \in \Sigma^*(X_{0k})$ the following hold. If $\nu, \nu' \in 3$ and $\nu \neq \nu'$, then $[\bigcup(\sigma^*)^S(\nu)] \cap [\bigcup(\sigma^*)^S(\nu')] = \emptyset$.*

Proof. Let X_{0k} be crowded and $\sigma(\nu) = \{V(\nu) : V \in \mathcal{B}(\sigma^*)\}$. Then $[\bigcup \sigma(\nu)] \cap [\bigcup \sigma(\nu')] = \emptyset$ easily implies our statement.

Let $X_{\gamma_0 k}$ be crowded for some $\gamma_0 > 0$. Then

$$(\sigma^*)^S(\nu) = \bigcup_{(U,n)\in\sigma} \{U(\nu) \times V(\nu) : V \in \mathcal{B}_n(Y_k)\} =$$

$$\bigcup_{(U,n)\in\sigma} (\{U(\nu)\} \times \mathcal{W}_n(Y_k)(\nu)) = \bigcup_{(U,n)\in\sigma} \left(\{U(\nu)\} \times \prod_{\gamma\in\tau_k\setminus\{0\}} \mathcal{W}_n(X_{\gamma k})(\nu) \right).$$

In $X_{\gamma_0 k}$, we have by our construction

$$[\bigcup \mathcal{W}_i(X_{\gamma_0 k})(\nu)] \cap [\bigcup \mathcal{W}_{i)}(X_{\gamma_0 k})(\nu')] = \emptyset.$$

Since, additionally, σ is nice in X_{0k} and $[U(\nu)] \subset U$ for any $U \in \sigma$, our proof is complete. □

For any family of sets $\mathcal{D} \subset \exp(S)$ and $O \subset S$, we denote $\mathcal{D}(O) = \{U \in \mathcal{D} : U \cap O \neq \emptyset\}$.

Lemma 4.2. *Let O and O' be any open sets in S such that $[O'] \subset O$. Then, for any $k < \tau$ and $x \in X_{0k}$, we can find $(U,n) \in \mathcal{P}^*(X_{0k})$ so that $x \in U$ and $\bigcup((U,n)^S(O')) \subset O$.*

Proof. For any point $t \in Y_x$ and $\gamma < \tau_k$, we can find

$$(U_{t\gamma}, n_{t\gamma}) \in \mathcal{P}^*_{n_{t\gamma}}(X_{\gamma k})$$

so that $O_t = \prod_{\gamma<\tau_k} U_{t\gamma}$ is an open in S neighborhood of t with the following property: $O_t \subset O$ whenever $O_t \cap O' \neq \emptyset$. Then almost always $U_{t\gamma} = X_{\gamma k}$ implies $n_{t\gamma} = 0$. We can find finitely many points $t_0, ..., t_m$ so that $\{O_{t_i} = \prod_{\gamma<\tau_k} U_{t_i\gamma} : i \leq m\}$ is a cover of Y_x by its compactness. Any $(U,n) \in \mathcal{P}^*(X_{0k})$ with $x \in U \subset \bigcap_{i\leq m} U_{t_i 0}$ and $n > n_{t_i\gamma}$ for all $i \leq m$ and $\gamma < \tau_k$ is as required.

Indeed, let $W = U \times \prod_{\gamma\in\tau_k\setminus\{0\}} V_\gamma$ belong to $(U,n)^S$. Then W intersects some O_{t_i}, i.e., $V_\gamma \cap U_{t_i\gamma} \neq \emptyset$ for all $\gamma \in \tau_k \setminus \{0\}$. But then $V_\gamma \in \mathcal{B}_n(X_{\gamma k})$, $U_{t_i\gamma} \in \mathcal{P}_{n_{t_i\gamma}}(X_{\gamma k})$, and $n > n_{t_i\gamma}$ imply $V_\gamma \subset U_{t_i\gamma}$ by our construction and, so, $W \subset O_{t_i}$. If $W \cap O' \neq \emptyset$, then $O_{t_i} \cap O' \neq \emptyset$ implies $W \subset O_{t_i} \subset O$. □

Lemma 4.3. *Let O and O' be any open sets in S such that $[O'] \subset O$. Then there is $\sigma^* \in \Sigma^*(X)$ with $\bigcup((\sigma^*)^S(O')) \subset O$.*

Proof. For any point $x \in X$ let $(U_x, n_x) \in \mathcal{P}^*(X)$ be as constructed in the previous lemma and $\pi^* = \{(U_x, n_x) : x \in X\}$. Then $\sigma^* = \mathcal{B}^*(\pi^*)$ is as required.

Indeed, by our construction, σ^* is nice. Any $(V, n) \in \sigma^*$ belongs $\mathcal{B}^*(U, n)$ for some $(U, n) \in \pi^*$. Hence, $(V, n)^S \succeq (U, n)^S$ implies

$$\bigcup(V, n)^S(O') \subset \bigcup(U, n)^S(O') \subset O,$$

and the proof is complete. □

5. Butterfly-point

Let $p \in S^*$ and $\sigma^*, \delta^* \in \Sigma^*(X)$. We modify the notion of a p-filter on σ^* as follows: A collection $\mathcal{F}$ of subfamilies $F^* \subset \sigma^*$ is called a *p-filter on* σ^* if, for any of its finite subcollections $\{F_0^*, ..., F_n^*\}$, the following hold: p is in the closure of $\bigcap_{i \leq n}(F_i^*)^S = (\bigcap_{i \leq n} F_i^*)^S$. As above, any p-filter $\mathcal{F}$ on σ^* may be embedded into a p-ultrafilter. From now on we write $\sigma^* \succ_{\mathcal{F}} \delta^*$ if there is $F^* \in \mathcal{F}$ with $F^* \succ_S \delta^*$. We define a *projection*

$$f_{\delta^*}^{\sigma^*} : \exp(\sigma^*) \to \exp(\delta^*)$$

in every point $F^* \in \exp(\sigma^*)$ by the following rule:

$$f_{\delta^*}^{\sigma^*}(F^*) = \{(V, j) \in \delta^* : U \cap V \neq \emptyset \text{ for some } (U, i) \in F^*\}.$$

Lemma 5.1. *For any $\sigma^*, \delta^* \in \Sigma^*(X)$, $\sigma^* \succ_S \delta^*$ if and only if $(\sigma^*)^S \succ (\delta^*)^S$.*

Proof. Let $\sigma^* \succ_S \delta^*$ and $U_\sigma \times V_\sigma \in (\sigma^*)^S$ intersect $U_\delta \times V_\delta \in (\delta^*)^S$. The following conditions hold:

(1) $U_\sigma \times V_\sigma \in (U_\sigma, n_\sigma)^S$ for unique $(U_\sigma, n_\sigma) \in \sigma^*$;
(2) $V_\sigma = \prod_{\gamma \in \tau_k \setminus \{0\}} V_{\gamma k}^\sigma$ for some $V_{\gamma k}^\sigma \in \mathcal{B}_{n_\sigma}(X_{\gamma k})$;
(3) $U_\delta \times V_\delta \in (U_\delta, n_\delta)^S$ for unique $(U_\delta, n_\delta) \in \delta^*$;
(4) $V_\delta = \prod_{\gamma \in \tau_k \setminus \{0\}} V_{\gamma k}^\delta$ for some $V_{\gamma k}^\delta \in \mathcal{B}_{n_\delta}(X_{\gamma k})$;

By our construction, $(U_\sigma, n_\sigma) \succ_S (U_\delta, n_\delta)$ if and only if $U_\sigma \subset U_\delta$ and $n_\sigma > n_\delta$. If X_{0k} is crowded, then U_σ is a proper subset of U_δ. Analogously, $V_{\gamma k}^\sigma \cap V_{\gamma k}^\delta \neq \emptyset$ implies both $V_{\gamma k}^\sigma \subset V_{\gamma k}^\delta$ for every $\gamma \in \tau_k \setminus \{0\}$ and $V_{\gamma_0 k}^\sigma$ is a proper subset of $V_{\gamma_0 k}^\delta$ if $X_{\gamma_0 k}$ is crowded.

Let $(\sigma^*)^S \succ (\delta^*)^S$, $(U_\sigma, n_\sigma) \in \sigma^*$, $(U_\delta, n_\delta) \in \delta^*$, and $U_\sigma \cap U_\delta \neq \emptyset$. Since every $\mathcal{B}_n(X_{\gamma k})$ is nice, any $V_{\gamma k}^\sigma \in \mathcal{B}_{n_\sigma}(X_{\gamma k})$ intersects some $V_{\gamma k}^\delta \in \mathcal{B}_{n_\delta}(X_{\gamma k})$. Then $V_\sigma = \prod_{\gamma \in \tau_k \setminus \{0\}} V_{\gamma k}^\sigma$ intersects $V_\delta = \prod_{\gamma \in \tau_k \setminus \{0\}} V_{\gamma k}^\delta$ and $U_\sigma \times V_\sigma \in (\sigma^*)^S$ intersects $U_\delta \times V_\delta \in (\delta^*)^S$. Hence, $U_\sigma \times V_\sigma$ is a proper subset of $U_\delta \times V_\delta$ by our assumption; i.e., either U_σ is a proper subset of U_δ or $U_\sigma = U_\delta$ and V_σ is a proper subset of V_δ. In both of these cases, $n_\sigma > n_\delta$ implies $(U_\sigma, n_\sigma) \succ_S (U_\delta, n_\delta)$ and $\sigma^* \succ_S \delta^*$. □

Lemma 5.2. *Let $p \in S^*$. There is a cardinal $\lambda = \lambda(p)$ and, for every $\alpha < \lambda$, there are a nice family $\sigma^*_\alpha \in \Sigma^*(X)$ and a p-ultrafilter $\mathcal{F}_\alpha$ on σ^*_α, which satisfy, for all $\alpha < \beta < \lambda$ and projections $f^\alpha_\beta = f^{\sigma^*_\alpha}_{\sigma^*_\beta}$, the following conditions:*

1. *$p \notin [\bigcup (U,n)^S]_{\beta S}$ for each member (U,n) of σ^*_0;*
2. *$f^\alpha_\beta \mathcal{F}_\alpha \subset \mathcal{F}_\beta$;*
3. *$\sigma^*_\alpha \prec_{\mathcal{F}_\alpha} \sigma^*_\beta$;*
4. *for any $\sigma^* \in \Sigma^*(X) \setminus \{\sigma^*_\alpha : \alpha < \lambda\}$, there is $\alpha < \lambda$ with $\neg(\sigma^*_\alpha \prec_{\mathcal{F}_\alpha} \sigma^*)$.*

Proof. Let

$$\pi^* = \bigcup_{k<\tau} \{(U,n) \in \mathcal{P}^*(X_{0k}) : p \notin [U \times Y_k]_{\beta S}\}.$$

Then $\{U : (U,n) \in \pi^*\}$ is a cover of X by compactness of every Y_k and standard arguments. Denote $\sigma_0 = \mathcal{B}^*(\pi^*)$ and let $\mathcal{F}_0$ be any p-ultrafilter on σ_0.

For any ordinal β assume that σ^*_α and p-ultrafilter $\mathcal{F}_\alpha$ on σ^*_α have been constructed for all $\alpha < \beta$. If some $\sigma^* \in \Sigma^*(X) \setminus \{\sigma^*_\alpha : \alpha < \beta\}$ satisfies the condition $\sigma^*_\alpha \prec_{\mathcal{F}_\alpha} \sigma^*$ for all $\alpha < \beta$, then we put $\sigma^*_\beta = \sigma^*$ and embed the p-filter $\bigcup_{\alpha<\beta} f^\alpha_\beta \mathcal{F}_\alpha$ into some p-ultrafilter $\mathcal{F}_\beta$ on σ^*_β. Otherwise, our construction is complete. □

For every $\alpha < \lambda(p)$, we denote

$$\mathcal{G}_\alpha = \{(F^*)^S(Op) : F^* \in \mathcal{F}_\alpha \text{ and } Op \subset \beta S\} \text{ and } B_\alpha = \bigcap_{G \in \mathcal{G}_\alpha} [\bigcup G]_{\beta S}.$$

Lemma 5.3. *$B_0 \subset S^*$.*

Proof. Let $x \in X$ be an arbitrary point. Then $F^* = \{(U,n) \in \sigma_0 : x \notin [U]\}$ belongs to $\mathcal{F}^*_0$, $(F^*)^S = (F^*)^S(\beta S)$ belongs to $\mathcal{G}_0$, and $Y_x \cap [\bigcup (F^*)^S] = \emptyset$. □

Lemma 5.4. *If $\alpha < \beta < \lambda$, then $B_\beta \subset B_\alpha$.*

Proof. There is $F^* \in \mathcal{F}_\alpha$ with $\sigma^*_\beta \succ_S F^*$. For any $G^* \in \mathcal{F}_\alpha$, $G^* \cap F^* \in \mathcal{F}_\alpha$ is refined by $f^\alpha_\beta(G^* \cap F^*) \in \mathcal{F}_\beta$. Hence, $(G^* \cap F^*)^S \in \mathcal{G}_\alpha$ is refined by $(f^\alpha_\beta(G^* \cap F^*))^S \in \mathcal{G}_\beta$ by Lemma 7. For any neighborhood $Op \subset \beta S$, we obtain

$$B_\beta \subset [\bigcup (f^\alpha_\beta(G^* \cap F^*))^S(Op)]_{\beta S} \subset [\bigcup (G^* \cap F^*)^S(Op)]_{\beta S} \subset [\bigcup (G^*)^S(Op)]_{\beta S}.$$ □

Lemma 5.5. *For any neighborhood $O \subset \beta S$ of p, there is $\alpha < \lambda$ with $B_\alpha \subset O$.*

Proof. Let $[O']_{\beta S} \subset O$ for a neighborhood O' of p and let $\sigma^* \in \Sigma^*$ be constructed in Lemma 6 so that $\bigcup(\sigma^*)^S(O') \subset O$. There is $\alpha < \lambda$ with $\neg(\sigma^*_\alpha \prec_{\mathcal{F}_\alpha} \sigma^*)$. It means that $F^* = \{(V, n) \in \sigma^*_\alpha : V \subset W$ and $n \geq m$ for some $(W, m) \in \sigma^*\}$ belongs $\mathcal{F}_\alpha$. But then, by Lemma 7,

$$B_\alpha \subset [\bigcup(F^*)^S(O')]_{\beta S} \subset [\bigcup(\sigma^*)^S(O')]_{\beta S} \subset [O]_{\beta S}. \qquad \square$$

Lemma 5.6. *For any $\nu \in 3$, a finite sequence of indices $\alpha < \beta_0 < ... < \beta_m < \lambda$, $F^* \in \mathcal{F}_\alpha$ and a neighborhood $O \subset \beta S$ of p the following holds:*

$$\bigcup(F^*)^S(O) \cap \bigcap_{i \leq m} (\bigcup(\sigma^*_{\beta_i})^S(\nu)) \neq \emptyset.$$

Proof. Our first task is to construct in $\mathcal{B}^*(X)$ a sequence

$$(*)(U, n) \prec_S (U_0, n_0) \prec_S (U_1, n_1) \prec_S (U_2, n_2) \prec_S ... \prec_S (U_\gamma, n_\gamma)$$

for some $\gamma \leq m$ with the following properties:

(1) $(U, n) \in F^*$;
(2) $U \times V \in (F^*)^S(O)$ for some $V \in \mathcal{B}_n(Y)$;
(3) For any $i \leq m$, $(U_j, n_j) \in \sigma^*_{\beta_i}$ for some $j \leq \gamma$;
(4) $U \supset U_0 \supset U_0(\nu) \supset U_1 \supset U_1(\nu) \supset U_2 \supset U_2(\nu) \supset ... \supset U_t \supset U_t(\nu)$

$$= U_{t+1} = U_{t+1}(\nu) = ... = U_\gamma,$$

where the last line, consisting of singletons, may be empty.

We may assume $F^* \prec_S \sigma^*_{\beta_0}$. For any $i < m$, we can find $G^*_i \in \mathcal{F}^*_{\beta_i}$ with $G^*_i \prec_S \sigma^*_{\beta_{i+1}}$. We denote $F^*_0 = f^\alpha_{\beta_0} F^* \cap G^*_0$ and $F^*_{i+1} = f^{\beta_i}_{\beta_{i+1}} F^*_i \cap G^*_{i+1}$. Then $F^*_i \in \mathcal{F}_{\beta_i}$, $F^*_i \prec_S F^*_{i+1}$, and $\bigcup \mathcal{B}(F^*_i) \supset \bigcup \mathcal{B}(F^*_{i+1})$. Denote

$$W^* = \{(U, n) \in F^* : \text{ there is } (V, k) \in F^*_m \text{ with } V \subset U\}.$$

Since $F^*_m \in \mathcal{F}_{\beta_m}$ and $\bigcup \mathcal{B}(W^*) \supset \bigcup \mathcal{B}(F^*_m)$, then $W^* \in \mathcal{F}_\alpha$. Hence, $p \in [\bigcup(W^*)^S]_{\beta S}$ and we can choose $(U, n) \in W^*$ and $V \in \mathcal{B}_n(Y)$ so that $(U \times V) \cap O \neq \emptyset$. For any $i \leq m$, by our construction, we can choose $(U^0_i, n^0_i) \in F^*_i$ so that $U^0_i \subset U$ and $\bigcap_{i \leq m} U^0_i \neq \emptyset$. Then, in this step of induction, we obtain an s-embedded sequence satisfying (1)–(4) and

$$(*)_0 : (U, n) \prec_S (U^0_0, n^0_0) \prec_S (U^0_1, n^0_1) \prec_S ... \prec_S (U^0_m, n^0_m).$$

In other words, let $\Delta_0 = \{\sigma^*_{\beta_0}, ..., \sigma^*_{\beta_m}\}$. For any $\sigma^* \in \Delta_0$, we can choose $(U^0_{\sigma^*}, n^0_{\sigma^*}) \in \sigma^*$ so that $\{(U^0_{\sigma^*}, n^0_{\sigma^*}) : \sigma^* \in \Delta_0\}$ is an s-embedded sequence, whose s-minimal member (U_0, n_0) s-refines some $(U, n) \in F^*$, satisfying (2). For $\Delta_1 = \Delta_0 \setminus \{\sigma^* \in \Delta_0 : (U^0_{\sigma^*}, n^0_{\sigma^*}) =_S (U_0, n_0)\}$, we have the following sequence

$$(*)_1 : (U, n) \prec_S (U_0, n_0) \prec_S (U^0_{\sigma^*}, n^0_{\sigma^*}) \prec_S ...(\sigma^* \in \Delta_1).$$

Two cases are possible for $(*)_1$:

Case 1^0: There is $\sigma^* \in \Delta_1$ such that $U^0_{\sigma^*} = U_0$ and $n^0_{\sigma^*} > n_0$.

Then the sequence $(*)_1$ looks as follows:

$$(U, n) \prec_S (\{a\}, n_0) \prec_S (\{a\}, n_1) \prec ... \prec_S (\{a\}, n_\gamma)$$

for some $a \in X$, where $n < n_0 < n_1 < ... < n_\gamma$. Since $\{a\}(\nu) = \{a\}$ for all $\nu < 3$, (4) also holds, and the first step is complete.

Case 2^0: For any $\sigma^* \in \Delta_1$, $U^0_{\sigma^*}$ is a proper subset of U_0.

Since all σ^* are nice, we can choose $(U^1_{\sigma^*}, n^1_{\sigma^*}) \in \sigma^*$ so that $U^1_{\sigma^*} \subset U_0(\nu)$ and $\bigcap_{\sigma^* \in \Delta_1} U^1_{\sigma^*} \neq \emptyset$. Then $U^1_{\sigma^*}$ is also a proper subset of U_0 by our construction. Hence, $n^1_{\sigma^*} > n_0$. Let (U_1, n_1) be s-minimal in the s-embedded sequence $\{(U^1_{\sigma^*}, n^1_{\sigma^*}) : \sigma^* \in \Delta_1\}$. For $\Delta_2 = \Delta_1 \setminus \{\sigma^* \in \Delta_1 : (U^1_{\sigma^*}, n^1_{\sigma^*}) =_S (U_1, n_1)\}$, we obtain

$$(*)_2 : (U, n) \prec_S (U_0, n_0) \prec_S (U_1, n_1) \prec_S (U^1_{\sigma^*}, n^1_{\sigma^*}) ... (\sigma^* \in \Delta_2),$$
$$\text{and } U \supset U_0 \supset U_0(\nu) \supset U_1 \supset U^1_{\sigma^*} ... (\sigma^* \in \Delta_2).$$

Two cases are possible for $(*)_2$:

Case 1^1: There is $\sigma^* \in \Delta_2$ with $U^1_{\sigma^*} = U_1$ and $n^1_{\sigma^*} > n_1$.

Then the sequence $(*)_2$ looks as follows:

$$(U, n) \prec_S (U_0, n_0) \prec_S (\{a\}, n_1) \prec ... \prec_S (\{a\}, n_\gamma),$$

for some $a \in X$, where $n < n_0 < n_1 < ... < n_\gamma$, and the first step is complete.

Case 2^1: For any $\sigma^* \in \Delta_2$, $U^1_{\sigma^*}$ is a proper subset of U_1.

Since all σ^* are nice, we can choose $(U^2_{\sigma^*}, n^2_{\sigma^*}) \in \sigma^*$ so that $U^2_{\sigma^*} \subset U_1(\nu)$ and $\bigcap_{\sigma^* \in \Delta_2} U^2_{\sigma^*} \neq \emptyset$. Then $U^2_{\sigma^*}$ is also a proper subset of U_1 and $n^2_{\sigma^*} > n_1$. Let (U_2, n_2) be s-minimal in the s-embedded sequence $\{(U^2_{\sigma^*}, n^2_{\sigma^*}) : \sigma^* \in \Delta_2\}$. For $\Delta_3 = \Delta_2 \setminus \{\sigma^* \in \Delta_2 : (U^2_{\sigma^*}, n^2_{\sigma^*}) =_S (U_2, n_2)\}$, we obtain

$$(*)_2 : (U, n) \prec_S (U_0, n_0) \prec_S (U_1, n_1) \prec_S (U_2, n_2) \prec_S (U_{\sigma^*}, n_{\sigma^*}) ... (\sigma^* \in \Delta_3),$$
$$\text{and } U \supset U_0 \supset U_0(\nu) \supset U_1 \supset U_1(\nu) \supset U_2 \supset U^2_{\sigma^*} ... (\sigma^* \in \Delta_3),$$

and so on until we have (*), satisfying (1–4).

Now we are working with (*).

We can find $V_0 \in \mathcal{B}_{n_0}(Y)$ so that $V_0 \cap V \neq \emptyset$. Then $n_0 \geq n$ implies $V_0 \subset V$ and $U_0 \times V_0 \subset U \times V$.

We can find $V_1 \in \mathcal{B}_{n_1}(Y)$ so that $V_1 \cap V_0(\nu) \neq \emptyset$. Then $n_1 > n_0$ implies $V_1 \subset V_0(\nu)$ and $U_1 \times V_1 \subset U_0(\nu) \times V_0(\nu)$.

We can find $V_2 \in \mathcal{B}_{n_2}(Y)$ so that $V_2 \cap V_1(\nu) \neq \emptyset$. Then $n_2 > n_1$ implies $V_2 \subset V_1(\nu)$ and $U_2 \times V_2 \subset U_1(\nu) \times V_1(\nu)$ and so on until we have the following sequence:

$$U \times V \prec_S U_0 \times V_0 \supset U_0(\nu) \times V_0(\nu) \supset U_1 \times V_1 \supset U_1(\nu) \times V_1(\nu)$$

$$\supset U_2 \times V_2 \supset ... \supset U_\gamma(\nu) \times V_\gamma(\nu).$$

Finally, $U_\gamma(\nu) \times V_\gamma(\nu)$ is a non-empty subset of the set in our lemma. □

It easily implies the following.

Lemma 5.7. *For any $\alpha < \lambda$ and $\nu \in 3$, the following holds:*

$$B_\alpha \cap \bigcap_{\beta \in \lambda \setminus \alpha} [\bigcup (\sigma_\beta^*)^S(\nu)]_{\beta S} \neq \emptyset.$$

Lemma 5.8. *A point p is a butterfly-point in βS.*

Proof. For any $\nu \in 3$, denote $F_\nu = \{p_\alpha(\nu) : \alpha < \lambda\}$, where $p_\alpha(\nu)$ is any point of the set in the previous lemma. By our construction, $F_\nu \subset B_0 \subset S^*$ and, for any neighborhood O of p, there is $\alpha < \lambda$ with $\{p_\beta(\nu) : \beta \in \lambda \setminus \alpha\} \subset B_\alpha \subset O$. Then the condition $\{p_\beta(\nu) : \beta < \alpha\} \subset [\bigcup (\sigma_\alpha^*)^S(\nu)]_{\beta S}$ implies by Lemma 4.1 both that the sets $[F_\nu]_{\beta S} \setminus \{p\}$ are pairwise disjoint and that $p \in F_\nu$ for no more than one unique F_ν. The other two ensure that p is a b-point in βS. Our proof is complete. □

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References

[1] A. Błaszczyk and A. Szymański, *Some non-normal subspaces of the Čech-Stone compactification of a discrete space* in Abstracta. Proceedings of 8th Winter School on Abstract Analysis. Ed. Zdeněk Frolík. Praha: Czechoslovak Academy of Sciences, 1980. 35–38.

[2] A. Gryzlov, *Some types of points in N^** in Proceedings of the 12th Winter School on Abstract Analysis (Srní, 1984). Rend. Circ. Mat. Palermo (2) 1984, Suppl. No. 6, 137–138.

[3] Sergei Logunov, *On hereditary normality of compactifications*, Topology Appl. **73** (1996), no. 3, 213–216.

[4] ———, *On hereditary normality of zero-dimensional spaces*, Topology Appl. **102** (2000), no. 1, 53–58.

[5] ———, *Strong non-remote points*, Topology Appl. **149** (2005), no. 1-3, 285–288.

[6] ———, *On non-normality points and metrizable crowded spaces*, Comment. Math. Univ. Carolin. **48** (2007), no. 3, 523–527.

[7] ———, *On remote points and butterfly-points* (Russian), Izvestia instituta matematiki i informatiki, Udmurt State University, Izhevsk **3 (26)** (2002), 115–120.

[8] Jan van Mill, *A remark on the Rudin–Keisler order of ultrafilters*, Houston J. Math. **9** (1983), no. 1, 125–129.

[9] B. È Šapirovskiĭ, *The imbedding of extremally disconnected spaces in bicompacta, b-points and weight of pointwise normal spaces* (Russian), Dokl. Akad. Nauk SSSR **223** (1975), no. 5, 1083–1086.

[10] Jun Terasawa, $\beta X\{p\}$ *are non-normal for non-discrete spaces* X, Topology Proc. **31** (2007), no. 1, 309–317.

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