Geography of Symplectic 4- and 6-Manifolds

by

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ABSTRACT. The geography of minimal symplectic 4-manifolds with arbitrary fundamental group and symplectic 6-manifolds with abelian fundamental group of small rank, and with arbitrary fundamental group are addressed.

1. INTRODUCTION

The geography problem for simply connected irreducible symplectic 4-manifolds with negative signature has been settled in its entirety [20], [16], [36], [37], [1], [15], [8], [9], [3], [4]. Much less is known about the non simply connected realm. It has been observed that the new construction techniques [20], [32], [5] prove to be rather effective when one wishes to study the symplectic geography problem for a variety of choices of fundamental groups [20], [9], [3], [4], [46], [45]. Concerning symplectic 4-manifolds with prescribed fundamental group, we obtained the following theorem.

Theorem 1.1. *Let G be a group with a presentation consisting of g generators and r relations. Let e and σ be integers that satisfy $2e + 3\sigma \geq 0, e + \sigma \equiv 0 \mod 4$, and $e + \sigma \geq 8$, and assume $\sigma \leq -1$. Then there exists a minimal symplectic 4-manifold $M(G)$ with fundamental group $\pi_1(M(G)) \cong G$, with characteristic numbers*

$$(c_1^2(M(G)), \chi_h(M(G))) = (2e + 3\sigma + 4(g + r), \tfrac{1}{4}(e + \sigma) + (g + r)),$$

and with odd intersection form over $\mathbb{Z}$. If G is finite, then $M(G)$ admits exotic smooth structures.

Assume $s \geq 1$ and $n \in \mathbb{N}$. For each of the following pairs of integers

$$(c, \chi) = (8n - 8 + 8(g + r), 2s + n - 1 + (g + r)),$$

there exists a spin symplectic 4-manifold $X(G)$ with $\pi_1(X(G)) \cong G$ and characteristic numbers

$$(c_1^2(X(G)), \chi_h(X(G))) = (c, \chi).$$

Moreover, there is an infinite set $\{X_m(G)) : m \in \mathbb{N}\}$ of pairwise non-diffeomorphic symplectic and nonsymplectic manifolds homeomorphic to $X(G)$.

If G is a residually finite group, the work of M. J. D. Hamilton and D. Kotschick [24] implies that the manifolds constructed are irreducible. Theorem 1.1 extends earlier results [20], [7], [38], [9]; we give a brief account of them in order to make clear the contribution of the previous theorem. Robert E. Gompf proved the well-known result that any finitely presented group can be the fundamental group of a symplectic $2n$-manifold for $n \geq 2$ [20, §6]. Scott Baldridge and Paul Kirk took on the endeavor of minimizing the Euler characteristic with respect to the generators and relations of a presentation of the fundamental group in [7], using the elliptic surface $E(1)$ and the manifold described in §2.2 as building blocks. They constructed a minimal symplectic manifold with Euler characteristic $12 + 12(g + r)$, and signature $-8 - 8(g + r)$. Jongil Park systematically studied the geography in [38, Theorem 1] using the construction in [7] as a building block. Moreover, his result also addresses the existence of a myriad of exotic smooth structures by establishing a homeomorphism on the manifolds he constructed, using the results of Steven Boyer [11] on homeomorphisms of simply connected 4-manifolds with a given boundary.

Using a minimal symplectic manifold homeomorphic to $3\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$ as a building block instead of $E(1)$, Baldridge and Kirk constructed a smaller minimal symplectic example with Euler characteristic $10 + 6(g + r)$, and signature $-2 - 2(g + r)$ in [9, Theorem 24] ($(c_1^2, \chi_h) = (14 + 6(g + r), 2 + (g + r))$ under the coordinates used in the statement of Theorem 1.1). In [48], the second author of the present paper improved the known bounds by producing a minimal symplectic example with Euler characteristic $10 + 4(g + r)$, and signature -2 ($(c_1^2, \chi_h) = (14 + 8(g + r), 2 + (g + r))$). His construction uses as a building block the symplectic manifold sharing the homology of $S^2 \times S^2$ built by Ronald Fintushel, B. Doug Park, and Ronald J. Stern in [15, §4]; see §2.2. Theorem 1.1 offers an improvement and extends the results in [7], [38], [9], [48].

For known fundamental-group-dependent bounds on the geographical regions that can be populated by minimal symplectic manifolds with

prescribed fundamental group and comparisons to fundamental-group-dependent results on the existence of smooth manifolds in dimension four, the reader is directed to the work of Clifford Henry Taubes [43], of Baldridge and Kirk [7], of Kotschick [29], and of Kirk and Charles Livingston [27], [28].

We now direct our attention to high dimensional manifolds. The geography of simply connected symplectic 6-manifolds was settled by Mihai Halic in [22] (see also [26], [35]), where he proved that any triple $(a, b, c) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ that satisfies $a \equiv c \equiv 0 \bmod 2$, and $b \equiv 0 \bmod 24$ can be realized by a simply connected symplectic 6-manifold with Chern numbers $(c_1^3, c_1 c_2, c_3) = (a, b, c)$. Although there is no established definition of minimality in dimension six yet (see [31, Definition 5.2]), the existence result of Halic relies heavily on blow ups along points and along surfaces [22, Lemma 4.1]. This scenario motivates interest for constructions of examples that do not involve blow ups. In this direction, we obtained the following two results.

Theorem 1.2. *Let G be a group among the choices $\{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}_p \oplus \mathbb{Z}, \mathbb{Z}, \pi_1(\Sigma_g), F_n\}$, where Σ_g is a closed oriented surface of genus g and F_n is the free group of rank n. Let e_i, σ_i, $i = 1, 2$ be integers that satisfy $2e_i + 3\sigma_i \geq 0, e_i + \sigma_i \equiv 0 \bmod 4$, and $e_i + \sigma_i \geq 8$. There exist symplectic 6-manifolds $W_0(G), W_1(G), W_2(G)$ with fundamental group $\pi_1(W_i(G)) \cong G$, for $i = 0, 1, 2$, and the following Chern numbers:*

(1) $c_1^3(W_0(G)) = 18 \cdot (\sigma_1 + \sigma_2) + 12 \cdot (e_1 + e_2)$
    $c_1 c_2(W_0(G)) = 6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2)$, *and*
    $c_3(W_0(G)) = 2 \cdot (e_1 + e_2)$;

(2) $c_1^3(W_1(G)) = 0$,
    $c_1 c_2(W_1(G)) = 0$, *and*
    $c_3(W_1(G)) = 0$;

(3) $c_1^3(W_2(G)) = -18 \cdot (\sigma_1 + \sigma_2) - 12 \cdot (e_1 + e_2) - 48$,
    $c_1 c_2(W_2(G)) = -6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) - 24$, *and*
    $c_3(W_2(G)) = -2 \cdot (e_1 + e_2) - 8$.

Assume $s_1, s_2 \geq 1$ and $n_1, n_2 \in \mathbb{N}$. There exist spin symplectic 6-manifolds $Y_i(G)$ for $i = 0, 1, 2$ with fundamental group $\pi_1(Y_i(G)) \cong G$ and whose Chern numbers are

(1) $c_1^3(Y_0(G)) = 48 \cdot (n_1 + n_2 - 2)$,
    $c_1 c_2(Y_0(G)) = 48 \cdot (s_1 + s_2) + 24 \cdot (n_1 + n_2 - 2)$, *and*
    $c_3(Y_0(G)) = 48 \cdot (s_1 + s_2) + 8 \cdot (n_1 + n_2 - 2)$;

(2) $c_1^3(Y_1(G)) = 0$,
$\quad c_1 c_2(Y_1(G)) = 0$, and
$\quad c_3(Y_1(G)) = 0$;

(3) $c_1^3(Y_2(G)) = -48 \cdot (n_1 + n_2 - 1)$,
$\quad c_1 c_2(Y_2(G)) = -48 \cdot (s_1 + s_2) - 24 \cdot (n_1 + n_2 - 1)$, and
$\quad c_3(Y_2(G)) = -48 \cdot (s_1 + s_2) - 8(n_1 + n_2 - 1)$.

Theorem 1.2 is obtained by building upon the aforementioned work on the geography of simply connected 4-manifolds. Next, the current understanding on the geography of minimal symplectic 4-manifolds with arbitrary fundamental group ([20], [7], [38], [48], Theorem 1.1) is used to study the existence of symplectic 6-manifolds with prescribed fundamental group. Such enterprise had been pursued previously by Gompf in his seminal paper [20, Theorem 7.1].

Theorem 1.3. *Let G be a group with a presentation that consists of g generators and r relations. Let e_i, σ_i, for $i = 1, 2$, be integers that satisfy $2e_i + 3\sigma_i \geq 0$, $e_i + \sigma_i \equiv 0 \bmod 4$, and $e_i + \sigma_i \geq 8$. There exist symplectic 6-manifolds $W_0(G), W_1(G), W_2(G)$ with fundamental group $\pi_1(W_i(G)) \cong G$ for $i = 0, 1, 2$, and the following Chern numbers:*

(1) $c_1^3(W_0(G)) = 18 \cdot (\sigma_1 + \sigma_2) + 12 \cdot (e_1 + e_2) + 48 \cdot (g + r)$,
$\quad c_1 c_2(W_0(G)) = 6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) + 24 \cdot (g + r)$, and
$\quad c_3(W_0(G)) = 2 \cdot (e_1 + e_2) + 8 \cdot (g + r)$;

(2) $c_1^3(W_1(G)) = 0$,
$\quad c_1 c_2(W_1(G)) = 0$, and
$\quad c_3(W_1(G)) = 0$;

(3) $c_1^3(W_2(G)) = -18 \cdot (\sigma_1 + \sigma_2) - 12 \cdot (e_1 + e_2) - 48 \cdot (g + r) - 48$,
$\quad c_1 c_2(W_2(G)) = -6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) - 24 \cdot (g + r) - 24$, and
$\quad c_3(W_2(G)) = -2 \cdot (e_1 + e_2) - 8 \cdot (g + r)$.

Assume $s_1, s_2 \geq 1$ and $n_1, n_2 \in \mathbb{N}$. There exist spin symplectic 6-manifolds $Y_i(G)$ for $i = 0, 1, 2$ with fundamental group $\pi_1(Y_i(G)) \cong G$ and whose Chern numbers are

(1) $c_1^3(Y_0(G)) = 48 \cdot (n_1 + n_2 - 2) + 48(g + r)$,
$\quad c_1 c_2(Y_0(G)) = 48 \cdot (s_1 + s_2) + 24 \cdot (n_1 + n_2 - 2) + 24 \cdot (g + r)$, and
$\quad c_3(Y_0(G)) = 48 \cdot (s_1 + s_2) + 8 \cdot (n_1 + n_2 - 2) + 8 \cdot (g + r)$;

(2) $c_1^3(Y_1(G)) = 0$,
$\quad c_1 c_2(Y_1(G)) = 0$, and
$\quad c_3(Y_1(G)) = 0$;

(3) $c_1^3(Y_2(G)) = -48 \cdot (n_1 + n_2 - 2) - 48 \cdot (g + r) - 48$,
$c_1 c_2(Y_2(G)) = -24 \cdot (n_1 + n_2 - 2) - 48 \cdot (s_1 + s_2) - 24 \cdot (g + r) - 24$,
and
$c_3(Y_2(G)) = -48 \cdot (s_1 + s_2) - 8 \cdot (n_1 + n_2 - 2) - 8 \cdot (g + r) - 8$.

Recent constructions of symplectic manifolds with canonical class $c_1 = 0$ are related to our geographical studies (see [40], [12], [13], [10], [2], [14]). There are manifolds of Theorem 1.2 and Theorem 1.3(2) that satisfy $c_1 = 0$. The first examples of manifolds $Y_1(\{1\})$ and $Y_1(\mathbb{Z})$, in Theorem 1.3(2) with $c_1 = 0$ appeared in [2]. These manifolds have large third Betti number. In [14, Theorem 3], Joel Fine and Dmitri Panov constructed such symplectic manifolds with arbitrary fundamental group and $b_3 = 0$ [14, Theorem 3].

Finally, we recover and extend the main result in [22] to the geography of symplectic 6-manifolds for various choices of fundamental groups. The case of arbitrary fundamental group was studied previously by Gompf in [20, §7], and by Federica Benedetta Pasquotto in [39, Proposition 4.31]. The statement is the following.

Corollary 1.4. *Let $X(G)$ be a symplectic 6-manifold. Then*
$$c_1^3(X(G)) \equiv c_3(X(G)) \equiv 0 \bmod 2,$$
$$c_1 c_2(X(G)) \equiv 0 \bmod 24, \text{ and } \pi_1(X(G)) \cong G.$$
Conversely, any triple (a, b, c) of integers satisfying $a \equiv c \equiv 0 \bmod 2$ and $b \equiv 0 \bmod 24$ occurs as a triple $(c_1^3, c_1 c_2, c_3)$ of Chern numbers of a symplectic 6-manifold $X(G)$ with $\pi_1(X(G)) \cong G$, where G is a finitely presented group.

Unlike the manifolds of Theorem 1.2 and Theorem 1.3, the construction of the manifolds of Corollary 1.4 involve blow ups along points and along surfaces. The fundamental tools of construction used to obtain all these results are Gompf's symplectic sum [20] and Luttinger surgery [32] (see also [5]).

The paper is organized as follows. The second section contains a description of our 4-dimensional building blocks. In §2.1, we present the building blocks that allow us to produce manifolds for several choices of fundamental groups. The building block used to construct manifolds with arbitrary fundamental group is described in §2.2. A useful explanation on the fundamental group computations of Theorem 1.1 is given in §2.3. The fundamental group computations are located in §5. Sections 2.4 and 2.5 contain the building blocks we use to populate geographic regions. The geography of symplectic 4-manifolds with prescribed fundamental group

is studied in §3. The proof of Theorem 1.1 is given in sections 3.1 and 3.2. We then use these results to study 6-manifolds in §4. The formulae needed to compute Chern numbers are given in §4.1. In §4.2, we build symplectic sums that are fundamental for our purposes. The tool that allows us to prove Corollary 1.4 is presented in §4.3, where we work out an example to demonstrate the utility of these techniques for several fundamental groups. Section 5.1 provides a detailed description on our choices of gluing maps used in the 6-dimensional symplectic sums. The fundamental group computations involved in the statements of several of our results, including Theorem 1.2 and Theorem 1.3, are presented in §5.2. Finally, §5.3 contains the proofs of Theorem 1.3 and Corollary 1.4.

2. Building Blocks

2.1. Symplectic 4-manifolds used to obtain several of our choices for π_1.

The following building block in our symplectic sums will allow us to vary our choices of fundamental groups without contributing anything to the characteristic numbers. Take the product of a torus and a genus two surface $T^2 \times \Sigma_2$. The characteristic numbers are given by $c_2(T^2 \times \Sigma_2) = 0 = \sigma(T^2 \times \Sigma_2)$. Endow $T^2 \times \Sigma_2$ with the symplectic product form. The torus $T^2 \times \{x\}$ and the genus two surface $\{x\} \times \Sigma_2$ are symplectic submanifolds of $T^2 \times \Sigma_2$, which are geometrically dual to each other. Let F denote a parallel copy of the surface of genus two. Furthermore, $T^2 \times \Sigma_2$ contains 4 pairs of geometrically dual Lagrangian tori available to perform Luttinger surgery [32], [5] on them; see also [25, §2]. These tori are of the form $S^1 \times S^1 \subset T^2 \times \Sigma_2$.

Regarding the very relevant fundamental group computations, we have the following result. Let $\{x, y\}$ be the generators of $\pi_1(T^2)$ and let $\{a_1, b_1, a_2, b_2\}$ be the standard set of generators of $\pi_1(\Sigma_2)$.

Proposition 2.1 (Baldridge and Kirk [9, Proposition 7]). *The fundamental group*

$$\pi_1(T^2 \times \Sigma_2 - (T_1 \cup T_2 \cup T_3 \cup T_4 \cup F))$$

is generated by the loops x, y, a_1, b_1, a_2, and b_2. Moreover, with respect to certain paths to the boundary of the tubular neighborhoods of the T_i and F, the meridians and two Lagrangian push offs of the surfaces are given by

- $T_1 : m_1 = x, l_1 = a_1,\ \mu_1 = [b^{-1}, y^{-1}],$
- $T_2 : m_2 = y, l_2 = b_1 a_1 b^{-1},\ \mu_2 = [x^{-1}, b_1],$
- $T_3 : m_3 = x, l_3 = a_2,\ \mu_3 = [b_2^{-1}, y^{-1}],$

- $T_4 : m_4 = y, l_4 = b_2 a_2 b_2^{-1}, \mu_4 = [x^{-1}, b_2]$,
- $\mu_F = [x, y]$.

The loops a_1, b_1, a_2, and b_2 lie on the genus 2 surface and form a standard set of generators; the relation $[a_1, b_1][a_2, b_2] = 1$ holds.

The reader is referred to the lovely paper [9] for details on constructions of symplectic 4-manifolds using Luttinger surgery. We finish this section by pointing out that the description above yields the building blocks needed to produce the manifolds of Theorem 1.1, Theorem 1.2, and Corollary 1.4 with surface fundamental groups $\pi_1(\Sigma_g)$. Take the product symplectic manifold $T^2 \times \Sigma_g$ of the 2-torus with a surface of genus $g \geq 3$. Endowed with the product symplectic form, the submanifold $T^2 \times \{x\} \subset T^2 \times \Sigma_g$ is symplectic. Given that $\{x\} \times \Sigma_g$ is geometrically dual to it, it is straightforward to see that the meridian μ_{T^2} in $T^2 \times \Sigma_g - T^2 \times \{x\}$ can be expressed in terms of the generators and relations of $\pi_1(\{x\} \times \Sigma_g)$ see [8] and [9] for details.

2.2. **4-DIMENSIONAL BUILDING BLOCK WITH PRESCRIBED FUNDAMENTAL GROUP.**

In [48, Theorem 1.1], the second author proved the following result.

Theorem 2.2. *Let G be a group with a presentation with g generators and r relations. There exists a minimal symplectic 4-manifold X with fundamental group $\pi_1(X(G)) \cong G$ and characteristic numbers*

$$c_1^2(X(G)) = 14 + 8(g + r),\ c_2(X(G)) = 10 + 4(g + r),\ and$$
$$\sigma(X(G)) = -2,\ \chi_h(X(G)) = 2 + (g + r).$$

The manifold $X(G)$ contains a homologically essential Lagrangian torus T, and a symplectic surface of genus two F with trivial normal bundle, and

$$\pi_1(X(G) - T) \cong \pi_1(X(G) - F) \cong \pi_1(X(G)) \cong G\ .$$

Let us describe the materials used to prove this result. The fundamental building block for these constructions is a symplectic manifold built in [7], which allows the manipulation on the number of generators and relations on fundamental groups. We proceed to describe it.

Baldridge and Kirk take a 3-manifold Y that fibers over the circle, and build the 4-manifold $N := Y \times S^1$. It admits a symplectic structure ([7, p. 856]). Its Euler characteristic and its signature are both zero. The fundamental group $\pi_1(Y \times S^1)$ has classes $s, t, \gamma_1, \ldots, \gamma_{g+r}$ so that

$$G \cong \pi_1(Y \times S^1)/N(s, t, \gamma_1, \ldots, \gamma_{g+r}),$$

where $N(s, t, \gamma_1, \ldots, \gamma_{g+r})$ is the normal subgroup generated by the aforementioned classes.

A result of W. P. Thurston [44] implies that the manifold N is symplectic. There are $g+r+1$ symplectically imbedded tori $T_0, T_1, \ldots, T_{g+r} \subset N$, which have the following crucial traits for our purposes concerning fundamental group computations.

- The generators of $\pi_1(T_0)$ represent s and t.
- The generators of $\pi_1(T_i)$ represent s and γ_i for $i \geq 1$.

The curve s has the form $\{y\} \times S^1 \subset Y \times S^1$, with $y \in Y$, and γ_i has the form $\gamma_i \times \{x\} \subset Y \times \{x\}, x \in S^1$. We point out that the role of the curves γ_i is to provide the r relations in a presentation of G [7, §4].

The details of the construction depend on the presentation of our group; we give a description of the construction with the purpose of making the paper a bit more self-contained. Let G be a finitely presented group: $G = \langle x_1, \ldots, x_g | w_1, \ldots, w_r \rangle$ with g generators $x_1, \ldots, x_g$ and r relations $w_1, \ldots, w_r$. Let n_i denote the length of the relation w_i, and set

$$n = 1 + \sum_{i=1}^{r} n_i \,.$$

Let Σ_{gn} denote a surface of genus gn. We will define an automorphism R on Σ_{gn}, and so we consider the following description of Σ_{gn} to aid with specifying R. Consider a round 2-sphere $S^2 \subseteq \mathbb{R}^3$ centered at the origin, and let $\bar{R}$ denote a rotation of S^2 by an angle of $\frac{2\pi}{ng}$ through the axis passing through the points $(0, 0, \pm 1)$. The orbit of $(1, 0, 0)$ under $\langle \bar{R} \rangle \subseteq SO(3)$ gives ng distinct points on S^2, and performing a connected sum of a torus with S^2 at each point in $\langle \bar{R} \rangle \{(1, 0, 0)\}$ in an equivariant way gives a surface Σ_{gn} so that the action of $\bar{R}$ on S^2 naturally gives an action R of $\mathbb{Z}/(gn)$ on Σ_{gn}.

Let Y be a surface bundle over S^1 given by the monodromy R^g. That is, let

$$Y = \frac{F \times I}{(x, 0) \sim (R^g(x), 1)} \,.$$

Let $p = (0, 0, 1)$, and let τ denote the loop in Y that is the image in Y of $\{p\} \times I$ under $\sim$. This is a loop since $R^g(p) = p$. Let $s, t \in \pi_1(Y \times S^1)$ be given as follows: t is represented by a parametrization of $\tau \times \{1\}$ and s is a generator of $\{1\} \times \mathbb{Z} \subseteq \pi_1(Y) \times \pi_1(S^1)$.

We now use the presentation of G given previously to get a new presentation of G. We add g generators $y_1, \ldots, y_g$; we replace the relations $w_1, \ldots, w_r$ with new relations $\tilde{w}_1, \ldots, \tilde{w}_r$; and we add g more relations $x_1 y_1, \ldots, x_g y_g$. Hence, we have a new presentation of the same group G:

(2.1) $\qquad G = \langle x_1, \ldots, x_g, y_1, \ldots, y_g | \tilde{w}_1, \ldots, \tilde{w}_r, x_1 y_1, \ldots, x_g y_g \rangle \,.$

The relation $\tilde{w}_i$ is obtained from the relation w_i by replacing each occurrence of x_k^{-1} with y_k. Hence, each relation is expressed as a product of positive powers of the generators.

Each relation in the presentation of G given in (2.1) gives an immersed curve γ_i on the surface Σ_{gn}. Identify Σ_{gn} with a fiber of Y, and then take the product of these curve with a circle to obtain tori $T_i := \gamma_i \times S^1 \subseteq Y \times S^1$. Figure 1 below is taken from [7], and here we see examples of how occurrences of x_i and y_i give us curves on this surface.
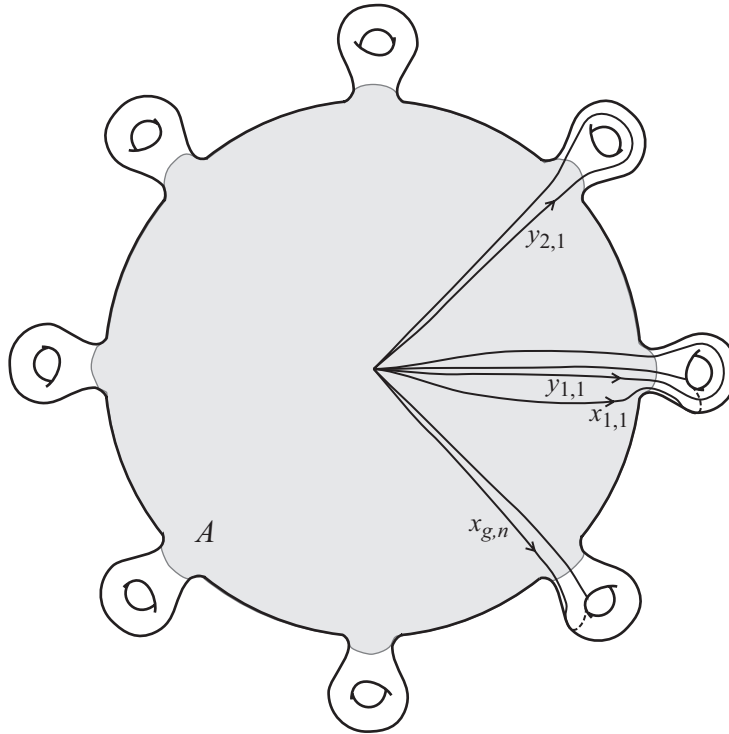

FIGURE 1

In order to make the construction work, the tori T_i need to be disjoint but immersed and the curves γ_i need to be mapped to different parts of the surface. The details are provided in [7]; here we only briefly mention that, by expressing the $r + g$ relations as only positive powers of the $2g$

generators, we ensure that we can perturb the tori T_i so that they are symplectic. A final torus that we consider is T_0, which is the product of the circle corresponding to $s, t \in \pi_1 \left(Y \times S^1 \right)$.

The presentation for $\pi_1 \left(Y \times S^1 \right)$ is

$$(2.2) \qquad \left\langle \pi_1 \Sigma_{gn}, t | R_*^g(x) = txt^{-1} \text{ for } x \in \pi_1 \Sigma_{gn} \right\rangle \times \langle s \rangle,$$

where $\pi_1 \Sigma_{gn}$ has the presentation

$$(2.3) \qquad\qquad\qquad\qquad \pi_1 \Sigma_{gn} =$$

$$\left\langle \begin{array}{c} x_{1,1}, y_{1,1}, \ldots, x_{g,1}, y_{g,1}, x_{1,2}, y_{1,2}, \ldots, x_{g,2}, y_{g,2}, \ldots, \\ x_{1,n}, y_{1,n}, \ldots, x_{g,n}, y_{g,n} \end{array} \middle| \prod_{l=1}^{n} \prod_{k=1}^{g} [x_{k,l}, y_{k,l}] \right\rangle$$

We symplectic sum along the tori $T_0, \ldots, T_{r+g}$ with other symplectic manifolds to kill the push offs of the curves that generate $\pi_1(T_i)$ while not introducing additional generators to the fundamental group of our manifold. For example, if we symplectic sum in such a manner along T_0, then we will kill the generators s and t, and the presentation of (2.2) reduces to

$$\left\langle \pi_1 \Sigma_{gn} | R_*^g(x) = x \text{ for } x \in \pi_1 \Sigma_{gn} \right\rangle.$$

Notice that in the notation of (2.2), $R_*^g(x_{k,l}) = x_{k,l+1}$ and $R_*^g(y_{k,l}) = y_{k,l+1}$, where the addition on the second subscript is taken modulo n (with $n+1 = 1$), and so setting $x_k = x_{k,l}$ and $y_k = y_{k,l}$, the presentation of (2.2) actually reduces further to

$$(2.4) \qquad\qquad \left\langle x_1, y_1, \ldots, x_g, y_g \middle| \prod_{l=1}^{n} \prod_{k=1}^{g} [x_k, y_k] \right\rangle.$$

Summing along the tori $T_{r+1}, \ldots, T_{r+g}$ introduced the relations $x_k y_k = 1$, and so the commutators $[x_k, y_k]$ are trivial, so that the surface relation

$$\prod_{l=1}^{n} \prod_{k=1}^{g} [x_k, y_k]$$

of (2.4) is trivial. Finally, performing a symplectic sum along the tori $T_1, \ldots, T_r$ introduces the relations $\tilde{w}_1, \ldots, \tilde{w}_r$, and the presentation of (2.4) is now

$$\langle x_1, y_1, \ldots, x_g, y_g | x_1 y_1, \ldots, x_g y_g, \tilde{w}_1, \ldots, \tilde{w}_r \rangle$$

which is just the presentation of G in (2.1). Hence, we have shown that fiber summing $Y \times S^1$ with the appropriate symplectic manifolds along the tori $T_0, \ldots, T_{r+g}$ is a symplectic manifold with fundamental group G.

For a proof of Theorem 2.2, the second author used a spin symplectic 4-manifold V with $c_2(V) = 4, \sigma = 0$ that shares the cohomology ring of

$2(S^2 \times S^2) \# S^3 \times S^1$ and contains a symplectic torus T_V of self-intersection zero carrying a generator of $H_1(V; \mathbb{Z}) \cong \mathbb{Z}$. This manifold was constructed in [15, §4]; its construction consists of applying seven Luttinger surgeries ([32], [5]) to the product of two copies of a genus two surface $\Sigma_2 \times \Sigma_2$. There exists a symplectic surface of genus two and self-intersection zero $F \hookrightarrow V$.

A careful analysis of fundamental groups computations under cut-and-paste constructions was done by the second author in [48, §3 and §4]. The main ingredient that yields an improvement of [9, Theorem 24] is the symplectic sum [20, Theorem 1.3] of N and $g + r$ copies of V along the symplectic tori T_i and T_{V_i} for $1 \le i \le g+r$; see [48] for details. Denote this building block by BK. Its characteristic numbers are computed using the well-known formulae in [20, p. 535] to be $c_2(BK) = 4(g+r), \sigma(BK) = 0$.

2.3. Computations of π_1 in Theorem 1.1.

Suppose there is a minimal symplectic 4-manifold X which contains a symplectic torus $T \hookrightarrow X$ of self-intersection zero such that $\pi_1(X - T) \cong \{1\}$ [48, §4]. Build the symplectic sum along tori

$$Z := BK_{T_0 = T} X.$$

In [48, §3 and §4], it is proven that there exists an isomorphism

$$\pi_1(Z) \cong \langle g_1, \ldots, g_g | r_1, \ldots, r_r \rangle \cong G.$$

The 4-manifold X used in the proof of Theorem 2.2 is a minimal symplectic manifold homeomorphic to $3\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$ constructed in [9, Theorem 18]. To prove Theorem 1.1, we will vary the choice of X used in the construction described previously, where the choices come from Proposition 2.3 and Proposition 2.7 in §2.4. The fundamental group computations follow verbatim from the work of the second author in [48].

2.4. 4-dimensional building blocks used to populate geographic regions I.

We proceed to describe the building blocks that have been successfully used by topologists to understand the geography and botany problems of symplectic 4-manifolds (see for example [20], [3], [9], [4], [46], and the references there).

Proposition 2.3. *Let $b^- \in \{4, 5, \ldots, 17, 18, 19\}$. There exists a minimal symplectic simply connected 4-manifold X_{3,b^-} with second Betti number given by $b_2^+(X_{3,b^-}) = 3$ and $b_2^-(X_{3,b^-}) = b^-$, which contains two homologically essential Lagrangian tori T_1 and T_2 such that*

$$\pi_1(X_{3,b^-} - (T_1 \cup T_2)) = \{1\}.$$

The characteristic numbers are $c_2(X_{3,b^-}) = 5 + b_2^-$, $\sigma = 3 - b^-$, *and* $c_1^2(X_{3,b^-}) = 10 + 2b^- + 9 - 3b_2^- = 19 - b^-$.

Table 1 provides a blueprint on the construction of the manifolds in the statement of Proposition 2.3, modulo the number of Luttinger surgeries to be performed and the sources of the building blocks [46]. The notation used is similar to that of Proposition 2.3; the manifolds $X_{1,n}$ are homeomorphic to $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$. The elliptic surface is denoted by $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. The symplectic sum of X and Y along Z is denoted by $X \#_Z Y$.

TABLE 1. Minimal symplectic 4-manifolds with $b_2^+ = 3$.

b_2^-	Symplectic sum/manifold.	Reference.
4	$X_{1,2} \#_{\Sigma_2} T^2 \times \Sigma_2$	[4, Section 9]
5	$X_{3,5}$	[9, Theorem 18]
6	$X_{1,2} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	[4, Section 9]
7	$X_{3,7}$	[9, Corollary 15]
8	$X_{1,5} \#_{\Sigma_2} (T^4 \# \overline{\mathbb{CP}^2})$	[4, Sections 3 and 4].
9	$X_{1,5} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	[9, Corollary 16].
10	$X_{1,6} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	[4, Lemma 15]
11	$E(1) \#_{T^2} S$	[3, Lemma 16]
12	$E(1) \#_{T^2} A$	Theorem 2.6
13	$E(1) \#_{T^2} B$	Theorem 2.6
14	$\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2} \#_{\Sigma_2} T^2 \times \Sigma_2$	[20, Building Block 5.6]
15	$\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2} \#_{\Sigma_2} T^2 \times \Sigma_2$	[20, Building Block 5.6]
16	$\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	[20, Building Blocks 5.6 and 5.6]
17	$\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	[20, Building Blocks 5.6 and 5.7]
18	$S_{1,1}$	[20, Example 5.4]
19	$E(1) \#_{T^2} T^4 \#_{T^2} E(1)$	[41]

The following building block was used in [3] to fill in vast regions of the geography.

Definition 2.4 ([3, Definition 2]). An ordered triple (X, T_1, T_2) consisting of a symplectic 4-manifold X and two disjointly embedded Lagrangian tori T_1 and T_2 is called a *telescoping triple* if

   (1) the tori T_1 and T_2 span a 2-dimensional subspace of $H_2(X; \mathbb{R})$;
   (2) the fundamental group is given by $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$. In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2))$;

(3) the image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \to \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$;
(4) the homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

A smooth 4-manifold is minimal if it does not contain any sphere of self-intersection -1. If X is minimal, then the telescoping triple is said to be minimal. The meridians μ_{T_1} and μ_{T_2} in $\pi_1(X - (T_1 \cup T_2))$ are trivial and the relevant fundamental groups are abelian. The push off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of T_i represents a well-defined element of $\pi_1(X - (T_1 \cup T_2))$ which is independent of the choices of framing and base-point. The first condition allows for the symplectic form on X to be slightly perturbed so that one of the tori T_i remains Lagrangian while the other becomes symplectic [20, Lemma 1.6]. For our purposes, we will require for T_1 to be symplectic.

Proposition 2.5 (see [3, Proposition 3]). *Let (X, T_1, T_2) and (X', T_1', T_2') be two telescoping triples. Then for an appropriate gluing map the triple*
$$(X \#_{T_2, T_1'} X', T_1, T_2')$$
is again a telescoping triple. If X and X' are minimal symplectic 4-manifolds, then the resulting telescoping triple is minimal. The Euler characteristic and the signature of $X \#_{T_2, T_1'} X'$ are given by $c_2(X) + c_2(X')$ and $\sigma(X) + \sigma(X')$.

The proof of Proposition 2.5 follows from [20], [3], and [4], and the claim about minimality follows from [47, Theorem 1.1]. The results concerning the existence of telescoping triples are gathered in the following theorem, which was proven in [3, §5] and [45, Lemma 6 and Lemma 7].

Theorem 2.6. *Existence of telescoping triples.*
- *There exists a minimal telescoping triple (A, T_1, T_2) satisfying $c_2(A) = 5$, $\sigma(A) = -1$.*
- *For each $g \geq 0$, there exists a minimal telescoping triple (B_g, T_1, T_2) satisfying $c_2(B_g) = 6 + 4g$, $\sigma(B_g) = -2$.*
- *There exists a minimal telescoping triple (C, T_1, T_2) satisfying $c_2(C) = 7$, $\sigma(C) = -3$.*
- *There exists a minimal telescoping triple (D, T_1, T_2) satisfying $c_2(D) = 8$, $\sigma(D) = -4$.*
- *There exists a minimal telescoping triple (F, T_1, T_2) satisfying $c_2(F) = 10$, $\sigma(F) = -6$.*

2.5. **4-dimensional building blocks used to populate geographic regions II.**

The building blocks of the previous sections are used to produce the manifolds involved in the proofs of our main results. The following result is due to [3].

Proposition 2.7 (see [3, Theorem A] and [4, Theorem 1]). *Let e and σ be integers that satisfy $2e + 3\sigma \geq 0$, $e + \sigma \equiv 0 \mod 4$, and $e + \sigma \geq 8$, and assume $\sigma \leq -1$. Then there exist simply connected minimal symplectic 4-manifolds $Z_{1,1}$ and $Z_{1,2}$, both with Euler characteristic e, signature σ, and with odd intersection form over $\mathbb{Z}$. Moreover, the 4-manifold $Z_{1,1}$ contains two homologically essential Lagrangian tori T_1 and T_2 such that*
$$\pi_1(Z_{1,1} - (T_1 \cup T_2)) \cong \{1\}.$$
The 4-manifold $Z_{1,2}$ contains a homologically essential Lagrangian torus T_1 and a symplectic surface of genus two F such that
$$\pi_1(Z_{1,2} - (T_1 \cup F)) \cong \{1\}.$$

Remark 2.8. At this point we emphasize that Proposition 2.3 and Proposition 2.7 are fruits of the efforts of several topologists (see [8], [9], [3], [4], [46], and the references provided there). The observation regarding the existence of the submanifolds is crucial for our purposes, and it has not appeared in the literature previously.

Proof. The construction of the 4-manifold $Z_{1,1}$ is obtained by an iterated use of the telescoping triples of Theorem 2.6, Proposition 2.3, Proposition 2.1 (which cover most of the geographic points), and the results of [9], [3], [4], and [15, §4]. The details are left to the reader (see [46] for details).

The construction of the 4-manifold $Z_{1,2}$ goes as follows. Build the symplectic sum $Z := Z_{1,1} \#_{T_1 = T^2} T^2 \times \Sigma_2$. Apply four Luttinger surgeries to obtain a simply connected symplectic 4-manifold $Z_{1,2}$; see Proposition 2.1 for the fundamental group computations. The torus T_2 is contained in $Z_{1,2}$ and so is a parallel copy of the genus two surface $\{x\} \times \Sigma_2 \subset T^2 \times \Sigma_2$ that we now call F [20, Corollary 1.7]. The meridian μ_F is trivial in the complement by Proposition 2.1.   □

3. Geography of Symplectic 4-Manifolds with Prescribed Fundamental Group

In this section we give a proof of Theorem 1.1. The proof is split into two parts.

3.1. Manifolds with odd intersection form and negative signature.

The first part of Theorem 1.1 is proven in this section. We build greatly upon the progress on the geography problem for nonspin simply connected symplectic 4-manifolds with negative signature ([20], [38], [8], [9], [15], [3], [4]), and the study of minimal symplectic 4-manifolds with arbitrary fundamental group ([7], [48]).

Theorem 3.1. *Let G be a group with a presentation consisting of g generators and r relations. Let e and σ be integers that satisfy $2e + 3\sigma \geq 0$, $e + \sigma \equiv 0 \mod 4$, and $e + \sigma \geq 8$, and assume $\sigma \leq -1$. Then there exists a minimal symplectic 4-manifold $M(G)$ with fundamental group $\pi_1(M(G)) \cong G$ and characteristic numbers*

$$c_2(M(G)) = e + 4(g + r), \text{ and } \sigma(M(G)) = \sigma.$$

The intersection form over $\mathbb{Z}$ of $M(G)$ is odd. If the group G is residually finite, then $M(G)$ is irreducible. If the group G is finite, then $M(G)$ admits exotic smooth structures.

Proof. The proof of the existence claim is verbatim to the proof of [48, Theorem 1]. It consists of building the symplectic sum [20, Theorem 1.3] $M(G) := Z_{1,1} \#_T BK$ along a symplectic torus of the manifold BK described in §2.2 and a manifold $Z_{1,1}$ from Proposition 2.7. If the group G is residually finite, then $M(G)$ is irreducible by [24].

The claim concerning the existence of exotic smooth structures on 4-manifolds follows from a pigeonhole argument (see [48, Remark 1.2]). An infinite family of pairwise nondiffeomorphic manifolds sharing the same fundamental group and characteristic numbers is constructed using the techniques in [16], [17], and [15] and keeping track of the Seiberg-Witten invariants using [42] and [34]; see [9, p. 343: Remark]. It is proven in [23, Corollary 1.5] that there are only finitely many homeomorphism types of closed oriented 4-manifolds with finite fundamental group, and a given Euler characteristic. Thus, there exist exotic smooth structures as it was claimed.   □

3.2. Spin manifolds with negative signature.

We now prove the second part of Theorem 1.1. B. D. Park and Zoltán Szabó settled the geography of simply connected spin symplectic 4-manifolds in [36] (see [37] as well). We extend their results to finitely presented fundamental group. We then follow J. Park's work [38] to build exotic smooth structures on the manifolds constructed by using Fintushel and Stern's knot surgery [16] and using Boyer's work [11] on homeomorphisms of simply connected 4-manifolds with a given boundary.

We recall Boyer's [11] results on homeomorphisms of simply connected 4-manifolds with a given boundary following the exposition in [38]. Let V_1 and V_2 be compact connected simply connected 4-manifolds with connected boundary $\partial V_1 \cong \partial V_2$, and let $f : \partial V_1 \to \partial V_2$ be an orientation-preserving homeomorphism. There are two obstructions that need be overcome in order to extend f to a homeomorphism $F : V_1 \to V_2$. First, an isometry $\Lambda : (H_2(V_1; \mathbb{Z}), Q_{V_1}) \to (H_2(V_2; \mathbb{Z}), Q_{V_2})$ for which the following diagram commutes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\partial V_1) & \longrightarrow & H_2(V_1) & \longrightarrow & H_2(V_1, \partial V_1) & \longrightarrow & H_1(\partial V_1) & \longrightarrow & 0 \\
& & \downarrow f_* & & \downarrow \Lambda & & \uparrow \Lambda^* & & \downarrow f_* & & \\
0 & \longrightarrow & H_2(\partial V_2) & \longrightarrow & H_2(V_2) & \longrightarrow & H_2(V_2, \partial V_2) & \longrightarrow & H_1(\partial V_2) & \longrightarrow & 0
\end{array}
$$

must exist. In this context, an isometry is an isomorphism $\Lambda : H_2(V_1; \mathbb{Z}) \to H_2(V_2; \mathbb{Z})$ that preserves intersection forms Q_{V_i}. The morphism Λ^* in the diagram is the adjoint of Λ with respect to the identification of $H_2(V_i, \partial V_i)$ with $Hom(H_2(V_i); \mathbb{Z})$ that arises from Lefschetz duality. This information is encoded in the pair (f, Λ), which is called a *morphism* and is denoted symbolically by $(f, \Lambda) : V_1 \to V_2$.

Second, one must find a homeomorphism $F : V_1 \to V_2$ such that the morphism satisfies $(f, \Lambda) = (F|_{\partial V_1}, F_*)$. A morphism (f, Λ) is *geometrically realized* if this condition is satisfied. In [11], Boyer studied conditions under which a given morphism can be geometrically realized.

Theorem 3.2 ([11, Theorem 0.7, Proposition 0.8]). *If $(f, \Lambda) : V_1 \to V_2$ is a morphism between two simply connected compact smooth 4-manifolds V_1 and V_2 with boundary $\partial V_1 \cong \partial V_2$, there is an obstruction $\theta(F, \Lambda) \in I^1(\partial V_1)$ such that (f, Λ) is realized geometrically if and only if $\theta(f, \Lambda) = 0$. Moreover, if $H_1(\partial V_1; \mathbb{Q}) = 0$, then $\theta(f, \Lambda) = 0$.*

Particularly useful is the following corollary which appears in J. Park's work [38].

Corollary 3.3 ([38, Corollary 2.3]). *Assume $\partial V_1 \cong \partial V_2$ is a homology 3-sphere. For any morphism $(f, \Lambda) : V_1 \to V_2$ between simply connected smooth 4-manifolds V_1 and V_2, there is a homeomorphism $F : V_1 \to V_2$ such that $(f, \Lambda) = (F|_{\partial V_1}, F_*)$.*

We are now ready to prove our first result. The precise statement of Theorem 1.1 reads as follows.

Theorem 3.4. *Let $s \geq 1$, $n \in \mathbb{N}$ and let G be a finitely presented group. For each of the following pairs of integers*

$$(c, \chi) = (8n - 8 + 8(g + r), 2s + n - 1 + (g + r)),$$

there exists a symplectic spin 4-manifold $X(G)$ with

$$\pi_1(X(G)) = G \ and \ (c_1^2(X(G)), \chi_h(X(G))) = (c, \chi).$$

If the group G is residually finite, then $X(G)$ is irreducible. Moreover, there is an infinite set $\{X_m(G)) : m \in \mathbb{N}\}$ of pairwise nondiffeomorphic symplectic and nonsymplectic manifolds homeomorphic to $X(G)$ for every $m \in \mathbb{N}$.

The characteristic numbers of the examples of Theorem 3.4 can be stated as $c_2(X(G)) = 4(n - 4 + 6s + (g + r))$ and $\sigma(X(G)) = -16s$. The following proof is a modification of J. Park's work done in [38].

Proof. Let Z be an irreducible simply connected symplectic spin 4-manifold of [36] (see also [37].) B. D. Park and Szabó constructed Z using simply connected minimal elliptic surfaces without multiple fibers $E(n)$, which are known to contain Gompf nuclei $C_n \subset E(n)$ [19]. In particular, there exists a symplectic torus $T \hookrightarrow Z$ of self-intersection zero, with $\pi_1(Z - T) \cong \{1\}$. Build the symplectic sum [20, Theorem 1.3]

$$X(G) := Z \#_T BK,$$

where BK is the manifold described in §2.2. By [20, Proposition 1.2], the manifolds $X(G)$ are spin manifolds. The computations of the characteristic numbers are straightforward, since $c_1(T^2) = 0$, $c_2(BK) = 4(g+r)$, and $\sigma(BK) = 0$. The claim $\pi_1(X(G)) \cong G$ follows from the computations in §2.3. If the group G is residually finite, then $M(G)$ is irreducible by [24]. The existence claim follows from a variation of the choice of symplectic manifold Z one takes from [36] in the construction of the symplectic sum $X(G)$.

From the existence of Gompf nuclei inside the building block Z, one concludes that there exists a symplectic torus $T' \hookrightarrow X(G)$ such that the inclusion induces the trivial map $\pi_1(T') \to \pi_1(X(G) - T') \cong G$. The infinite set of pairwise nondiffeomorphic symplectic and nonsymplectic 4-manifolds $\{X_m(G) : m \in \mathbb{N}\}$ is constructed using Fintushel and Stern's knot surgery [16], varying the choice of knot.

To prove the claim on the homeomorphism type of the manifolds constructed, we use an argument due to J. Park [38, Proposition 2.2]. Let the 4-manifold with arbitrary fundamental group $X_i(G) := Z_{K_i} \#_T BK \in \{X_m(G) : m \in \mathbb{N}\}$ be the result of applying knot surgery with knot K_i. We claim that there exists a homeomorphism between the manifolds $X_i(G) = Z_{K_i} \#_T BK$ and $X_j(G) = Z_{K_j} \#_T BK$ for any $i, j \in \mathbb{N}$. In this notation, we are specifying that the surgery occurs in the Z block of the construction and calling Z_{K_i} the homeomorphic simply connected manifolds produced using knot surgery with knot K_i. The spin manifold Z_{K_i} contains a Gompf nucleus as a codimensional zero submanifold, independent of the knot K_i. Let C_n be the Gompf nucleus, let Σ be a homology

3-sphere, and set $Z_{K_i}^{\circ} := Z_{K_i} - C_n$. The symplectic sum $Z_{K_i} \#_T BK$ can be decomposed as

$$Z_{K_i}^{\circ} \cup_{\Sigma} (BK \#_T C_n).$$

The endeavor at hand is to construct a morphism (f, Λ) and then use Boyer's results to show that it is geometrically realized. The choice of homeomorphism f is canonical by setting $g := id : BK \#_T C_n \to BK \#_T C_n$, so that its restriction yields $g|_{\partial(BK \#_T C_n)} = f : \Sigma \to \Sigma$. Take an isomorphism $\Lambda : H_2(Z_{K_i}^{\circ}; \mathbb{Z}) \to H_2(Z_{K_j}^{\circ}; \mathbb{Z})$. Note that the intersection forms $Q_{Z_{K_i}^{\circ}}$ and $Q_{Z_{K_i}^{\circ}}$ are unimodular, indefinite forms over $\mathbb{Z}$ that have the same rank, same signature, and same type. By the classification of unimodular indefinite integral forms, $Q_{Z_{K_i}^{\circ}} \cong Q_{Z_{K_i}^{\circ}}$. We have that $\Lambda : (H_2(Z_{K_i}^{\circ}; \mathbb{Z}), Q_{Z_{K_i}^{\circ}}) \to (H_2(Z_{K_j}^{\circ}; \mathbb{Z}), Q_{Z_{K_j}^{\circ}})$ is an isomorphism that preserves intersection form, i.e., an isometry. Therefore, we have a required morphism $(f, \Lambda) : Z_{K_i}^{\circ} \to Z_{K_j}^{\circ}$.

Since $H_1(\partial(Z_{K_i}^{\circ}); \mathbb{Q}) = H_1(\partial(Z_{K_i}^{\circ}); \mathbb{Q}) = H_1(\Sigma; \mathbb{Q}) = 0$, then $\theta(f, \Lambda) = 0$ by Theorem 3.2. It follows from Corollary 3.3 that there exists a homeomorphism $F : Z_{K_i}^{\circ} \to Z_{K_j}^{\circ}$ such that $(f, \Lambda) = (F|_{\Sigma}, F_*)$. Thus, for $i, j \in \mathbb{N}$, any two manifolds $X_i(G) = Z_{K_i} \#_T BK$ and $X_j(G) = Z_{K_j} \#_T BK$ are homeomorphic. $\qquad\square$

4. GEOGRAPHY OF SYMPLECTIC 6-MANIFOLDS

4.1. COMPUTING CHERN NUMBERS.

Concerning the 6-manifolds arising as products of 4- and 2-manifolds, we have the following lemma.

Lemma 4.1. *Let Y be a symplectic 4-manifold and Σ_g a symplectic surface of genus g. Let $e := e(Y)$ be the Euler characteristic and $\sigma := \sigma(Y)$ the signature. The Chern numbers of the product symplectic 6-manifold $X := Y \times \Sigma_g$ are given as follows:*

- *$c_1^3(X) = (2e + 3\sigma) \cdot (6 - 6g)$,*
- *$c_1 c_2(X) = (e + \sigma) \cdot (6 - 6g)$, and*
- *$c_3(X) = e \cdot (2 - 2g)$.*

Proof. We need to identify the Chern classes. For such a purpose, consider the projections

$$\pi_1 : X = Y \times \Sigma_g \to Y, \ \pi_2 : X = Y \times \Sigma_g \to \Sigma_g.$$

The total Chern class can be written as $c(X) = \pi_1^*(c(Y)) \cdot \pi_2^*(c(\Sigma_g))$. The lemma follows by integrating over $X = Y \times \Sigma_g$. $\qquad\square$

The change on the Chern numbers of 6-manifolds after blowups at points and along surfaces is given by the following formulas.

Lemma 4.2. *Let X be a symplectic 6-manifold. Let X' be the symplectic 6-manifold obtained by blowing up a point in X. The Chern numbers of X' are as follows:*

- $c_1^3(X') = c_1^3(X) - 8$,
- $c_1 c_2(X') = c_1 c_2(X)$,
- $c_3(X') = c_3(X) + 2$.

Let $\Sigma_g \hookrightarrow X$ be a symplectic surface of genus g and denote by $N(\Sigma_g)$ its normal bundle. Let X'_g be the symplectic 6-manifold obtained by blowing up X along Σ_g. The Chern numbers of X'_g are as follows:

- $c_1^3(X'_g) = c_1^3(X) + 6(g-1) - 2\langle c_1(N(\Sigma_g)), [\Sigma_g]\rangle$,
- $c_1 c_2(X'_g) = c_1 c_2(X)$,
- $c_3(X'_g) = c_3(X) - 2(g-1)$.

See [22, Lemma 2.1] for a proof. Regarding the 6-manifolds obtained as symplectic sums, we have the following lemma.

Lemma 4.3. *Let X_1 and X_2 be symplectic 6-manifolds that contain 4-dimensional symplectic submanifolds $Y_i \subset X_i$ for $i = 1, 2$ with trivial normal bundle such that $Y_1 \cong Y_2$. Set $Y := Y_i$. The Chern numbers of the symplectic sum $X := X_1 \#_Y X_2$ are*

$$c_1^3(X) = c_1^3(X_1) + c_1^3(X_2) - 6c_1^2(Y),$$
$$c_1 c_2(X) = c_1 c_2(X_1) + c_1 c_2(X_2) - 2(c_1^2(Y) + c_2(Y)), \text{ and}$$
$$c_3(X) = c_3(X_1) + c_3(X_2) - 2c_2(Y).$$

See [22, Proposition 1.3] for a proof (see also [39, Lemma 3.23]). The existence of a symplectic structure induces an almost-complex structure on the underlying manifold. The possible restrictions on the Chern numbers of almost-complex 6-manifolds are computed in [35, Proposition 9] and [18]; see [30] for interesting 6-dimensional phenomena.

4.2. **6-DIMENSIONAL SYMPLECTIC SUMS.**

We proceed to prove Theorem 1.2. The fundamental group computations are carried out in §5.

Proof. Let us construct the manifolds $W_i(G)$ for $i = 0, 1, 2$; the method of construction of the manifolds $Y_i(G)$ is similar. Let X_1 and X_2 be minimal symplectic 4-manifolds of Proposition 2.7. Moreover, using the procedure described in §5.2 on X_1, we can produce a minimal symplectic 4-manifold $X_1(G)$ such that $\pi_1(X_1(G)) \cong G$. The manifold $X_1(G)$ contains a homologically essential Lagrangian torus $T := T_1$ such that $\pi_1(X_1(G) - T) \cong G$. Perturb the symplectic sum on $X_1(G)$ so that T becomes symplectic [20,

Lemma 1.6]. We make use of the following prototypes of symplectic sums according to the item in the statement of the theorem:

$$(1): W_0(G) := X_1(G) \times S^2 \#_{T^2 \times S^2} X_2 \times S^2$$

$$(2): W_1(G) := X_1(G) \times T^2 \#_{T^2 \times T^2} X_2 \times T^2$$

$$(3): W_2(G) := X_1(G) \times \Sigma_2 \#_{\Sigma_2 \times \Sigma_2} X_2 \times \Sigma_2.$$

The reader should notice that the manifolds X_1 and X_2 are to be chosen from the manifolds $Z_{1,1}$ and $Z_{1,2}$ of Proposition 2.7 appropriately, so that they contain the necessary symplectic submanifolds for the symplectic sums performed.

A description of the choices of gluing maps for the symplectic sums is given in §5.1. The computations of Chern numbers are straightforward and follow from Lemma 4.1 and Lemma 4.3. Concerning the claim about fundamental groups, we refer the reader to §5. The manifolds $Y_i(G)$ are constructed in a similar way by taking the manifolds X_1 and X_2 in the procedure described above from [36]. The details are left to the reader.   □

4.3. Realization of all possible combinations of Chern numbers.

The geography of simply connected symplectic 6-manifolds was settled by Halic in [22]. The examples given by him rely heavily on the blow-ups of some basic building blocks. His main tool to realize all possible combinations of Chern numbers with a symplectic 6-manifold is stated in the following result.

Lemma 4.4 ([22, Lemma 4.1]). *Let X be a symplectic 6-manifold whose Chern numbers are given by $(c_1^3, c_1c_2, c_3) = (2a, 24b, 2c)$ and which has the following properties. We denote by $N(E_X)$ the normal bundle of a projective line $E \subset X$.*

- *The manifold X contains a symplectically embedded product $U \times D$, where U is an open subset of a symplectic 4-manifold and D is a disc. Assume that U contains a projective line E having the property that*
$$-\alpha := \langle c_1(N(E_X)), [E] \rangle \leq -1.$$
- *There is a symplectic genus two surface $\Sigma_2 \subset X$, which is disjoint from the exceptional sphere E, and $[\Sigma_2]^2 = 0$.*

Then, by blowing up p points, r distinct copies of E, and z distinct copies of Σ_2, one obtains all triples of Chern numbers of the form $(2a', 24b, 2c')$, where a' and c' are arbitrary integers.

Example 4.5. Let $X(G)$ be a symplectic 6-manifold. Then

$$c_1^3(X(G)) \equiv c_3(X(G)) \equiv 0 \bmod 2 \text{ and}$$
$$c_1 c_2(X(G)) \equiv 0 \bmod 24.$$

Conversely, let $G \in \{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \pi_1(\Sigma_g), F_n\}$. Any triple (a, b, c) of integers satisfying $a \equiv c \equiv 0 \bmod 2$ and $b \equiv 0 \bmod 24$ occurs as a triple $(c_1^3, c_1 c_2, c_3)$ of Chern numbers of a symplectic 6-manifold $X(G)$ with $\pi_1(X(G)) \cong G$.

Proof. Let X be a symplectic 6-manifold. Then X admits an almost-complex structure, and [35, Proposition 9] implies that the Chern numbers of X satisfy $c_1^3(X) = c_3(X) \equiv 0 \bmod 2$ and $c_1 c_2(X) \equiv 0 \bmod 24$.

Now, let $X_1(G)$ be the manifold as in the proof of Theorem 1.2. There are two things to be checked in order to establish the converse. First, one needs to see that the result of blowing up at a point the manifolds constructed in Theorem 1.2 satisfies the hypothesis of Lemma 4.4. As a consequence, we are then able to conclude that all the possible values of c_1^3 and c_3 are realized by a symplectic 6-manifold with the given fundamental group. Second, it needs to be seen that such manifolds fulfill all the possible values of $c_1 c_2$ (values of b in the statement of Lemma 4.4). Notice that by Lemma 4.2, the value $c_1 c_2$ is invariant under blowups of points and under blowups along surfaces.

By Proposition 2.7, there exists a symplectic surface of genus two F inside the submanifold $X_1(G) \times \{x\} \subset X_1(G) \times \Sigma_g$ for every 6-manifold obtained as a symplectic sum in Theorem 1.2. By [20], every manifold of Theorem 1.2 contains a symplectic surface of genus two. The manifold obtained by blowing up a point that is disjoint from the surface of genus two satisfies the hypothesis of Lemma 4.4.

We now consider the possible values of $c_1 c_2$. The claim is that all possible values are obtained by the manifolds constructed in Theorem 1.2. The case $c_1 c_2 = 0$ is clear. The claim is equivalent to finding $e_i, \sigma_i, e_i', \sigma_i'$, $i = 1, 2$ that produce integers k_+ and k_- that satisfy

$$6 \cdot (e_1 + \sigma_1 + e_2 + \sigma_2) = 24k_+ \text{ and } -6 \cdot (e_1' + \sigma_1' + e_2' + \sigma_2') = 24k_-.$$

The numbers $e_i, \sigma_i, e_i', \sigma_i'$, $i = 1, 2$ are as in the statement of the theorem, and they correspond to the Euler characteristic and signatures of minimal 4-manifolds of Proposition 2.7, from which $X_1(G)$ is constructed. The corresponding manifolds to these integer numbers can be chosen so that the equalities $(e_1 + \sigma_1 + e_2 + \sigma_2) = 4k_+$ and $(e_1' + \sigma_1' + e_2' + \sigma_2') = 4k_-$ are satisfied. Thus, all possible values of $c_1 c_2$ are realized by manifolds from Theorem 1.2. The corollary now follows from Lemma 4.4. $\square$

5. Fundamental Group Computations

5.1. Choices of gluing maps for symplectic sums in §4.2.

An important piece of data in the 6-dimensional symplectic sum constructions of §4.2 are the gluing maps. Here we proceed to describe them in detail in order to conclude the fundamental group computations claimed in the statements of Theorem 1.2, Corollary 1.4, and Theorem 1.3. The setting is as follows. Let X and Z be symplectic 4-manifolds containing symplectic tori $T \subset X$ and $T' \subset Z$ or symplectic surfaces of genus two $F \subset X$ and $F' \subset Z$ (see Proposition 2.7); notice that, according to Proposition 2.7 or Theorem 2.2, one might need to perturb the symplectic form of the ambient 4-manifold using [20, Lemma 1.6] for the homologically essential Lagrangian tori to be assumed symplectic.

Regarding the homotopy computations, we assume

$$\pi_1(X - T) \cong \pi_1(X) \text{ and } \pi_1(Z - T') \cong \pi_1(Z) \cong \{1\},$$
$$\pi_1(X - F) \cong \pi_1(X) \text{ and } \pi_1(Z - F') \cong \pi_1(Z) \cong \{1\}.$$

Take a symplectic surface of genus g, and build the 6-manifolds $X \times \Sigma_g$ and $Z \times \Sigma_g$ equipped with the product symplectic form. For $g = 0, 1$, these symplectic 6-manifolds contain a symplectic submanifold $T^2 \times \Sigma_g$: $T \times \Sigma_g \subset X \times \Sigma_g$ and $T' \times \Sigma_g \subset Z \times \Sigma_g$. Similarly, we have the symplectic submanifolds (case $g = 2$) $F \times \Sigma_2 \subset X \times \Sigma_2$ and $F' \times \Sigma_2 \subset Z \times \Sigma_2$. In all our constructions, Σ_g is either a 2-sphere S^2 ($g = 0$), a 2-torus T^2 ($g = 1$), or a surface of genus two Σ_2 ($g = 2$); surfaces of higher genus can be used with a small modification to our arguments.

For surfaces of genus $g = 0, 1$, the 6-manifolds are symplectic sums of the form

$$W_g := X \times \Sigma_g \#_{T \times \Sigma_g = T' \times \Sigma_g} Z \times \Sigma_g,$$

where the gluing map is a diffeomorphism

$$\phi_g : \partial(T \times \Sigma_g \times D^2) \to \partial(T' \times \Sigma_g \times D^2) \cong T^2 \times \Sigma_g \times S^1.$$

Similarly, for surfaces of genus two, we built

$$W_2 := X \times \Sigma_2 \#_{F \times \Sigma_2 = F' \times \Sigma_2} Z \times \Sigma_2,$$

where the gluing map given by a diffeomorphism

$$\phi_2 : \partial(F \times \Sigma_2 \times D^2) \to \partial(F' \times \Sigma_2 \times D^2) \cong \Sigma_2 \times \Sigma_2 \times S^1.$$

Lemma 5.1. $\pi_1(W_g) \cong \pi_1(X)$ for $g = 0, 1, 2$.

Proof. We use the Seifert–van Kampen Theorem to compute the fundamental group of W_g for $g = 0, 1, 2$. In order to obtain $\pi_1(W_g) \cong \pi_1(X)$, we choose a diffeomorphism ϕ_g that maps the generators of the fundamental groups involved as follows; remember we can assume $\pi_1(Z - T') \cong$

$\pi_1(Z) \cong \{1\}$ (see Proposition 2.7). The case $g = 0$ is clear, since there are isomorphisms $\pi_1(X \times S^2 - T \times S^2) \cong \pi_1(X)$ and $\pi_1(Z \times S^2 - T' \times S^2) \cong \{1\}$.

We work out the case $g = 1$; the case $g = 2$ follows verbatim from our argument. In this case, the symplectic 4-manifold, along which the gluing in the symplectic sum [20] is performed, is the 4-torus

$$T \times T^2 \cong T^4 \cong T' \times T^2.$$

Let the groups $\pi_1(X \times T^2 - T \times T^2) \cong \pi_1(X) \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_1(Z \times T^2 - T' \times T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ be generated by $\{x_1, x_2, \ldots, x_n, x = 1, y = 1, w, z\}$ and $\{x' = 1, y' = 1, w', z'\}$, respectively, where $\{x_1, x_2, \ldots, x_n\}$ are generators of $\pi_1(X)$, $\{x, y\}$ are push offs of generators of $\pi_1(T)$, and each element $\{w', z'\}$ generates a $\mathbb{Z}$. Let μ and μ' be the respective (based) meridians of the submanifolds. Notice that in both cases the meridians $\mu = 1 = \mu'$ are trivial; we assume μ gets identified with μ'^{-1} during the gluing employed to construct W_1. Choose ϕ_1 to be the diffeomorphism that identifies the generators of the fundamental groups as follows:

$$x \mapsto w', y \mapsto z', w \mapsto x', z \mapsto y'.$$

This choice of diffeomorphism allows us to conclude that the extra generators w, z, w', z' are killed during the gluing. An application of the Seifert–van Kampen Theorem implies $\pi_1(W_1) \cong \pi_1(X)$, as desired.   □

5.2. Computations of π_1 for Theorem 1.2, Theorem 1.3, and Corollary 1.4.

Concerning the geography of symplectic 6-manifolds, the desired fundamental groups are obtained as follows. A minimal 4-manifold $Z_{1,1}$ of Proposition 2.7 contains two homologically essential Lagrangian tori with trivial meridians; in particular, there is an isomorphism $\pi_1(Z_{1,1} - (T_1 \cup T_2)) \to \pi_1(Z_{1,1}) \cong \{1\}$ (a similar argument works for $Z_{1,2}$).

The idea is to build the symplectic sum of $Z_{1,1}$ along T_1 and a chosen manifold in order to obtain a manifold P with the desired fundamental group G. The homologically essential Lagrangian torus $T_2 \subset P$ satisfies that the homomorphism induced by inclusion $T_2 \hookrightarrow P$ is an isomorphism $\pi_1(P - T_2) \to \pi_1(P) \cong G$. This is then used in our 6-dimensional symplectic sums in §4.2 to obtain the prescribed fundamental group. We proceed to prove the fundamental group claims on the four-dimensional building blocks employed.

Proof. Case: Abelian group of small rank.

To be precise, we construct manifolds with fundamental group among $\{\{1\}, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}\}$. Consider the 4-torus $T^2 \times T^2$ endowed with the product symplectic form. The 2-torus $\{x\} \times T^2 \subset T^2 \times T^2$ is

a symplectic submanifold of self-intersection zero. Start by building the symplectic sum

$$Z(\mathbb{Z} \oplus \mathbb{Z}) := Z_{1,1}{}_{T_1=T^2} T^2 \times T^2.$$

Since $\pi_1(Z_{1,1}) \cong \pi_1(Z_{1,1} - T_1) \cong \{1\}$, it follows from the Seifert–van Kampen Theorem that $\pi_1(Z(\mathbb{Z} \oplus \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}$. There are two pairs of geometrically dual Lagrangian tori inside $Z(\mathbb{Z} \oplus \mathbb{Z})$ on which one can perform Luttinger surgeries [32], [5]. These tori come from the Lagrangian tori inside the 4-torus used in the symplectic sum construction. Applying the appropriate surgery yields the desired fundamental groups; see [8, §2] and [9, Theorem 1] for the fundamental group computations and [46] for details.

Case: Surface groups.

Manifolds with fundamental group $\{1\}$ and $\mathbb{Z} \oplus \mathbb{Z}$ were constructed in the previous item. To construct a manifold S with $\pi_1(S) \cong \pi_1(\Sigma_g)$ for $g \geq 2$, we proceed as follows. Consider the product $T^2 \times \Sigma_g$ of a 2-torus and a genus $g \geq 2$ surface endowed with the product symplectic form. The submanifold $T^2 \times \{x\} \subset T^2 \times \Sigma_g$ is symplectic and its self-intersection number is zero. One builds the symplectic sum

$$S := Z_{1,1}{}_{T_1=T^2} T^2 \times \Sigma_g.$$

Since $\pi_1(Z_{1,1}) \cong \pi_1(Z_{1,1} - T_1) \cong \{1\}$, it follows from the Seifert–van Kampen Theorem that $\pi_1(S) \cong \pi_1(\Sigma_g)$ as desired.

Case: Free groups of rank $n \in \mathbb{N}$, F_n.

The case $n = 1$ was proven previously. Let Σ_n be a surface of genus n, and denote the standard generators of $\pi_1(\Sigma_n)$ by $x_1, y_1, \ldots, x_n, y_n$. Let $\phi : \Sigma_n \to \Sigma_n$ be a diffeomorphism consisting of the composition of $n-1$ Dehn twists along the loops x_i; denote by Y the mapping torus of ϕ that fibers over the circle.

Build the product $Y \times S^1$ which admits a symplectic structure [44]. This manifold contains a symplectic torus T' given by $T' = S \times S^1$, where S is a section of $Y \to S^1$. The fundamental group $\pi_1(Y \times S^1)$ is calculated by expressing $\pi_1(Y)$ as an HNN extension which yields

$$\pi_1(Y \times S^1) = \left\langle x_i, y_i, t : t x_i t^{-1} = x_i, t y_i t^{-1} = y_i x_i \right\rangle \oplus \mathbb{Z}s.$$

Build the symplectic sum

$$F(n) := Z_{1,1}{}_{T_1=T'} Y \times S^1.$$

Since $\pi_1(Z_{1,1}) \cong \pi_1(Z_{1,1} - T_1) \cong \{1\}$, it follows from the Seifert–van Kampen Theorem that the generators t and s are killed during the

gluing. The relations $t = 1 = s$ imply that $x_i = 1$ for all i. The Seifert–van Kampen Theorem implies that the group $\pi_1(F(n))$ is generated by the y_i's with no relations among them in the group presentation. That is, $\pi_1(F(n))$ is a free group on n generators. $\square$

5.3. Proofs of Theorem 1.3 and Corollary 1.4.

The paper finishes with the proofs of the results promised in the introduction.

Proof of Theorem 1.3. The argument is verbatim to the one employed in the proof of Theorem 1.2, modulo one of the building blocks. This new piece is going to be taken from the construction of Theorem 2.2. The changes go as follows. Take a minimal symplectic 4-manifold X_1 from Proposition 2.7 and perturb the symplectic form so that the torus T_1 becomes symplectic [20, Lemma 1.6]. Build the 4-dimensional symplectic sum [20]

$$X_1(G) := BK \#_{T = T_1} X_1,$$

where BK is the minimal symplectic 4-manifold with Euler characteristic given by $c_2(BK) = 4(g + r)$, signature $\sigma(BK) = 0$, and fundamental group $\pi_1(BK) \cong \langle g_1, \ldots, g_g | r_1, \ldots, r_r \rangle \cong G$ described in §2.2. A straightforward computation yields $c_2(X_1(G)) = c_2(X_1) + 4(g + r), \sigma(X_1(G)) = \sigma(X_1)$. The discussion in §2.3 implies $\pi_1(X_1(G)) \cong G$, which explains our choice of notation.

We proceed to build the 6-dimensional examples. Let X_2 be a minimal symplectic 4-manifold of Proposition 2.7. We make use of the following prototypes of symplectic sums according to the item in the statement of the theorem:

(1): $W_0(G) := X_1(G) \times S^2 \#_{T^2 \times S^2} X_2 \times S^2$

(2): $W_1(G) := X_1(G) \times T^2 \#_{T^2 \times T^2} X_2 \times T^2$

(3): $W_2(G) := X_1(G) \times \Sigma_2 \#_{\Sigma_2 \times \Sigma_2} X_2 \times \Sigma_2$.

A detailed description of the gluing maps chosen for the symplectic sums is given in §5.1. Varying our choices of manifolds X_1 and X_2 results in the realization of the Chern numbers claimed; under our notation, $c_2(X_i) = e_i, \sigma(X_i) = \sigma_i$ for $i = 1, 2$. The computations of the Chern numbers are straightforward and follow from Lemma 4.1 and Lemma 4.3.

The construction of the manifolds $Y_i(G)$ for $i = 0, 1, 2$ follows the same mechanism described above, using as building blocks the manifolds of Theorem 3.4 and [36]. $\square$

We now prove Corollary 1.4 (see Proof of Example 4.5).

Proof of Corollary 1.4. Let $X(G) := X_1(G)$ be as in the proof of Theorem 1.3. There are two things to be checked. First, we need to see that the

result of blowing up at a point the manifolds constructed in Theorem 1.2 satisfies the hypothesis of Lemma 4.4. As a consequence, we are then able to conclude that all the possible values of c_1^3 and c_3 are realized by a symplectic 6-manifold with the given fundamental group. Second, it is needed to be seen that such manifolds fulfill all the possible values of $c_1 c_2$ (values of b in the statement of Lemma 4.4). Notice that by Lemma 4.2, the value $c_1 c_2$ is invariant under blowups of points and under blowups along surfaces.

By Proposition 2.7, there exists a symplectic surface of genus two F inside the submanifold $X(G) \times \{x\} \subset X(G) \times \Sigma_g$ for every 6-manifold obtained as a symplectic sum in Theorem 1.2. Every manifold of Theorem 1.2 contains a symplectic surface of genus two [20]. The manifold obtained by blowing up a point that is disjoint from the surface of genus two satisfies the hypothesis of Lemma 4.4.

We now consider the possible values of $c_1 c_2$. The claim is that all possible values are obtained by the manifolds constructed in Theorem 1.2. The case $c_1 c_2 = 0$ is clear. The claim is equivalent to finding $e_i, \sigma_i, e_i', \sigma_i'$, $i = 1, 2$, that produce integers k_+ and k_- that satisfy

$$6 \cdot ((e_1 + 4(g + r)) + \sigma_1 + e_2 + \sigma_2) = 24k_+ \text{ and}$$
$$-6 \cdot ((e_1' + 4(g + r)) + \sigma_1' + e_2' + \sigma_2') = 24k_-.$$

The numbers $e_i, \sigma_i, e_i', \sigma_i'$, $i = 1, 2$, are as in the statement of Theorem 1.2 and they correspond to the Euler characteristic and signatures of minimal 4-manifolds of Proposition 2.7 used to construct $X(G)$. Such manifolds can be done so that their characteristic numbers satisfy the equalities $(e_1 + \sigma_1 + e_2 + \sigma_2) = 4k_+ + 4(g + r)$ and $(e_1' + \sigma_1' + e_2' + \sigma_2') = 4k_- + 4(g + r)$. Thus, all possible values of $c_1 c_2$ are realized by manifolds from Theorem 1.2 and Theorem 1.3. The corollary now follows from Lemma 4.4.   □

References

[1] Anar Akhmedov, *Small exotic 4-manifolds*, Algebr. Geom. Topol. **8** (2008), no. 3, 1781–1794.

[2] _____, *Simply connected symplectic Calabi-Yau 6-manifolds*. Preprint. Available at arXiv:1107.2623v3 [math.GT].

[3] Anar Akhmedov, Scott Baldridge, R. İnanç Baykur, Paul Kirk, and B. Doug Park, *Simply connected minimal symplectic 4-manifolds with signature less than -1*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 1, 133–161.

[4] Anar Akhmedov and B. Doug Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. **181** (2010), no. 3, 577–603.

[5] D. Auroux, S. K. Donaldson, and L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. **326** (2003), no. 1, 185–203.

[6] Scott Baldridge and Paul Kirk, *Symplectic 4-manifolds with arbitrary fundamental group near the Bogomolov–Miyaoka–Yau line*, J. Symplectic Geom. **4** (2006), no. 1, 63–70.

[7] _____ , *On symplectic 4-manifolds with prescribed fundamental group*, Comment. Math. Helv. **82** (2007), no. 4, 845–875.

[8] _____ , *A symplectic manifold homeomorphic but not diffeomorphic to* $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$, Geom. Topol. **12** (2008), no. 2, 919–940.

[9] _____ , *Constructions of small symplectic 4-manifolds using Luttinger surgery*, J. Differential Geom. **82** (2009), no. 2, 317–361.

[10] _____ *Coisotropic Luttinger surgery and some new symplectic 6-manifolds with vanishing canonical class*, Indiana Univ. Math. J. **62** (2013), no. 5, 1457–1471.

[11] Steven Boyer, *Simply-connected 4-manifolds with a given boundary*, Trans. Amer. Math. Soc. **298** (1986), no. 1, 331–357.

[12] Joel Fine and Dmitri Panov, *Symplectic Calabi-Yau manifolds, minimal surfaces and the hyperbolic geometry of the conifold*, J. Differential Geom. **82** (2009), no. 1, 155–205.

[13] _____ , *Hyperbolic geometry and non-Kähler manifolds with trivial canonical bundle*, Geom. Topol. **14** (2010), no. 3, 1723–1763.

[14] _____ , *The diversity of symplectic Calabi–Yau 6-manifolds*, J. Topol. **6** (2013), no. 3, 644–658.

[15] Ronald Fintushel, B. Doug Park, and Ronald J. Stern, *Reverse engineering small 4-manifolds*, Algebr. Geom. Topol. **7** (2007), 2103–2116.

[16] Ronald Fintushel and Ronald J. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), no. 2, 363–400.

[17] _____ , *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$*, J. Topol. **1** (2008), no. 1, 1–15.

[18] Hansjörg Geiges, *Chern numbers of almost complex manifolds*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3749–3752.

[19] Robert E. Gompf, *Nuclei of elliptic surfaces*, Topology **30** (1991), no. 3, 479–511.

[20] _____ , *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), no. 3, 527–595.

[21] Robert E. Gompf and András I. Stipsicz, *4-Manifolds and Kirby Calculus*. Graduate Studies in Mathematics, 20. Providence, RI: American Mathematical Society, 1999.

[22] Mihai Halic, *On the geography of symplectic 6-manifolds*, Manuscripta Math. **99** (1999), no. 3, 371–381.

[23] Ian Hambleton and Matthias Kreck, *On the classification of topological 4-manifolds with finite fundamental group*, Math. Ann. **280** (1988), no. 1, 85–104.

[24] M. J. D. Hamilton and D. Kotschick, *Minimality and irreducibility of symplectic four-manifolds*, Int. Math. Res. Not. **2006** Art. ID 35032, 1–13.

[25] Chung-I Ho and Tian-Ju Li, *Luttinger surgery and Kodaira dimension*, Asian J. Math. **16** (2012), no. 2, 299–318.

[26] Bruce Hunt, *Complex manifold geography in dimension 2 and 3*, J. Differential Geom. **30** (1989), no. 1, 51–153.

[27] Paul Kirk and Charles Livingston, *The Hausmann–Weinberger 4-manifold invariant of abelian groups*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1537–1546

[28] _____, *The geography problem for 4-manifolds with specified fundamental group*, Trans. Amer. Math. Soc. **361** (2009), no. 8, 4091–4124.

[29] D. Kotschick, *Minimizing Euler characteristics of symplectic four-manifolds*, Proc. Amer. Math. Soc. **134** (2006), no. 10, 3081–3083.

[30] Claude LeBrun, *Topology versus Chern numbers for complex 3-folds*, Pacific J. Math. **191** (1999), no. 1, 123–131.

[31] Tian-Jun Li and Yongbin Ruan, *Symplectic birational geometry* in New Perspectives and Challenges in Symplectic Field Theory. Ed. Miguel Abreu, François Lalonde, and Leonid Polterovich. CRM Proceedings & Lecture Notes, 49. Providence, RI: Amer. Math. Soc., 2009. 307–326.

[32] Karl Murad Luttinger, *Lagrangian tori in $\mathbb{R}^4$*, J. Differential Geom. **42** (1995), no. 2, 220–228.

[33] John D. McCarthy and Jon G. Wolfson, *Symplectic normal connect sum*, Topology **33** (1994), no. 4, 729–764.

[34] John W. Morgan, Tomasz S. Mrowka, and Zoltán Szabó, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), no. 6, 915–929.

[35] Ch. Okonek and A. Van de Ven, *Cubic forms and complex 3-folds*, Enseign. Math. (2) **41** (1995), no. 3-4, 297–333.

[36] B. Doug Park and Zoltán Szabó, *The geography problem for irreducible spin four-manifolds*, Trans. Amer. Math. Soc. **352** (2000), no. 8, 3639–3650.

[37] Jongil Park, *The geography of spin symplectic 4-manifolds*, Math. Z. **240** (2002), no. 2, 405–421.

[38] _____, *The geography of symplectic 4-manifolds with an arbitrary fundamental group*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2301–2307.

[39] Federica Benedetta Pasquotto, *On the geography of symplectic manifolds*. Dissertation, Leiden University, 2004.

[40] I. Smith, R. P. Thomas, and S.-T. Yau, *Symplectic conifold transitions*, J. Differential Geom. **62** (2002), no. 2, 209–242.

[41] Zoltán Szabó, *Simply-connected irreducible 4-manifolds with no symplectic structures*, Invent. Math. **132** (1998), no. 3, 457–466.

[42] Clifford Henry Taubes, *The Seiberg–Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), no. 6, 809–822.

[43] _____, *SW → Gr: From the Seiberg–Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), no. 3, 845–918.

[44] W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), no. 2, 467–468.

[45] Rafael Torres, *Irreducible 4-manifolds with abelian non-cyclic fundamental group of small rank*, Topology Appl. **157** (2010), no. 5, 831–838.

[46] _____, *Geography and botany of irreducible non-spin symplectic 4-manifolds with abelian fundamental group*, Glasg. Math. **56** (2014), no. 2, 261–281.

[47] Michael Usher, *Minimality and symplectic sums*, Int. Math. Res. Not. **2006**, Art. ID 49857, 1–17.

[48] Jonathan T. Yazinski, *A new bound on the size of symplectic 4-manifolds with prescribed fundamental group*, J. Symplectic Geom. **11** (2013), no. 1, 25–36.

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