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NOTES ON OXTOBY SPACES AND PSEUDOCOMPLETENESS

ROBERTO PICHARDO-MENDOZA

Abstract. This paper focuses on the notion of O-pseudocompleteness, introduced by John C. Oxtoby, and two modifications of it [Aaron R. Todd, Pacific J. Math. **95** (1981), no. 1, 233–250] and [Melvin Henriksen et al., Topology Appl. **100** (2000), no. 2-3, 119–132]. The following facts are known for these notions: (1) all of them imply Baireness, (2) they are productive, and (3) all completely metrizable spaces and locally compact Hausdorff spaces possess these properties.

We prove that these topological properties coincide in the class of spaces having a π-base of countable subsets. We also show that if a space X is a dense G_δ-subset of either a space possessing a dense completely metrizable subspace or a product of completely metrizable spaces, then X is O-pseudocomplete.

It is established that the notions of pseudocompleteness appearing in the above noted papers by Todd and by Henriksen et al. are inverse invariants of irreducible closed mappings.

Applications of elementary submodels to the study of these notions of pseudocompleteness are presented too.

1. Introduction

In [8, §5] John C. Oxtoby was able to isolate the key ingredient in the classical proof of the Baire Category Theorem. By using the notions of quasiregularity and π-base (see definitions below), he introduced a productive subclass of the class of Baire spaces which contains all completely metrizable and all Hausdorff locally compact spaces.

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As it is well known today, the class of Baire spaces is not productive [3, §4], so determining productive subclasses of it which are large enough became an interesting problem. Two successful attempts in finding such classes are Aaron R. Todd's weakening of Oxtoby's notion and the modification introduced by Melvin Henriksen et al. in [5, Definition 1.1], the so called *Oxtoby spaces*.

Given the three classes mentioned above, one may wonder what the connections are between them. For example, we know that Todd's class is precisely the class of Oxtoby spaces which are quasiregular [5, Theorem 2.6], but it is still an open question if Oxtoby's class is a proper subclass of Todd's.

On the other hand, how similar are these classes to the class of Baire spaces? Are they closed under open continuous images? How about under dense subsets of type G_δ?

All of the above questions are the main concern of this article.

Our paper is divided as follows, §2 establishes the basic definitions and notation we will use. Section 3 introduces formally the notions we discussed above and presents some classes of spaces in which some of the questions mentioned earlier have a positive answer (spaces having a π-base of countable sets or having a dense completely metrizable subspace). In §4, we study some applications of elementary submodels to the classes of spaces mentioned above.

In §5 we consider unions of Oxtoby spaces and show that the class of Oxtoby spaces is closed under finite unions and that all Baire spaces which can be written as a countable union of Oxtoby spaces are themselves Oxtoby.

Finally, sections 6 and 7 are dedicated to products and continuous mappings. It is proved that all dense G_δ-subsets of an arbitrary product of completely metrizable subspaces belong to Oxtoby's class and that Todd's class is closed under irreducible closed preimages.

2. Preliminaries

Given a topological space X and a set $A \subseteq X$, the closure and the interior of A will be denoted by $\overline{A}$ and $\operatorname{int} A$, respectively, unless we feel there is risk of confusion, in which case, the symbols $\operatorname{cl}_X A$ and $\operatorname{int}_X A$ will be used.

To make things simpler, we adopt the following convention: τ_X will represent the collection of all open subsets of X and τ_X^* will denote the family of all non-empty open subsets of X.

The space X will be called *quasiregular* if every non-empty open set in X contains the closure of a non-empty open subset of X; i.e., for each $U \in \tau_X^*$, there is $V \in \tau_X^*$ with $\overline{V} \subseteq U$. One easily verifies the following.

Remark 2.1. All dense subspaces of a quasiregular space are themselves quasiregular.

A π^0*-base for* X is a collection $\mathcal{P}$ of subsets of X with nonempty interior such that any nonempty open subset of X contains a member of $\mathcal{P}$. When all members of $\mathcal{P}$ are open in X, $\mathcal{P}$ will be called a π*-base for* X.

Recall that a topological space X is *Baire* if the intersection of any countable family of dense open subsets of X is dense in X.

Let Y be a subset of a set S. Given $\mathcal{A}$, a family of subsets of S, the *trace of* $\mathcal{A}$ *over* Y is $\mathcal{A} \upharpoonright Y := \{A \cap Y : A \in \mathcal{A}\}$.

As usual, ω denotes the first infinite ordinal and $\mathfrak{c}$ represents the cardinality of the continuum.

All topological notions whose definitions are not stated explicitly here should be understood as in [2]. A similar remark goes for set-theoretical notions and [7].

3. Three Notions of Pseudocompleteness

Given a topological space X, a sequence $\langle A_n : n \in \omega \rangle$ of subsets of X will be called a *nest* if $A_{n+1} \subseteq A_n$, for each $n \in \omega$. If the sequence satisfies $\overline{A_{n+1}} \subseteq \operatorname{int} A_n$, for all $n \in \omega$, then it will be called a *strong nest.* Also, if $\mathcal{A}$ is a family of subsets of X, a *nest in* $\mathcal{A}$ (a *strong nest in* $\mathcal{A}$) is a nest (a strong nest) for which all members of it belong to $\mathcal{A}$.

Given $\vec{\mathcal{P}} = \langle \mathcal{P}_n : n \in \omega \rangle$, a sequence of π^0-bases for X, we will say that a nest (a strong nest) $\vec{P} = \langle P_n : n \in \omega \rangle$ is an *associated nest for* (a *strong nest associated to*) $\vec{\mathcal{P}}$ if $P_n \in \mathcal{P}_n$ for each $n \in \omega$.

A quasiregular space X will be called *O-pseudocomplete* or *pseudocomplete according to Oxtoby* (see [8, §5]) if it possesses a countable sequence of π-bases, $\vec{\mathcal{P}}$, in such a way that each strong nest associated to $\vec{\mathcal{P}}$ has nonempty intersection. A sequence $\vec{\mathcal{P}}$ like the one described above will *witness that* X *is O-pseudocomplete.*

When $\vec{\mathcal{P}}$ is a sequence of π^0-bases for a quasiregular space and all strong nests associated to $\vec{\mathcal{P}}$ have nonempty intersection, the space is called *T-pseudocomplete* or *pseudocomplete according to Todd* (see [9, Definition 1.2]). In this case, we will say that $\vec{\mathcal{P}}$ *witnesses T-pseudocompleteness for* X.

After reading the previous definitions, one immediately wonders if the class of all O-pseudocomplete spaces differs from the class of T-pseudocomplete spaces. It is clear that all O-pseudocomplete spaces are T-pseudocomplete, but the reverse implication remains an open question (posed originally in [9]).

Question 3.1. Are O-pseudocompleteness and T-pseudocompleteness equivalent?

An *Oxtoby sequence* for X is a countable sequence of π^0-bases for X in such a way that each associated nest to it has nonempty intersection. Naturally, any space having an Oxtoby sequence will be called an *Oxtoby space* (see [5, Definition 1.1]).

The following was proved in [5, Theorem 2.6].

Remark 3.2. A quasiregular space is T-pseudocomplete if and only if it is Oxtoby. In particular, all T-pseudocomplete spaces are Oxtoby.

By [8, §5], all completely metrizable spaces, as well as all Hausdorff locally compact spaces, are O-pseudocomplete.

The following result is a summary of [5, Theorem 1.2] and [8, (5.1)].

Proposition 3.3. *Let X be an arbitrary topological space.*

(1) *If X is an Oxtoby space, X is Baire.*
(2) *If $\mathcal{P}$ is a π^0-base for X and D is dense in X, $\mathcal{P} \restriction D$ is a π^0-base for D.*
(3) *When X has a dense Oxtoby subspace, X itself is Oxtoby.*
(4) *If X is quasiregular and possesses a dense O-pseudocomplete subspace, then X is O-pseudocomplete.*

Moreover, (1) *remains true if one replaces Oxtoby by T-pseudocomplete or by O-pseudocomplete.*

The implication in (1) cannot be reversed. Indeed, it is argued in [5, §2] that any Bernstein subset of the real line is a counterexample (see also [2, Problem 5.5.4]). Thus, a natural question to ask is, how close are these properties to Baireness? Particularly, we ask the following question.

Question 3.4. If X is Oxtoby (O-pseudocomplete, T-pseudocomplete),

(1) are all its dense G_δ-subsets Oxtoby (O-pseudocomplete, T-pseudocomplete) too?
(2) are all its (quasiregular) open continuous images Oxtoby (O-pseudocomplete, T-pseudocomplete) as well?

The following result shows that, in a non-trivial class of topological spaces, Question 3.1 has a strong affirmative answer.

Proposition 3.5. *Let X be a T_1 space. If X has a π-base of countable sets (equivalently, a π^0-base of countable sets), then the following statements are equivalent.*

(1) *X is O-pseudocomplete.*
(2) *X is Oxtoby.*

(3) *X is Baire.*
(4) *X has a dense set of isolated points.*

Proof. By Remark 3.2, (2) follows from (1). On the other hand, Proposition 3.3 guarantees that (3) is a consequence of (2).

Let us denote by A the set of isolated points of X.

To show that (3) implies (4), assume that (4) fails. Then $U := X \setminus \overline{A}$ is a non-empty open subset of X, and so there is $V \in \tau_X^*$ such that $V \subseteq U$ and $|V| \leq \omega$. Thus, no point in V is isolated and since X is T_1, $\{X \setminus \{x\} : x \in V\}$ is a countable family of dense open subsets of X whose intersection misses V; i.e., X is not Baire, and so (3) fails too.

Now suppose (4) holds. Hence, A is dense open in X. As a consequence, if U is a non-empty open subset of X, there exists $z \in U \cap A$, and so $\{z\}$ is a non-empty open subset of X whose closure is contained in U, i.e., X is quasiregular. To complete the implication (4)→(1), set $\mathcal{P}_n := \{\{x\} : x \in A\}$, for each $n \in \omega$, to obtain $\langle \mathcal{P}_k : k < \omega \rangle$, a sequence witnessing O-pseudocompleteness for X. □

Since all indiscrete spaces with at least two points are Baire quasiregular and have no isolated points, the assumption on the space being T_1 cannot be dropped in Proposition 3.5.

Observe that the argument given in the proof of (4)→(1) shows that any space having a dense set of isolated points is Oxtoby. Also, given that all countable spaces have, trivially, a π-base of countable sets, we get the following.

Corollary 3.6. *If X is a countable T_1 topological space, then X is O-pseudocomplete if and only if X has a dense set of isolated points.*

Corollary 3.7. *Assume X is a T_1 space possessing a π-base of countable sets. If X is Baire, then all its dense subspaces and all its T_1 continuous open images are O-pseudocomplete.*

Proof. Suppose X is Baire and let us denote by A the set of isolated points of X.

When Y is a dense subspace of X, $A \subseteq Y$ and, by Proposition 3.3-(2), Y satisfies the assumptions of Proposition 3.5. Thus, Y is O-pseudocomplete.

Now let f be an open continuous map from X onto the T_1-space Z. Observe that if $\mathcal{B}$ is a π-base for X, the collection $\{f[B] : B \in \mathcal{B}\}$ is a π-base for Z. Hence, Z has a π-base of countable sets. Also, $f[A]$ is a dense subset of Z consisting of isolated points. Therefore, Z is O-pseudocomplete. □

According to [1, p. 4, Corollary], a metrizable space is O-pseudocomplete if and only if it has a dense completely metrizable subspace. Our following result establishes a connection between this kind of spaces and Question 3.4.

Proposition 3.8. *If X is a regular space containing a dense completely metrizable subspace, then all dense G_δ-subsets of X are O-pseudocomplete.*

Proof. Let Z be a dense completely metrizable subspace of X and let Y be a dense G_δ-subset of X. Since $Z \cap Y$ is a G_δ-subset of Y, Alexandroff's Theorem (see [2, Theorem 4.3.23]) implies that $Z \cap Y$ is completely metrizable. Hence, we only need to show that $Z \cap Y$ is dense in Y to apply Proposition 3.3(4) and conclude that Y is O-pseudocomplete.

Let us fix d, a complete metric compatible with the subspace topology of Z, and $\{U_n : n \in \omega\}$, a decreasing sequence of open subsets of X whose intersection is equal to Y. Thus, given $x \in Z$ and $r > 0$, we define $B(x, r) := \{y \in Z : d(x, y) < r\}$.

Assume that V is a non-empty open subset of X.

Claim. For each $n \in \omega$, there exist $z_n \in Z$, $r_n \in \mathbb{R}$, and $W_n \in \tau_X$ such that

(1) $0 < r_n < 2^{-n}$,
(2) $z_n \in Z \cap W_n \subseteq Z \cap \mathrm{cl}_X(W_n) \subseteq B(z_n, r_n) \subseteq V \cap U_n$, and
(3) $B(z_{n+1}, r_{n+1}) \subseteq W_n$.

Before we engage in the details of the construction, let us note that, for each integer n, the last two conditions give

$$(\star) \qquad z_{n+1} \in Z \cap W_{n+1} \subseteq B(z_{n+1}, r_{n+1}) \subseteq Z \cap W_n \subseteq B(z_n, r_n);$$

therefore, $d(z_n, z_{n+1}) < 2^{-n}$. Hence, $\langle z_k : k \in \omega \rangle$ is a Cauchy sequence in Z, and so it converges to some point $z \in Z$. On the other hand, a consequence of $(\star)$ is that $\{z_k : k \in \omega \setminus n\} \subseteq W_n$, which combined with condition (2) produces

$$z \in \bigcap_k (Z \cap \mathrm{cl}_X(W_k)) \subseteq \bigcap_k (V \cap U_k) = V \cap Y.$$

In other words, $V \cap Y$ has non-empty intersection with $Z \cap Y$.

To finish the argument, let us prove the claim by induction. Since U_0 is dense in X, we get $V \cap U_0 \in \tau_X^*$ and so there exists $z_0 \in Z \cap (V \cap U_0)$; fix $r_0 < 1$ such that $B(z_0, r_0) \subseteq V \cap U_0$ and use regularity of Z to obtain $W_0 \in \tau_X$ satisfying $z_0 \in Z \cap W_0 \subseteq Z \cap \mathrm{cl}_X(W_0) \subseteq B(z_0, r_0)$.

Now assume that z_n, r_n, and W_n have been defined accordingly for some $n \in \omega$. Hence, $z_n \in V \cap W_n$ and U_{n+1} is dense in X, so $V \cap W_n \cap U_{n+1}$ is a non-empty open subset of X. Since Z is dense, there exists $z_{n+1} \in Z \cap V \cap W_n \cap U_{n+1}$ and, by proceeding as we did in the previous paragraph, we obtain r_{n+1} and W_{n+1} as required. ∎

It is routine to verify that if $\langle \mathcal{P}_n : n \in \omega \rangle$ is an Oxtoby sequence for X and Y is an open subspace of X, then $\langle \{P \in \mathcal{P}_n : P \subseteq Y\} : n \in \omega \rangle$ is an Oxtoby sequence for Y. Thus, we get the following.

Remark 3.9. Any open subspace of an Oxtoby space is Oxtoby itself.

Following [10], we say that a point x in a topological space is *rare* if the closure of $\{x\}$ has void interior. Note that in a T_1 space, the rare points are precisely the non-isolated points.

A routine modification of the argument used to prove [10, Theorem 4] can be used to show the following.

Lemma 3.10. *If X is a non-empty topological space with a dense set of rare points, any of the following conditions imply that $|X| \geq \mathfrak{c}$.*

(1) *X is O-pseudocomplete.*
(2) *X is Oxtoby and T_2.*

From [4, Corollary 23B] we know that the statement *if X is a Hausdorff second countable topological space of size $< \mathfrak{c}$, then all its subsets are of type G_δ* is consistent with ZFC. Thus, a naïve approach to obtain the consistency of a negative answer to Question 3.4(1) would be to find a "pseudocomplete" space with the properties given above which has a non-pseudocomplete dense subspace. Unfortunately, that approach will not work.

Proposition 3.11. *If X is a T_2 Oxtoby space or an O-pseudocomplete space, then X has a dense set of isolated points or $|X| \geq \mathfrak{c}$.*

Proof. Assume X is T_2 Oxtoby and let A be the set of all isolated points of X. When A is not dense in X, the open subspace $Y := X \setminus \overline{A}$ is non-empty, Oxtoby, and has no isolated points (so, every point of it is a rare point). Therefore, $|X| \geq |Y| \geq \mathfrak{c}$. The same reasoning is used for the case when X is O-pseudocomplete. $\square$

4. Elementary Submodels

For the arguments used in this section we are assuming familiarity with the material contained in [7, Section III.8].

Given a topological space X, $\pi w(X)$ denotes the *π-weight of* X, i.e., the least cardinality of an infinite π-base for X. Note that $\pi w(X)$ is also the minimum size of an infinite π^0-base for X.

If σ and τ are a pair of topologies on a set X, we will say that σ is Π-*related* to τ if σ^* is a π^0-base for τ. It is proved in [9, Proposition 3.1] that "being Π-related" is an equivalence relation on the topologies for X. Also, [5, Example 3.7] exhibits a quasiregular topology which is Π-related to a non-quasiregular topology.

Proposition 4.1. *Let X be a topological space and set $\kappa := \pi w(X)$. If X is Oxtoby, then*

(1) *X has a dense Oxtoby subspace Y with $|Y| \leq \kappa^\omega$ and*
(2) *there exists $\sigma \subseteq \tau_X$, a topology on X which is Π-related to τ_X, such that (X, σ) is Oxtoby and the weight of (X, σ) is at most κ^ω.*

Moreover, the above statements remain true when one replaces "Oxtoby" with "T-pseudocomplete" ("O-pseudocomplete", respectively).

Proof. Let θ be a suitable large cardinal for which H_θ has an elementary submodel M of size κ^ω satisfying $M^\omega \cup (\kappa^\omega + 1) \cup \{X, \tau_X\} \subseteq M$. Hence, there is $\mathcal{B} \in M$ such that $\mathcal{B}$ is a π-base for X with $|\mathcal{B}| = \kappa$, so the assumption $\kappa^\omega + 1 \subseteq M$ gives $\mathcal{B} \subseteq M$.

In order to prove (1) we need some remarks. Suppose that $\mathcal{P} \in M$ is a π^0-base for X. Hence, given $U \in \tau_X^*$, there is $B \in \mathcal{B}$ with $B \subseteq U$, and so, by elementarity, $P \subseteq B$, for some $P \in M \cap \mathcal{P}$. Thus, $M \cap \mathcal{P}$ is a π^0-base for X as well.

Observe that the above argument proves that if $\mathcal{P} \in M$ is a π-base for X, then $M \cap \mathcal{P}$ is a π-base for X too.

Assume now that X is Oxtoby and set $Y := M \cap X$. We will show that Y is as needed in (1). Clearly, $|Y| \leq \kappa^\omega$ so let us fix $\vec{\mathcal{P}} := \langle \mathcal{P}_n : n \in \omega \rangle \in M$, an Oxtoby sequence for X, and let us set $\vec{\mathcal{P}}_M := \langle M \cap \mathcal{P}_n : n \in \omega \rangle$ to obtain a sequence of π^0-bases for X.

Claim. If $\vec{P} = \langle P_n : n \in \omega \rangle$ is an associated nest for $\vec{\mathcal{P}}_M$, then $Y \cap \bigcap_n P_n \neq \emptyset$.

Start by noticing that $\vec{P}$ is also an associated nest for $\vec{\mathcal{P}}$. Now, the assumption $M^\omega \subseteq M$ gives $\vec{P} \in M$, and so, by elementarity, M thinks that $\vec{P}$ has non-void intersection, i.e., $\emptyset \neq M \cap \bigcap_n P_n = Y \cap \bigcap_n P_n$.

Let $P_0 \in M \cap \mathcal{P}_0$ be arbitrary. Using the fact that $\vec{\mathcal{P}}_M$ is a sequence of π^0-bases, define $\langle P_n : n \in \omega \rangle$, an associated nest for $\vec{\mathcal{P}}_M$. Hence, according to the claim, $Y \cap P_0 \neq \emptyset$. In conclusion, Y is dense in X because it has non-void intersection with each member of the π^0-base $M \cap \mathcal{P}_0$.

Set $\vec{\mathcal{Q}} := \langle (\mathcal{P}_n \cap M) \restriction Y : n \in \omega \rangle$ to obtain a sequence of π^0-bases for Y (see Proposition 3.3) and let $\langle Q_n : n \in \omega \rangle$ be an associated nest for $\vec{\mathcal{Q}}$. For each $n \in \omega$ fix $P_n \in \mathcal{P}_n \cap M$ such that $Q_n = P_n \cap Y$. Then P_{n+1} and P_n are members of M satisfying $P_{n+1} \cap M \subseteq P_n \cap M$ so, by elementarity, $P_{n+1} \subseteq P_n$. In other words, $\langle P_n : n \in \omega \rangle$ is an associated nest for $\vec{\mathcal{P}}_M$, and therefore an application of the claim produces $\bigcap_n Q_n = Y \cap \bigcap_n P_n \neq \emptyset$.

We just proved that if X is Oxtoby, then Y is Oxtoby and dense in X. Moreover, when X is T-pseudocomplete, Y inherits quasiregularity from X, and so Y is T-pseudocomplete whenever X is.

Assume, for the rest of the argument, that X is O-pseudocomplete.

With the same notation as above: if $\vec{\mathcal{P}} \in M$ witnesses O-pseudocompleteness for X, then the claim holds when "nest" is replaced by "strong nest" (note that if $A \subseteq X$ and $A \in M$, then $\mathrm{cl}_X A \in M$) so one can use the quasiregularity of X to prove that Y has non-empty intersection with each member of $M \cap \mathcal{P}_0$. Hence, Y is dense in X and, as a consequence, quasiregular.

Now let $\vec{Q} = \langle Q_n : n \in \omega \rangle$ be a strong nest for $\vec{\mathcal{Q}}$ and for each $n < \omega$, fix $P_n \in M \cap \mathcal{P}_n$ with $Q_n = Y \cap P_n$. Observe that our assumption on $\vec{Q}$ gives $M \cap \mathrm{cl}_X P_{n+1} \subseteq M \cap P_n$ and since $\mathrm{cl}_X P_{n+1} \in M$, we conclude that $\mathrm{cl}_X P_{n+1} \subseteq P_n$. Therefore, $\langle P_n : n \in \omega \rangle \in M$ is a strong nest for $\vec{\mathcal{P}}$ so, by elementarity, $\vec{Q}$ has non-void intersection. In other words, $\vec{\mathcal{Q}}$ witnesses O-pseudocompleteness for Y. This completes the proof of (1) for all possible cases.

For (2), by elementarity, $M \cap \tau_X$ is a base for some topology σ on X. Note that σ is Π-related to τ_X (because $\mathcal{B} \subseteq \sigma$) and its weight does not exceed κ^ω. If (X, τ_X) is Oxtoby, [5, Corollary 3.6] implies that (X, σ) is Oxtoby too. On the other hand, if X has a sequence witnessing O-pseudocompleteness, then there is $\langle \mathcal{P}_n : n \in \omega \rangle \in M$ witnessing O-pseudocompleteness for X. Thus, by elementarity, $\vec{\mathcal{P}}_M := \langle M \cap \mathcal{P}_n : n \in \omega \rangle$ is a sequence of π-bases for σ and since τ_X is finer than σ, any strong nest for $\vec{\mathcal{P}}_M$ in (X, σ) is a strong nest for $\vec{\mathcal{P}}$ in X; therefore, we have proved that $\vec{\mathcal{P}}_M$ witnesses O-pseudocompleteness for (X, σ).

To complete the proof, note that if X is quasiregular and $U \in M \cap \tau_X^*$, then, by elementarity, there exist $F \in M$ and $V \in M \cap \tau_X^*$ satisfying $V \subseteq F \subseteq U$ and $X \setminus F \in M \cap \tau_X$. In other words, (X, σ) is quasiregular as well. □

Corollary 4.2. *Assume X is a T_1 non-empty topological space with no isolated points and satisfying $\pi w(X) \leq \mathfrak{c}$.*

1. *If X is Oxtoby and T_2, X has a dense Oxtoby subspace of size $\mathfrak{c}$.*
2. *When X is T-pseudocomplete (O-pseudocomplete, respectively), X has a dense T-pseudocomplete (O-pseudocomplete, respectively) subspace of size $\mathfrak{c}$.*

Proof. Note that, in either case, any dense subspace of X possesses a dense set of non-isolated points, so Lemma 3.10 implies that the cardinality of the dense subset of X guaranteed by the previous proposition is $\mathfrak{c}$. □

A construction which has received some attention lately is the following: given a topological space X and M, an elementary submodel of some H_θ for which $\{X, \tau_X\} \subseteq M$, the collection $(M \cap \tau_X) \restriction M$ is a base for some topology on $X \cap M$; the resulting topological space will be denoted by X_M. In a more colloquial way, X_M is what M "thinks" that X is.

Proposition 4.3. *If X, θ, and M are as in the previous paragraph, the following statements hold.*

(1) *X_M is quasiregular if and only if X is quasiregular.*
(2) *When $M^\omega \subseteq M$ and X is Oxtoby, X_M is Oxtoby as well.*

Also, (2) *remains true when "Oxtoby" is replaced by "T-pseudocomplete" or "O-pseudocomplete."*

Proof. To prove (1), assume that X is quasiregular and let $U \in M \cap \tau_X$ be so that $U \cap M \neq \emptyset$. Since M *knows* that X is quasiregular, there exists $V \in M \cap \tau_X$ with $V \cap M \neq \emptyset$ and $\mathrm{cl}_X V \subseteq U$. To complete the argument, let us show that $\mathrm{cl}_{X_M}(V \cap M) \subseteq \mathrm{cl}_X V$. By letting $W := X \setminus \mathrm{cl}_X V$, we obtain $W \in M \cap \tau_X$ and $W \cap V = \emptyset$, so $M \cap W$ is an open subset of X_M which misses $V \cap M$; thus

$$X_M \setminus \mathrm{cl}_X V = M \cap W \subseteq X_M \setminus \mathrm{cl}_{X_M}(V \cap M).$$

And vice versa: if X is not quasiregular, M knows it, and so there exists $U \in M \cap \tau_X^*$ in such a way that $M \models \forall V \in \tau_X^* \ (\mathrm{cl}_X V \nsubseteq U)$. As a consequence, given $V \in M \cap \tau_X^*$, there is $x \in M \cap \mathrm{cl}_X V \setminus U$, and therefore $W \cap V \cap M \neq \emptyset$ whenever $x \in W \in M \cap \tau_X$. In other words, $x \in \mathrm{cl}_{X_M}(V \cap M) \setminus U$. This proves that X_M is not quasiregular.

The proof of (2) will be broken down into several claims.

Claim 1. If $\mathcal{P} \in M$ is a π^0-base for X, then $(\mathcal{P} \cap M) \restriction M$ is a π^0-base for X_M.

First, by elementarity, M thinks that each $P \in \mathcal{P} \cap M$ has non-empty interior, so there is $U \in \tau_X^*$ with $U \cap M \neq \emptyset$ and $U \subseteq P$. Clearly, $U \cap M$ is a non-empty open set in X_M which is contained in $P \cap M$. To finish the proof of Claim 1, let $U \in M \cap \tau_X$ be so that $U \cap M \neq \emptyset$. Since M knows that $\mathcal{P}$ is a π^0-base, there exists $Q \in M \cap \mathcal{P}$ satisfying $Q \subseteq U$, and so $Q \cap M \subseteq U \cap M$.

Claim 2. Claim 1 is true for π-bases.

The argument is similar to the previous one, so we omit it.

Claim 3. If $\vec{\mathcal{P}} = \langle \mathcal{P}_n : n \in \omega \rangle \in M$ is an Oxtoby sequence for X, $\vec{\mathcal{Q}} := \langle (\mathcal{P}_n \cap M) \restriction M : n \in \omega \rangle$ is an Oxtoby sequence for X_M.

According to Claim 1, $\vec{\mathcal{Q}}$ is a sequence of π^0-bases for X_M so let $\langle Q_n : n \in \omega \rangle$ be an associated nest for $\vec{\mathcal{Q}}$. For each $n \in \omega$, fix $P_n \in \mathcal{P}_n \cap M$ with $Q_n = P_n \cap M$ and observe that the fact $M^\omega \subseteq M$ implies that $\langle P_k : k \in \omega \rangle \in M$. On the other hand, from $P_{n+1} \cap M = Q_{n+1} \subseteq Q_n = P_n \cap M$, we deduce that $P_{n+1} \subseteq P_n$; i.e., $\langle P_k : k \in \omega \rangle$ is an associated nest to $\vec{\mathcal{P}}$ and hence, by elementarity, $\bigcap_k Q_k = M \cap \bigcap_k P_k \neq \emptyset$.

Claim 4. Claim 3 remains true if we replace "Oxtoby sequence" by "sequence witnessing O-pseudocompleteness."

Keeping the notation from Claim 3, let $\vec{Q} = \langle Q_n : n \in \omega \rangle$ be a strong nest for $\vec{\mathcal{Q}}$. Fix, for each $n \in \omega$, $P_n \in M \cap \mathcal{P}_n$ in such a way that $Q_n = M \cap P_n$ and note that $\vec{P} := \langle P_k : k \in \omega \rangle \in M$. We will show that,

$$(\dagger) \qquad \forall\, n \in \omega \ (M \cap \operatorname{cl}_X P_{n+1} \subseteq P_n).$$

Since $\vec{Q}$ is a strong nest, $x \in X_M \setminus P_n$ implies $x \notin \operatorname{cl}_X P_{n+1}$, and so, by definition, there exists $U \in M \cap \tau_X$ with $x \in U$ and $U \cap (P_{n+1} \cap M) = \emptyset$. In other words, M thinks that U misses P_{n+1} which, by elementarity, gives $U \cap P_{n+1} = \emptyset$. Thus, $x \notin \operatorname{cl}_X P_{n+1}$.

Using (†) and elementarity, we deduce that $\vec{P}$ is a strong nest for $\vec{\mathcal{P}}$, so M knows that $\vec{P}$ has non-empty intersection, i.e., $\bigcap_k Q_k = M \cap \bigcap_k P_k \neq \emptyset$. Hence, the proof of the proposition is complete. □

Note that the assumption $M^\omega \subseteq M$ cannot be removed from the hypotheses: If M is a countable elementary submodel and $\mathbb{R} \in M$, then $\mathbb{R}_M$ is countable T_1 and has no isolated points so, according to Corollary 3.6, $\mathbb{R}_M$ is not O-pseudocomplete.

5. Unions

We start this section by proving that all topological spaces have a nice decomposition.

Proposition 5.1. *Given a topological space X, there exist X_O and X_N such that*

1. *X_O and X_N are disjoint open subsets of X whose union is dense in X,*
2. *the subspace X_O is Oxtoby, and*
3. *all Oxtoby subspaces of X_N are nowhere dense in X.*

Proof. Let us denote by $\mathcal{U}$ the collection of all open subspaces of X which are Oxtoby. We will show that $X_O := \bigcup \mathcal{U}$ and $X_N := X \setminus \overline{X_O}$ satisfy conditions (1)–(3).

A standard argument involving the Axiom of Choice produces $\mathcal{V}$, a maximal pairwise disjoint family in $\mathcal{U}$. Hence, $Y := \bigcup \mathcal{V}$ is a dense subspace of X_O so to verify that (2) holds, we will argue that Y is Oxtoby (see Proposition 3.3(4)). Start by fixing, for each $V \in \mathcal{V}$, an Oxtoby sequence for V, let us say, $\vec{\mathcal{P}}_V := \langle \mathcal{P}_n^V : n \in \omega \rangle$. By letting $\vec{\mathcal{P}} := \langle \bigcup \{\mathcal{P}_n^V : V \in \mathcal{V}\} : n \in \omega \rangle$, we obtain a sequence of π^0-bases for Y and, moreover, since $\mathcal{V}$ is pairwise disjoint, any associated nest for $\vec{\mathcal{P}}$ is an associated nest for $\vec{\mathcal{P}}_V$, for some $V \in \mathcal{V}$. Thus, $\vec{\mathcal{P}}$ is an Oxtoby sequence for Y.

Finally, let S be an Oxtoby subspace of X_N. By Proposition 3.3(3) and Remark 3.9, we get $U := \operatorname{int}_X \operatorname{cl}_X S \in \mathcal{U}$, and so $U \subseteq X_O$. Given that $U \cap S \subseteq X_O \cap X_N$ and $U \subseteq \operatorname{cl}_X S$, we conclude that $U = \emptyset$. □

Any space which can be written as the union of countably many Oxtoby subspaces of it will be called *σ-Oxtoby*.

Corollary 5.2. *For a σ-Oxtoby space, being Baire is equivalent to being Oxtoby.*

Proof. Assume that $\{Y_n : n \in \omega\}$ is a family of Oxtoby subspaces of a topological space X such that $X = \bigcup_n Y_n$.

Let X_O and X_N be as described in Proposition 5.1. Since each $Y_n \cap X_N$ is an open subspace of Y_n, we deduce that $Y_n \cap X_N$ is an Oxtoby subspace of X contained in X_N, and therefore it is nowhere dense. Hence, X_N is meager.

When X is Baire, we get $X_N = \emptyset$ and, as a consequence, X_O is dense in X; thus, X is Oxtoby. The remaining implication is Proposition 3.3. □

Corollary 5.3. *Any finite union of Oxtoby spaces is Oxtoby.*

Proof. It suffices to prove that the union of two Oxtoby spaces is Oxtoby, so let A and B be two Oxtoby subspaces of a topological space X with $X = A \cup B$. Then $Y := \overline{A}$ is Oxtoby. Also, $Z := X \setminus Y$ is Oxtoby because it is an open subspace of B.

According to the previous corollary, we only need to show that X is Baire so let $\mathcal{D}$ be a countable family of dense open subsets of X and let $U \in \tau_X^*$ be arbitrary. If $U \cap Z \neq \emptyset$, then $U \cap \bigcap \mathcal{D} \neq \emptyset$ because all members of $\mathcal{D} \restriction Z$ are dense open in Z. When U and Z are disjoint, $U \subseteq Y$, and therefore U is Baire; so we conclude that $U \cap \bigcap \mathcal{D} \neq \emptyset$. □

As expected, one cannot replace "Oxtoby" by "T-pseudocomplete" or "O-pseudocomplete" in the last result.

Example 5.4. Let $\mathbb{P}$ be the subspace of $\mathbb{R}$ consisting of all irrational numbers and let $\alpha\mathbb{P}$ be its Alexandroff extension, i.e., $\alpha\mathbb{P}$ denotes the topological space obtained by endowing the set $\mathbb{P} \cup \{0\}$ with the topology which has the collection

$$\tau_{\mathbb{P}} \cup \{U \subseteq \mathbb{P} \cup \{0\} : (0 \in U) \wedge (\mathbb{P} \setminus U \text{ is compact in } \tau_{\mathbb{P}})\}$$

as a base. Since $\mathbb{P}$ and $\{0\}$ are completely metrizable, we only need to verify that $\alpha\mathbb{P}$ is not quasiregular. To do so, note that $\mathbb{P}$ is a non-empty open subset of $\alpha\mathbb{P}$ and that $0 \in \operatorname{cl}_{\alpha\mathbb{P}} V$, for each $V \in \tau_{\alpha\mathbb{P}}^*$, because $\mathbb{P}$ is not locally compact.

It is pointed out in [6, (2)] that all G_δ-subsets of $\mathbb{S}$, Sorgenfrey's line, are Oxtoby. One way to see this is as follows, given $\{U_n : n \in \omega\}$, a family of open subsets of $\mathbb{S}$, obtain an Oxtoby sequence for $A := \bigcap_n U_n$ by defining $P \in \mathcal{P}_n$ if and only if either

(1) $P = A \cap [a, b]$ for some $a, b \in A$ satisfying $a < b$ and $[a, b] \subseteq U_n$ or
(2) $P = \{a\}$, where a is an isolated point of A.

Now consider the unit square $[0,1] \times [0,1]$ equipped with the topology induced by the lexicographic order and denote by X the subspace $[0,1] \times \{0,1\}$. X is called *Alexandroff double arrow space.* As one easily checks, the subspaces $S_0 := (0,1] \times \{0\}$ and $S_1 := [0,1) \times \{1\}$ are homeomorphic to $\mathbb{S}$ so, whenever A is a G_δ-subset of X, the previous paragraph guarantees that $A \cap S_i$ is Oxtoby. Thus, Corollary 5.3 gives the following.

Example 5.5. All G_δ-subsets of the double arrow space are Oxtoby.

6. Products

For the purposes of this section, all members of a cartesian product are choice functions, so if x is an element of a cartesian product and A is a set, $x \restriction A$ is the restriction of x to the set A. Also, given a set S, $[S]^{<\omega}$ will denote the collection of all finite subsets of S and $[S]^{\leq\omega}$ will stand for the set of all countable subsets of S.

Regarding Question 3.4(2), we have the following.

Proposition 6.1. *Let $\{X_\alpha : \alpha < \kappa\}$ be a family of non-empty completely metrizable topological spaces and let X be its topological product. If Y is a dense G_δ-subset of X, then Y is O-pseudocomplete.*

Proof. Let us start by fixing, for each $\alpha < \kappa$, a complete metric d_α which is compatible with the topology of X_α. Also, set $\mathbb{P} := \{x \restriction F : x \in X \wedge F \in [\kappa]^{<\omega}\}$.

Now, given $p \in \mathbb{P}$ and $m \in \omega$, define

$$[p; m] := \{x \in X : \forall\, \alpha \in \mathrm{dom}(p)\ (d_\alpha(p(\alpha), x(\alpha)) < 2^{-m})\}$$

and notice that $\{[q; k] : q \in \mathbb{P} \wedge k \in \omega\}$ is a base for X.

Assume that $\{U_n : n \in \omega\}$ is a family of open subsets of X with $Y = \bigcap_n U_n$. In order to prove that Y is O-pseudocomplete, define, for each integer n, the family $\mathcal{P}_n$ as follows: $P \in \mathcal{P}_n$ if and only if there are $p \in \mathbb{P}$ and $m \in \omega \setminus n$ satisfying $\mathrm{cl}_X[p; m] \subseteq U_n$ and $P = Y \cap [p; m]$.

Let $q \in \mathbb{P}$ and $k \in \omega$ be arbitrary. Since U_n is a dense open subset of X, there exist $p \in \mathbb{P}$ and $\ell \in \omega$ such that $\mathrm{cl}_X[p; \ell] \subseteq U_n \cap [q; k]$. Thus, by letting $m := \ell + n + 1$, we get $[p; m] \subseteq [p; \ell]$, and so $P := Y \cap [p; m]$ satisfies $P \in \mathcal{P}_n$ and $P \subseteq Y \cap [q; k]$. Also, the fact that Y is dense in

X implies that all members of $\mathcal{P}_n$ are non-empty open subsets of Y. In other words, $\mathcal{P}_n$ is a π-base for Y.

Suppose now that $\vec{P} = \langle P_n : n \in \omega \rangle$ is a strong nest associated to $\langle \mathcal{P}_n : n \in \omega \rangle$.

For each $n \in \omega$, let $p_n \in \mathbb{P}$ and $m_n \in \omega \setminus n$ be so that $\mathrm{cl}_X[p_n; m_n] \subseteq U_n$ and $P_n = Y \cap [p_n; m_n]$. Since Y is dense in X,

$$(\dagger) \quad \mathrm{cl}_Y P_n = Y \cap \mathrm{cl}_X P_n = Y \cap \mathrm{cl}_X(Y \cap [p_n; m_n]) = Y \cap \mathrm{cl}_X[p_n; m_n].$$

For all $\alpha < \kappa$ and $n \in \omega$, let B^n_α be the open ball $\{x \in X_\alpha : d_\alpha(x, p_n(\alpha)) < 2^{-m_n}\}$ and set $a_\alpha := \{i \in \omega : \alpha \in \mathrm{dom}(p_i)\}$. Hence, if $\pi_\alpha : X \to X_\alpha$ is the αth projection map, we get

$$(\star) \qquad \mathrm{cl}_X[p_n; m_n] = \bigcap \{\pi_\xi^{-1}[\mathrm{cl}_{X_\xi}(B^n_\xi)] : \xi \in \mathrm{dom}(p_n)\}.$$

Claim. For each $\alpha \in A := \bigcup_n \mathrm{dom}(p_n)$, there exists $z_\alpha \in \bigcap \{\mathrm{cl}_{X_\alpha}(B^n_\alpha) : n \in a_\alpha\}$.

Before we prove our assertion, assume it is true and fix $z \in X$ with $z(\alpha) = z_\alpha$, for each $\alpha \in A$.

Let us show, first, that $z \in \bigcap_i \mathrm{cl}_X[p_i; m_i]$. Indeed, given $n \in \omega$, if $\alpha \in \mathrm{dom}(p_n)$, then $n \in a_\alpha$, and so $\pi_\alpha(z) \in \mathrm{cl}_{X_\alpha}(B^n_\alpha)$; thus, by $(\star)$, $z \in \mathrm{cl}_X[p_n; m_n]$.

From the previous paragraph, we deduce that $z \in \bigcap_n U_n = Y$, and therefore, by $(\dagger)$, $z \in Y \cap \mathrm{cl}_X[p_{n+1}; m_{n+1}] = \mathrm{cl}_Y P_{n+1} \subseteq P_n \subseteq [p_n; m_n]$. In conclusion, $z \in \bigcap_i P_i$ and the proof would be complete. So we only need to show that our claim holds.

Fix, for each integer n, an arbitrary point $x_n \in P_n$ and let $\alpha \in A$ be arbitrary. If $k, \ell \in a_\alpha$ satisfy $k < \ell$, then $\{x_k, x_\ell\} \subseteq P_k \subseteq \mathrm{cl}_X[p_k; m_k]$; so, according to $(\star)$,

$$\max\{d_\alpha(p_k(\alpha), x_k(\alpha)), d_\alpha(p_k(\alpha), x_\ell(\alpha))\} \leq 2^{-m_k} \leq 2^{-k}$$

and, as a consequence, $d_\alpha(x_k(\alpha), x_\ell(\alpha)) \leq 2 \cdot 2^{-k}$.

The last inequality implies that if a_α is infinite, then $\langle x_n(\alpha) : n \in a_\alpha \rangle$ is a Cauchy sequence in X_α, and so it converges to some point z_α. Now note that if $n \in a_\alpha$ and $i \in a_\alpha \setminus n$, then $x_i \in P_n$, and therefore $x_i(\alpha) \in B^n_\alpha$; in other words, $\{x_i(\alpha) : i \in a_\alpha \setminus n\} \subseteq B^n_\alpha$, for all $n \in a_\alpha$. Hence, we conclude that z_α is as needed in the claim.

When a_α is finite, we let $k := \max a_\alpha$. Thus, for each $n \in a_\alpha$, we obtain $n \leq k$, and so $x_k \in P_n$; in particular, $x_k(\alpha) \in B^n_\alpha$. Therefore, $z_\alpha := x_k(\alpha)$ works. □

It is proved in [5, Theorem 3.2] that Σ-products of Oxtoby spaces are Oxtoby spaces themselves so it is natural to ask if the same holds for σ-products.

Proposition 6.2. *If κ is an infinite cardinal, the σ-product*

$$X := \{x \in 2^\kappa : |x^{-1}\{1\}| < \omega\}$$

is not Baire.

Proof. For each $n \in \omega$, $D_n := \{x \in 2^\kappa : |x^{-1}\{1\}| \geq n\}$ is a dense open subset of 2^κ, and therefore $\{X \cap D_n : n \in \omega\}$ is a family of dense open subsets of X. Also, if $x \in \bigcap_n D_n$, then $|x^{-1}\{1\}| \geq \omega$, and so $\bigcap_n (X \cap D_n) = \emptyset$. □

Given a topological space X, the collection of all G_δ-subsets of X is a base for a topology on the set X. When X is endowed with this topology, the resulting topological space PX is called *the G_δ-expansion of X*.

Our next result requires some preliminary notions. A topological space is *feebly compact* if every locally finite family of open subsets of it is finite. Thus, a space in which every point has a feebly compact neighborhood will be called *locally feebly compact*. It is well known that for Tychonoff spaces, feeble compactness is equivalent to pseudocompactness (see [2, Theorem 3.10.22]).

By a *clopen* subset of a topological space, we mean a subset which is, simultaneously, closed and open. A subset F of a topological space will be called *regular closed* if $F = \overline{\operatorname{int} F}$.

Proposition 6.3. *If X is a Σ-product of a family of non-empty regular locally feebly compact spaces, then PX has a base consisting of non-empty clopen subsets such that any nest in it has non-void intersection. In particular, PX is zero-dimensional and O-pseudocomplete.*

Proof. Let us assume that $\{X_\alpha : \alpha < \kappa\}$ is the family whose Σ-product is X. Set $Z := \prod_{\alpha<\kappa} X_\alpha$ and denote by $\pi_\alpha : Z \to X_\alpha$ the αth projection map. Also, suppose that X is the Σ-product about $z \in Z$.

Given $\alpha < \kappa$, let $\mathcal{B}_\alpha$ be the collection of all sets of the form $\bigcap_n C_n$, where $\langle C_n : n \in \omega\rangle$ is a strong nest of non-empty regular closed feebly compact subsets of X_α. The argument used to prove [10, Main Theorem 2] shows three things:

(1) $\mathcal{B}_\alpha$ consists of non-empty closed G_δ-subsets of X_α,
(2) $\mathcal{B}_\alpha$ is a base for PX_α, and
(3) every nest in $\mathcal{B}_\alpha$ has non-void intersection.

Define $\mathcal{B}$ as follows, $B \in \mathcal{B}$ if and only if there exist $F \in [\kappa]^{\leq\omega}$ and $\{B_\alpha : \alpha \in F\}$ in such a way that each B_α belongs to $\mathcal{B}_\alpha$ and $B = X \cap \bigcap_{\alpha\in F} \pi_\alpha^{-1}[B_\alpha]$. We will show that $\mathcal{B}$ is the collection whose existence is claimed in our proposition. Start by noting that, according to (1) above, all members of $\mathcal{B}$ are clopen subsets of PX.

To prove that $\mathcal{B}$ is a base for PX, assume that W is an open subset of PX and let $x \in W$ be arbitrary. Then there is a sequence $\{V_n : n \in \omega\}$ of canonical basic open subsets of Z with $x \in X \cap \bigcap_n V_n \subseteq W$. For each integer n, fix $F_n \in [\kappa]^{<\omega}$ and $\{V_\alpha^n : \alpha \in F_n\}$ in such a way that each V_α^n is an open subset of X_α and $V_n = \bigcap_{\alpha \in F_n} \pi_\alpha^{-1}[V_\alpha^n]$. Now, if $\alpha \in F := \bigcup_n F_n$, then $U_\alpha := \bigcap\{V_\alpha^n : n \in \omega \ \wedge \ \alpha \in F_n\}$ is a G_δ-subset of X_α containing $\pi_\alpha(x)$, so we apply (2) to obtain $B_\alpha \in \mathcal{B}_\alpha$ satisfying $\pi_\alpha(x) \in B_\alpha \subseteq U_\alpha$. Hence, $X \cap \bigcap_{\alpha \in F} \pi_\alpha^{-1}[B_\alpha]$ is a member of $\mathcal{B}$ which contains x and is a subset of $\bigcap_n V_n$.

Finally, let us consider $\langle B_n : n \in \omega\rangle$, a nest in $\mathcal{B}$. Given $n \in \omega$, there exist $F_n \in [\kappa]^{\leq\omega}$ and $\{B_\alpha^n : \alpha \in F_n\}$ in such a way that $B_\alpha^n \in \mathcal{B}_\alpha$, for each $\alpha \in F_n$, and $B_n = X \cap \bigcap_{\alpha \in F_n} \pi_\alpha^{-1}[B_\alpha^n]$. Define $F := \bigcup_n F_n$ and, for each $\alpha \in F$, set $S_\alpha := \{n \in \omega : \alpha \in F_n\}$.

We will argue that $\{B_\alpha^n : n \in S_\alpha\}$ is a decreasing sequence. Indeed, if there were $m, n \in S_\alpha$ with $m < n$ and $B_\alpha^n \setminus B_\alpha^m \neq \emptyset$, then, using (1), we would be able to get a choice function $e \in \prod_{\beta \in F_n} B_\beta^n$ with $e(\alpha) \notin B_\alpha^m$. Therefore, $e \cup (z \restriction (\kappa \setminus F_n))$ would end up being a member of $B_n \setminus B_m$. An absurdity.

From the previous paragraph and (3), we deduce that, for each $\alpha \in F$, there exists $y_\alpha \in \bigcap_{n \in S_\alpha} B_\alpha^n$. Now fix a point $y \in X$ satisfying $\pi_\alpha(y) = y_\alpha$, for all $\alpha \in F$. A straightforward argument gives $y \in \bigcap_n B_n$. In particular, $\emptyset \notin \mathcal{B}$, and this finishes the proof. □

Todd [10, Questions 7] asks for an example of a T-pseudocomplete space with a non-Baire G_δ-expansion. The previous result shows that such an example cannot be obtained from a Σ-product of regular locally feebly compact spaces.

Let us recall that those subsets of a topological space X which are dense in PX are called G_δ*-dense*.

Proposition 6.4. *Let $\langle \mathcal{B}_n : n \in \omega\rangle$ be an Oxtoby sequence for a topological space X. If each $\mathcal{B}_n$ consists of clopen subsets of X, then all G_δ-dense subsets of X are O-pseudocomplete.*

Proof. Clearly, our assumptions imply that X is quasiregular, so let Y be a G_δ-dense subset of X. Hence, $\vec{\mathcal{P}} := \langle \mathcal{B}_n \restriction Y : n \in \omega\rangle$ is a sequence of π-bases of Y.

Now assume that $\langle P_n : n \in \omega\rangle$ is an associated strong nest for $\vec{\mathcal{P}}$. Fix, for each n, $B_n \in \mathcal{B}_n$ such that $P_n = Y \cap B_n$. Thus, $Y \cap B_{n+1} = P_{n+1} \subseteq P_n \subseteq B_n$, and since Y is dense in X,

$$B_{n+1} \subseteq \overline{B_{n+1}} = \overline{Y \cap B_{n+1}} \subseteq \overline{B_n} = B_n.$$

Given that $\bigcap_n B_n$ is a non-empty G_δ-subset of X, $\bigcap_n P_n = Y \cap \bigcap_n B_n \neq \emptyset$. □

7. Mappings

A continuous map f of X onto Y is called *irreducible* if no proper closed subset of X is mapped by f onto Y.

Proposition 7.1. *If $f : X \to Y$ is irreducible and closed, the following hold.*

1. *For each $U \in \tau_X^*$, there is $V \in \tau_Y^*$ with $V \subseteq f[U]$ and $f^{-1}[V] \subseteq U$.*
2. *X is quasiregular if and only if Y is quasiregular.*
3. *X is an Oxtoby space whenever Y is an Oxtoby space.*

Proof. Let us start by proving (1): If $U \in \tau_X^*$, then $V := Y \setminus f[X \setminus U]$ turns out to be non-empty and open because f is irreducible and closed. Moreover, we have that $f^{-1}[V] \subseteq U$ and, since f is onto, $V \subseteq f[U]$.

To prove (2) assume, first, that X is quasiregular and let $W \in \tau_Y^*$. Given that $f^{-1}[W] \in \tau_X^*$, there is $U \in \tau_X^*$ with $\overline{U} \subseteq f^{-1}[W]$. Thus, according to (1), there exists $V \in \tau_Y^*$ with $V \subseteq f[U] \subseteq f[\overline{U}]$. Applying the fact that f is closed, we obtain $\overline{V} \subseteq f[\overline{U}] \subseteq W$. So Y is quasiregular.

Now suppose that Y is quasiregular and let $U \in \tau_X^*$. Use (1) to get $V \in \tau_Y^*$ with $f^{-1}[V] \subseteq U$ and let $W \in \tau_Y^*$ be so that $\overline{W} \subseteq V$. Then $f^{-1}[W]$ is a non-void open set in X whose closure is contained in $f^{-1}[\overline{W}] \subseteq f^{-1}[V] \subseteq U$. So X is quasiregular.

For (3), let us start by fixing $\vec{\mathcal{P}} = \langle \mathcal{P}_n : n \in \omega \rangle$, an Oxtoby sequence for Y. For each integer n, set $\mathcal{Q}_n := \{f^{-1}[B] : B \in \mathcal{P}_n\}$. We will argue that $\vec{\mathcal{Q}} := \langle \mathcal{Q}_n : n \in \omega \rangle$ is an Oxtoby sequence for X.

Let $n \in \omega$ be arbitrary. Clearly, each member of $\mathcal{Q}_n$ has non-empty interior. Now, given $U \in \tau_X^*$, there exists $V \in \tau_Y^*$ with $V \subseteq f[U]$ and $f^{-1}[V] \subseteq U$. Thus, for some $P \in \mathcal{P}_n$, we obtain $P \subseteq V$; hence, $f^{-1}[P] \in \mathcal{Q}_n$ and $f^{-1}[P] \subseteq U$. Therefore, $\mathcal{Q}_n$ is a π^0-base for X.

Finally, let us assume that $\langle Q_n : n \in \omega \rangle$ is an associated nest for $\vec{\mathcal{Q}}$. Our definition of $\vec{\mathcal{Q}}$ guarantees the existence of a sequence $\langle P_n : n \in \omega \rangle$ such that $P_n \in \mathcal{P}_n$ and $Q_n = f^{-1}[P_n]$, for all $n \in \omega$. Since f is onto, $f[Q_n] = P_n$, and so $\langle P_k : k \in \omega \rangle$ is an associated nest for $\vec{\mathcal{P}}$. Thus, $\bigcap_n Q_n = f^{-1}[\bigcap_n P_n] \neq \emptyset$. □

An immediate consequence of the previous result is that "being Oxtoby" and "being T-pseudocomplete" are inverse invariants of irreducible closed mappings. Thus, the following question seems natural.

Question 7.2. Is O-pseudocompleteness an inverse invariant of closed irreducible mappings?

Given a topological space X, let us denote by $A(X)$ the Alexandroff duplicate of X, in other words, the topological space which results in endowing the set $X \times 2$ with the topology which has the collection

$$\{\{(x,1)\} : x \in X\} \cup \{(U \times 2) \setminus \{(x,1)\} : (U \in \tau_X) \wedge (x \in U)\}$$

as a base. Standard arguments show that (a) the subspace $X \times \{0\}$ is closed in $A(X)$ and homeomorphic to X; (b) $X \times \{1\}$ is a discrete subspace of $A(X)$; and (c) when X has no isolated points, $X \times \{1\}$ is dense in $A(X)$. Hence, we note the following.

Remark 7.3. If X is a topological space with no isolated points, then X embeds as a closed subspace of $A(X)$ and all dense subspaces of $A(X)$ are Oxtoby. In particular, when X is not Oxtoby, $A(X)$ is an Oxtoby space with a closed non-Oxtoby subspace.

Let $p : A(X) \to X$ be the map given by $p(x,i) = x$, for all $x \in X$ and $i < 2$. It is straightforward to prove that p is perfect, and therefore we conclude the following.

Remark 7.4. Being Oxtoby is not an invariant of perfect mappings.

References

[1] J. M. Aarts and D. J. Lutzer, *Pseudo-completeness and the product of Baire spaces*, Pacific J. Math. **48** (1973), 1–10.

[2] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[3] W. G. Fleissner and K. Kunen, *Barely Baire spaces*, Fund. Math. **101** (1978), no. 3, 229–240.

[4] D. H. Fremlin, *Consequences of Martin's Axiom*. Cambridge Tracts in Mathematics, 84. Cambridge: Cambridge University Press, 1984.

[5] Melvin Henriksen, Ralph Kopperman, Marlon Rayburn, and Aaron R. Todd, *Oxtoby's pseudocompleteness revisited*, Topology Appl. **100** (2000), no. 2-3, 119–132.

[6] ———, *Corrections to: "Oxtoby's pseudocompleteness revisited"* [Topology Appl. 100 (2000), no. 2-3, 119–132], Topology Appl. **114** (2001), no. 1, 115–116.

[7] Kenneth Kunen, *Set Theory*. Studies in Logic (London), 34. London: College Publications, 2011.

[8] John C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math. **49** (1960/1961), 157–166.

[9] Aaron R. Todd, *Quasiregular, pseudocomplete, and Baire spaces*, Pacific J. Math. **95** (1981), no. 1, 233–250.

[10] ———, *The G_δ-topology, light compactness, and pseudocompleteness* in Papers on General Topology and Applications (Amsterdam, 1994). Ed. Eva Coplakova and Klaas Pieter Hart. Annals of the New York Academy of Sciences, 788. New York: New York Acad. Sci., 1996. 197–202.

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