http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 159–179

http://topology.nipissingu.ca/tp/

ℓ^2-Betti Numbers and the Genus of a Graph

by

Timothy A. Schroeder

Electronically published on June 2, 2014

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 46 (2015)
Pages 159-179

E-Published on June 2, 2014

ℓ^2-BETTI NUMBERS AND THE GENUS OF A GRAPH

TIMOTHY A. SCHROEDER

ABSTRACT. Associated to a Coxeter system (W, S) is a labeled simplicial complex L, and a complex Σ on which W acts. Theorems and conjectures regarding the (reduced) ℓ^2-homology of Σ provide avenues along which to approach questions regarding the genus of a graph, where the graph is understood as a subcomplex of the simplicial complex L. This paper explains this connection, describes a program for estimating the genus of a graph, and uses this connection to provide examples of some nice embeddings of complete graphs in higher genus surfaces.

1. Introduction

In several papers (e.g., [1], [2], and [4]), Michael W. Davis describes a construction which associates to any Coxeter system (W, S), a simplicial complex $\Sigma(W, S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and cocompactly. This is the Davis complex. Associated to any Coxeter system is also a finite simplicial complex called the nerve of (W, S), denoted L. Now, it is not the case that every simplicial complex can arise as the nerve of a Coxeter system, but, for simplicial complexes that can arise as nerves, we can reverse this correspondence. That is, we can take a simplicial complex L as the initial data and then develop corresponding Coxeter systems and Davis complexes.

Indeed, a *Coxeter labeling* of a simplicial complex L is a labeling of the edges of L with integers ≥ 2 so that if m_{st} is the label on edge $\{s, t\}$, then L corresponds to the nerve of a Coxeter system (W_L, S) where S denotes the vertex set of L and W_L has the presentation

$$\left\langle S \mid s^2 = 1 \,\forall\, s \in S, \text{ and } (st)^{m_{st}} = 1 \text{ whenever } \{s, t\} \text{ is an edge of } L \right\rangle.$$

2010 *Mathematics Subject Classification.* 05C10, 05C25, 20F55, 55N35.
Key words and phrases. Coxeter groups, ℓ^2-homology theory, planar graphs.

We denote by Σ_L the Davis complex associated to (W_L, S). For details of this correspondence, see §2.2.

The complex Σ_L admits a cellulation under which the link of each vertex is L. Thus, when L is a triangulation of $\mathbb{S}^{n-1}$, Σ_L is an aspherical n-manifold. In this case, the authors of [5], [3], [14], and [15] prove various results regarding the (reduced) ℓ^2-homology of Σ_L, $\mathcal{H}_*(\Sigma_L)$. These results are largely sub-cases of Singer's Conjecture applied to these Davis manifolds.

Conjecture 1.1 (Singer's Conjecture for Coxeter Groups)**.** *Let* (W, S) *be a Coxeter system whose nerve* L *is a triangulation of* $\mathbb{S}^{n-1}$*. Then* $\mathcal{H}_i(\Sigma_L) = 0$ *for all* $i \neq \frac{n}{2}$.

With proper edge labeling, any graph Γ can be the nerve of a corresponding Coxeter system (see §2.2); that is, any graph can be treated as the (labeled) nerve of a related Coxeter system. This observation is the key to connecting the ℓ^2-homology of Davis complexes with the study of the *genus of a graph*, i.e., the minimal integer n such that Γ can be embedded in a surface of genus n. For example, in [15], the author proves Conjecture 1.1 is true in dimension 3, i.e., for Coxeter systems whose nerves are triangulations of $\mathbb{S}^2$. Hence, ℓ^2-homological statements about 1-dimensional subcomplexes of these triangulations can be translated into statements about genus 0 (planar) graphs, see [17]. The purpose of this paper is to use the following generalization of Singer's Conjecture to higher genus nerves as a means of studying the genus of graphs.

Conjecture 1.2 (Generalized Singer Conjecture for Surfaces)**.** *Let* (W, S) *be a Coxeter system with nerve* L*, a genus* g *surface. Then* $\beta_i(\Sigma_L) = 0$ *for* $i \neq 2$ *and* $\beta_2(\Sigma_L) = g$.

In [5], the authors provide evidence that the right-angled version of Conjecture 1.2 is true, and here in section 5, the author provides specific examples for which it is true. This survey paper will then have two facets: (1) Assuming Conjecture 1.2, provide an ℓ^2-homological program which approximates the genus of any graph Γ. (2) Provide a family of examples of Conjecture 1.2.

2. Background

Let (W, S) be a Coxeter system. Given a subset U of S, define W_U to be the subgroup of W generated by the elements of U. A subset T of S is *spherical* if W_T is a finite subgroup of W. In this case, we will also say that the subgroup W_T is spherical. Denote by $\mathcal{S}$ the poset of spherical subsets of S, partially ordered by inclusion. Given a subset V of S, let $\mathcal{S}_{\geq V} := \{T \in \mathcal{S} | V \subseteq T\}$. Similar definitions exist for $<, >$ and $\leq$. For any $w \in W$ and $T \in \mathcal{S}$, we call the coset wW_T a *spherical coset*. The poset of all spherical cosets we will denote by $W\mathcal{S}$.

The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex. This simply means that if $T \in \mathcal{S}_{>\emptyset}$ and T' is a nonempty subset of T, then $T' \in \mathcal{S}_{>\emptyset}$. Denote this simplicial complex by L and call it the *nerve* of (W, S). The vertex set of L is S and a non-empty subset of vertices T spans a simplex of L if and only if T is spherical.

2.1. The Davis complex.

Let $K = |\mathcal{S}|$, the geometric realization of the poset $\mathcal{S}$. It is a finite simplicial complex. Denote by $\Sigma(W, S)$, or simply Σ when the system is clear, the geometric realization of the poset $W\mathcal{S}$. This is the Davis complex. The natural action of W on $W\mathcal{S}$ induces a simplicial action of W on Σ which is proper and cocompact. K includes naturally into Σ via the map induced by $T \to W_T$. So we view K as a subcomplex of Σ and note that K is a strict fundamental domain for the action of W on Σ. Σ has a coarser cell structure: its cellulation by "Coxeter cells." (References include [2] and [5].) The features of the Coxeter cellulation are summarized by [2, Proposition 7.3.4]. We point out here that under this cellulation the link of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, then Σ is a topological n-manifold.

2.2. Metric flag simplicial complexes.

Given a Coxeter system (W, S) with nerve L, we define a labeling on the edges of L by the map $m : \mathrm{Edge}(L) \to \{2, 3, \ldots\}$, where $\{s, t\} \mapsto m_{st}$. This labeling accomplishes two things: (1) the Coxeter system (W, S) can be recovered (up to isomorphism) from L and (2) the 1-skeleton of L inherits a natural piecewise spherical structure in which the edge $\{s, t\}$ has length $\pi - \pi/m_{st}$. L is then a *metric flag* simplicial complex (see [2, Definition I.7.1]).

Our idea is to use a given simplicial complex as starting data and, from it, define Coxeter systems and corresponding Davis complexes. To that end, given a finite simplicial complex L with vertex set S, we consider all labelings, with integers ≥ 2, of the edges of L. For each labeling, define the Coxeter group W_L with presentation

(2.1)
$$\left\langle S \mid s^2 = 1 \,\forall\, s \in S, \text{ and } (st)^{m_{st}} = 1 \text{ whenever } \{s, t\} \text{ is an edge of } L \right\rangle .$$

If the labeled L is isomorphic to the labeled nerve of the Coxeter group defined by equation (2.1), we call this labeling a *Coxeter labeling* of L and refer to L as a metric flag simplicial complex. In this case, we denote the associated Davis Complex Σ_L with fundamental chamber K_L.

For the purpose of this paper, the term "metric flag triangulation" refers to both a simplicial complex and a particular Coxeter labeling of that complex. Note that for a labeling to define a simplicial complex L as a metric flag simplicial complex, it must have the property that a set of vertices spans a simplex if and only if the generators corresponding to that set of vertices, with the prescribed relations, generate a finite subgroup of W_L, as defined in equation (2.1).

Recall that a simplicial complex L is *flag* if every nonempty, finite set of vertices that are pairwise connected by edges spans a simplex of L. Thus, it is clear that any flag simplicial complex is metric flag if the edges are labeled with 2's. But if, for instance, L contains pairwise connected vertices r, s, and t, but not the 2-simplex spanned by r, s, and t, that is, L is not flag, then L cannot be labeled with 2's and be metric flag. For then $\{r, s, t\}$ is a spherical subset (generating $\mathbb{Z}_2^3$), but does not span a simplex in L. In this case, in order for L to be metric flag, we must have the labels on these edges satisfy

$$\frac{1}{m_{rs}} + \frac{1}{m_{st}} + \frac{1}{m_{rt}} \leq 1.$$

That is, we must ensure that the subgroup generated by $\{r, s, t\}$ is infinite. In general, to achieve a Coxeter labeling for a given simplicial complex L, one must label the edges so that whenever the $(k-1)$-skeleton of a k-simplex is in L but the k-simplex is not, then the simplices in the $(k-1)$-skeleton must each correspond to a finite Coxeter group, but the vertices of the k-simplex must generate an infinite Coxeter group. The classification of finite Coxeter groups is well known, see [2] or [9].

As stated in the §1, it is possible that a given simplicial complex cannot arise as the nerve of a corresponding Coxeter system, regardless of the labeling scheme. For example, the 2-skeleton of a 6-simplex cannot arise as the nerve of a Coxeter system (see [6]). It is also possible for a given simplicial complex to have some labelings define it as metric flag, and others not. For an example, see Figure 1. The 1-dimensional complex (graph) on the left is not metric flag, for the vertices, with the prescribed relations, do generate a finite group, yet the corresponding 2-simplex is not present. But the graph on the right is metric flag, because the corresponding Coxeter group is infinite. This example (and the discussion above applied to 1- and 2-simplices) also makes it clear that, with proper edge labeling, any graph can arise as the (labeled) nerve of a Coxeter system. Indeed, since a graph contains no 2-simplices, labeling all the edges of any graph with 3 results in a Coxeter labeling. But note that there are, in general, many Coxeter labelings for a given graph.

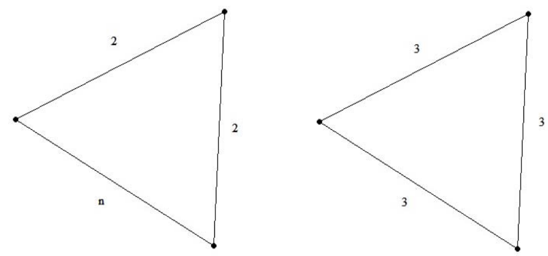

FIGURE 1. The graph on the left is not metric flag. The graph on the right is.

2.2.1. **FULL SUBCOMPLEXES**. A *full* subcomplex A of a simplicial complex L has the property that whenever the vertices of a simplex σ of L are contained in A, then σ is a simplex of A. When A is a full subcomplex of the metric flag simplicial complex L, then A is the nerve for the subgroup generated by the vertex set of A. We will denote this subgroup by W_A. Let Σ_A denote the Davis complex associated to (W_A, A^0) with fundamental domain K_A. The inclusion $W_A \hookrightarrow W_L$ induces an inclusion of posets $W_A\mathcal{S}_A \hookrightarrow W_L\mathcal{S}_L$ and thus an inclusion of Σ_A as a subcomplex of Σ_L. Note that W_A acts on Σ_A and that if $w \in W_L - W_A$, then Σ_A and $w\Sigma_A$ are disjoint copies of Σ_A in Σ_L. Denote by $W_L\Sigma_A$ the union of all translates of Σ_A in Σ_L.

2.3. USEFUL ℓ^2-HOMOLOGY.

Let L be a metric flag simplicial complex, and let A be a full subcomplex of L. The following notation will be used throughout.

$$\mathfrak{h}_i(L) := \mathcal{H}_i(\Sigma_L) \tag{2.2}$$

$$\mathfrak{h}_i(A) := \mathcal{H}_i(W_L\Sigma_A) \tag{2.3}$$

$$\mathfrak{h}_i(L, A) := \mathcal{H}_i(\Sigma_L, W_L\Sigma_A) \tag{2.4}$$

$$\beta_i(A) := \dim_{W_L}(\mathfrak{h}_i(A)) \tag{2.5}$$

$$\beta_i(L, A) := \dim_{W_L}(\mathfrak{h}_i(L, A)). \tag{2.6}$$

Here, $\dim_{W_L}(\mathfrak{h}_i(A))$ is the von Neumann dimension of the Hilbert W_L-module $W_L\Sigma_A$ and $\beta_i(A)$ is the i^{th} ℓ^2-Betti number of $W_L\Sigma_A$. The notation in equations (2.3) and (2.5) will not lead to confusion since $\dim_{W_L}(W_L\Sigma_A) = \dim_{W_A}(\Sigma_A)$. We now present several useful results from ℓ^2-homology theory. For references to the above notation and the following results, see [5], [7], and [15].

2.3.1. 0-DIMENSIONAL HOMOLOGY. Let Σ_A be the Davis complex constructed from a Coxeter system with nerve A, so W_A acts geometrically on Σ_A. The reduced ℓ^2-homology groups of Σ_A can be identified with the subspace of harmonic i-cycles (see [7] or [5]). That is, $x \in \mathfrak{h}_i(A)$ is an i-cycle and i-cocycle. 0-dimensional cocycles of Σ_A must be constant on all vertices of Σ_A. It follows that if W_A is infinite, and therefore the 0-skeleton of Σ_A is infinite, then $\beta_0(A) = 0$.

2.3.2. SINGER CONJECTURE IN DIMENSIONS 1, 2, AND 3. Conjecture 1.1 is true for elementary reasons in dimensions 1 and 2. Indeed, let L be $\mathbb{S}^0$ or $\mathbb{S}^1$, the nerve of a Coxeter system (W, S). Then W is infinite and so, as stated above, $\beta_0(L) = 0$. Poincaré duality then implies that the top-dimensional ℓ^2-Betti numbers are also 0, giving the result. In [15, Corollary 4.4], the author proves that Conjecture 1.1 holds for arbitrary Coxeter systems with nerve $\mathbb{S}^2$.

2.3.3. ORBIHEDRAL EULER CHARACTERISTIC. Σ_L is a geometric W_L-complex. So there is only a finite number of W_L-orbits of cells in Σ_L, and the order of each cell stabilizer is finite. The *orbihedral Euler characteristic* of Σ_L/W_L, denoted $\chi^{\mathrm{orb}}(\Sigma_L/W_L)$, is the rational number defined by

$$\chi^{\mathrm{orb}}(\Sigma_L/W_L) = \sum_{\sigma} \frac{(-1)^{\dim \sigma}}{|\operatorname{Stab}_{W_L}(\sigma)|}, \tag{2.7}$$

where the summation is over the simplices of K_L and $|\operatorname{Stab}_{W_L}(\sigma)|$ denotes the order of the stabilizer of σ in W_L. Then, if the dimension of L is $n-1$, a standard argument (see [7]) proves Atiyah's formula:

$$\chi^{\mathrm{orb}}(\Sigma_L/W_L) = \sum_{i=0}^{n} (-1)^i \beta_i(L). \tag{2.8}$$

2.3.4. RIGHT-ANGLED JOINS. For the following, please reference [5, §7]. If $L = L_1 * L_2$, the join of L_1 and L_2, where each edge connecting a vertex of L_1 with a vertex of L_2, is labeled 2, we write $L = L_1 *_2 L_2$ and then $W_L = W_{L_1} \times W_{L_2}$ and $\Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2}$. We may then use the Künneth formula to calculate the (reduced) ℓ^2-homology of Σ_L, and the following equation from [5, Lemma 7.2.4] extends to our situation:

$$\beta_k(L_1 *_2 L_2) = \sum_{i+j=k} \beta_i(L_1)\beta_j(L_2). \tag{2.9}$$

If $L' = P *_2 L$, where P is one point, then we call L' a *right-angled cone*, and denote it CL. $\Sigma_P = [-1, 1]$, so there are no 1-cycles in Σ_P and $\beta_1(P) = 0$. But $\chi^{\text{orb}}(\Sigma_P/W_P) = 1/2$. So by equation (2.8), $\beta_0(P) = 1/2$. Thus, equation (2.9) implies that

$$\beta_i(CL) = \frac{1}{2}\beta_i(L). \tag{2.10}$$

If $L' = P_2 *_2 L$, where P_2 is two disjoint points, then we call L' a *right-angled suspension* and denote it SL. $\mathfrak{h}_i(P_2) = 0$ for $i = 1, 2$, as noted in 2.3.2. So by equation (2.9), we have that

$$\beta_i(SL) = 0 \quad \text{for all } i. \tag{2.11}$$

It then follows from equations (2.10) and (2.11), excision, and the long exact sequence of the pair (CL, L) that

$$\beta_{i+1}(CL, L) = \frac{1}{2}\beta_i(L) \quad \text{for all } i \tag{2.12}$$

(see [5, Lemma 7.3.3]).

3. Surfaces of Higher Genus

We now consider cases is which the simplicial complex L is a metric flag triangulation of a surface of genus g, $g \geq 0$. We begin with a conjecture. It is a generalization of [5, Conjecture 11.5.1].

Conjecture 3.1. *Let L be a metric flag triangulation of a genus g surface. Then $\beta_i(L) = 0$ for $i \neq 2$ and $\beta_2(L) = g$.*

There is evidence to believe Conjecture 3.1, particularly in the right-angled case (see [5]). Herein, §5 provides specific examples of this conjecture. For this section, we will assume Conjecture 3.1 and describe a program that uses it to approximate the genus of a graph.

3.2. Triangulating a surface.

Suppose we are given a connected, finite graph Γ with a particular Coxeter labeling. Since we are concerned with the genus of Γ, and since it is known that trees are planar graphs, we assume that Γ is not a tree. Next, suppose Γ is embedded in a surface M_g of minimal genus g; that is, Γ does not embed in a surface of genus $g - 1$. The idea, then, is to develop a metric flag triangulation of M_g in which Γ, with its given labeling, is a full subcomplex. Then, assuming Conjecture 3.1, we will apply ℓ^2-homological results to the corresponding Coxeter system and Davis Complex to approximate g.

The first issue is to develop a metric flag triangulation of M_g. In doing this, we generalize Γ to any connected, metric flag simplicial complex A, not a tree, and we suppose A is embedded (topologically) in a surface M_g of minimal genus g. Introduce a vertex in the interior of each complementary region and cone off the boundary of each region. This produces a triangulation L of M_g. Preserve the labels of the edges of A and label each cone edge with 2. Now, it could be the case that there exists an n-cycle in A bounding a complementary region, $n \geq 4$, in which non-adjacent vertices x and y are connected by an edge in A. If A contains no such cycle, then we have a Coxeter labeling of L in which A is a full subcomplex. But if we do have such a cycle, then, with c representing the cone point introduced in this complementary region and with the edges $\{c, x\}$ and $\{c, y\}$ labeled 2, no matter the label on $\{x, y\}$, the vertices x, y, and c generate a finite Coxeter group. As a result, the nerve of the corresponding Coxeter system would not be L. In fact, the nerve would not only fail to triangulate a manifold, since it would triangulate the surface of M_g, but also include the 2-simplex $\{x, y, c\}$. See Figure 2.

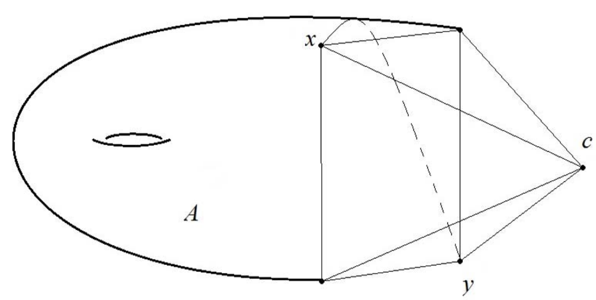

FIGURE 2. The nerve of W_L is not a triangulation of M_g.

Now, to avoid this issue, remove the labels of 2 on each edge of $L - A$ (preserve the labels on A) and take the barycentric subdivision bL of L, relative to A, as in [10, Definition 2.5.7]. That is, we do not subdivide any edge or 2-simplex in A, preserving the structure of A and W_A See Figure 3. It is clear that, in general, A is a full subcomplex of bL. Finally, label each edge of $bL - A$ with 2, and we have our desired metric flag triangulation of M_g.

Lemma 3.2.1. *The labeled triangulation bL of M_g, described above, is metric flag.*

Proof. We need to show that bL, with the described labeling, is the nerve of the Coxeter group W_{bL}, generated by the vertices of bL, with relations

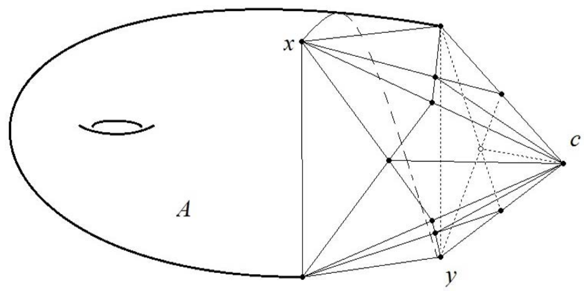

FIGURE 3. The barycentric subdivision of L relative to A.

determined by the labeling of the edges (see equation (2.1)). Since bL is 2-dimensional and since any two vertices generate a finite group if and only if they are connected by an edge, we only need to show that three vertices a, b, and c of bL span a 2-simplex of bL if and only if $\{a, b, c\}$ generate a finite (Coxeter) subgroup of W_{bL}. Where appropriate, we denote by m_{ab}, m_{bc}, and m_{ac} the labels on the edges $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$, respectively. We consider four cases.

Case 1: All three vertices are in A. Then, since A is metric flag, we know $\{a, b, c\}$ spans a 2-simplex if and only if $\{a, b, c\}$ generates a finite Coxeter group.

Case 2: Exactly two vertices are in A. Without loss of generality, take a and b as vertices in A. Since bL is the barycentric subdivision of L, relative to A, if $\{a, b, c\}$ spans a 2-simplex, then c must be the barycenter of a 2-simplex of L, and thus the edge $\{a, b\}$ is in the boundary of the complementary region containing c. Then, with the labeling scheme described above, we know $m_{ac} = m_{bc} = 2$, and so regardless of the label on $\{a, b\}$, $\{a, b, c\}$ generates a finite Coxeter group. Conversely, if $\{a, b, c\}$ generates a finite Coxeter group in W_{bL}, with $a, b \in A$, then a, b, and c must be pairwise connected by edges in bL. But again, since bL is the barycentric subdivision of L, relative to A, we know c is the barycenter of a 2-simplex of L, and thus $\{a, b, c\}$ spans a 2-simplex of bL.

Case 3: Exactly one vertex is in A. Again, without loss of generality, take a to be a vertex in A. If $\{a, b, c\}$ spans a 2-simplex in bL, then b and c are both barycenters of L, one of a 2-simplex, the other of an edge of L. Thus, $m_{ab} = m_{bc} = m_{ac} = 2$, and therefore $\{a, b, c\}$ generates a finite subgroup. Conversely, if $\{a, b, c\}$ generates a finite subgroup, then a, b, and c must be pairwise connected by edges in bL. But, with only a in A, these edges are part of the barycentric subdivision of L relative to A, and so $\{a, b, c\}$ spans a 2-simplex.

Case 4: No vertices are in A. If $\{a, b, c\}$ spans a 2-simplex in bL, then each edge label is 2, and thus $\{a, b, c\}$ generates a finite subgroup. Conversely, if $\{a, b, c\}$ generates a finite subgroup, then just as in the second half of Case 3, it follows that $\{a, b, c\}$ spans a 2-simplex of bL. □

Definition 3.2.2. We say a full subcomplex A of a metric flag simplicial complex L has a *right-angled complement* if the label on all edges not in A is 2.

In this metric flag triangulation, it is clear that A satisfies Definition 3.2.2. Thus, Lemma 3.2.1 and the initial paragraphs of §3.2 provide the proof of the following.

Proposition 3.2.3. *Let A be a connected, metric flag simplicial complex, not a tree, embedded in a surface M_g of minimal genus g. Then there exists a metric flag triangulation L of M_g which preserves the labeling of A and for which A is a full subcomplex with right-angled complement.*

3.3. ℓ^2-HOMOLOGY.

Now let L be a metric flag simplicial complex and let $A \subseteq L$ be a full subcomplex with a right-angled complement. Let B be a full subcomplex of L such that $A \subseteq B$ and let $v \in B - A$ be a vertex. Then B_v, the link of v in B, is a full subcomplex of L with a right-angled complement. Moreover, with $B' = B - v$, $B = B' \cup CB_v$. Indeed, it is clear that B_v has a right-angled complement. Next, to see that B_v is full in L, let T be a subset of vertices contained in B_v and the vertex set of a simplex σ of L. Then T defines a spherical subset of the corresponding Coxeter system, and since $v \notin A$, and since the elements of T are in B_v, v commutes with each vertex of T. Thus, $T \cup \{v\}$ is a spherical subset, and therefore σ is in B_v, and so B_v is a full subcomplex of L. This observation allows us to apply Mayer-Vietoris arguments to subcomplexes of L, decomposing them in terms of right-angled cones and allowing us to use results from §2.3.

Lemma 3.3.1. *Let L be a metric flag triangulation of $\mathbb{S}^1$ and let A be a full subcomplex of L. Then $\beta_i(A) = 0$ for $i > 1$.*

Proof. Consider the long exact sequence of the pair $(\Sigma_L, W\Sigma_A)$:

$$0 \to \mathfrak{h}_2(A) \to \mathfrak{h}_2(L) \to \mathfrak{h}(L, A) \to ...$$

Since Conjecture 1.1 is true in dimension 2, $\mathfrak{h}_2(L) = 0$ and exactness imply the result. □

Lemma 3.3.2. *Suppose that L is a metric flag simplicial complex of a genus g surface. Suppose that A is a full subcomplex with right-angled complement. Then $h_3(L, A) = 0$.*

Proof. Let (L, A) be as in the statement and let $A \subseteq B \subseteq L$. We induct on the number of vertices of $L - B$, the case $L = B$ being trivial. Assume $\mathfrak{h}_i(L, B) = 0$ for $i > k$ and let v be a vertex of $B - A$ and set $B' = A - v$. Then $B = B' \cup CB_v$ and $B' = B - v$, and $\beta_3(B, B') = \beta_3(CB_v, B_v) = \frac{1}{2}\beta_2(B_v)$, the first equality by excision, the second by 2.12. Since L is a surface, B_v is a subcomplex of $\mathbb{S}^1$, and so by Lemma 3.3.1, $\beta_2(B_v) = 0$. Hence, by induction, if we assume the lemma holds for (L, B), then it also holds for (L, B'). □

Then, assuming Conjecture 3.1, we have the following statement on subcomplexes embedded in surfaces.

Proposition 3.3.3. *If a finite, metric flag simplicial complex A, not a tree, can be embedded as a subcomplex in a surface of genus g, then $\beta_2(A) \leq g$.*

Proof. By Proposition 3.2.3, we can assume A is a full subcomplex, with a right-angled complement of some metric flag triangulation L of a genus g. By Lemma 3.3.2, $\mathfrak{h}_3(L, A) = 0$; hence, the map $\mathfrak{h}_2(A) \to \mathfrak{h}_2(L)$ is injective. Since we are assuming $\beta_2(L) = g$, the result follows. □

3.4. ℓ^2-HOMOLOGY AND GRAPHS.

Proposition 3.3.3 gives us the following program for testing the genus of any graph.

Indeed, suppose Γ is a simple, connected graph, not a tree. We consider all possible Coxeter labelings of Γ. Since Γ is not a tree, we know Γ is not a single point nor a single edge, so we know $\beta_0(\Gamma) = 0$ (see §2.3.1), so by equation (2.8), we know that $\chi^{\mathrm{orb}}(\Sigma_\Gamma/W_\Gamma) \leq \beta_2(\Gamma)$. So, using this formula, we have the following corollary of Proposition 3.3.3.

Corollary 3.4.1. *Suppose Γ is a simple, connected graph, not a tree, and suppose Γ embeds in a genus g surface. Then for all Coxeter labelings, $\chi^{orb}(\Sigma_\Gamma/W_\Gamma) \leq g$.*

3.4.2. **CALCULATING χ^{ORB}.** Key to this program is a calculation of the orbihedral Euler characteristic for a graph. So, consider a Coxeter labeling of a graph Γ with V vertices and E edges in which n_e is the label on the edge e. Let Σ_Γ denote the corresponding Davis complex with fundamental domain K_Γ, and consider the simplicial decomposition of K_Γ, in which simplices correspond to linearly ordered (with respect to containment) chains of spherical subsets. Then K_Γ has one 0-simplex with trivial stabilizer, corresponding to the empty set; V 0-simplices with stabilizers of order 2; and, for each edge e, a 0-simplex with a stabilizer of order $2n_e$. K_Γ has $E+V$ 1-simplices with trivial stabilizers, corresponding to chains of the form $\emptyset \subset \{r\}$ or $\emptyset \subset \{r,s\}$, where $r \neq s$ are vertices Γ, and $2E$ 1-simplices with stabilizers of order 2, corresponding to chains of the form $\{r\} \subset \{r,s\}$, where $r \neq s$ are vertices of Γ. Finally, K_Γ has $2E$ 2-simplices with trivial stabilizers corresponding to chains of the form $\emptyset \subset \{r\} \subset \{r,s\}$, where $r \neq s$ are vertices of Γ. So we have that

(3.1)

$$\begin{aligned}\chi^{\mathrm{orb}}(\Sigma_\Gamma/W_\Gamma) &= \left(1 + \frac{V}{2} + \left(\sum_e \frac{1}{n_e}\right)\frac{1}{2}\right) - \left(V + E + \frac{2E}{2}\right) + (2E)\\ &= 1 - \frac{V}{2} + \left(\sum_e \frac{1}{n_e}\right)\frac{1}{2}.\end{aligned}$$

So, with $\gamma(\Gamma)$ denoting the genus of a graph, Corollary 3.4.1 can be restated as follows.

Corollary 3.4.3. *Let Γ be a simple, connected graph, with $V > 2$ vertices. If Γ admits a Coxeter labeling where n_e (an integer ≥ 2) is the label on the edge e with*

$$1 - \frac{V}{2} + \left(\sum_{edges\ e} \frac{1}{n_e}\right)\frac{1}{2} > g,$$

for some non-negative integer g, then $\gamma(\Gamma) > g$.

In [15], the author proves Conjecture 3.1 is true for $g = 0$. So, the above does give an ℓ^2-homological test for planar graphs. See [17] for a complete treatment of planar graphs.

3.5. Complete graphs.

Now consider a complete graph on n vertices; we denote this graph by Γ_n. A uniform labeling of 3 on each edge is a Coxeter labeling. So, in this case

$$\chi^{\mathrm{orb}}(\Sigma_{\Gamma_n}/W_{\Gamma_n}) = 1 - \frac{n}{2} + \left(\frac{n(n-1)}{6}\right)\frac{1}{2} = \frac{(n-4)(n-3)}{12}.$$

That is, by Corollary 3.4.3,

$$\gamma(\Gamma_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil . \tag{3.2}$$

Now let $\Gamma_{m,n}$ denote the complete bipartite graph on $m+n$ vertices. Since $\Gamma_{m,n}$ does not contain 3-cycles, a uniform labeling of 2 results in a metric flag complex. In this case, we have

$$\chi^{\text{orb}}(\Sigma_{\Gamma_{m,n}}/W_{\Gamma_{m,n}}) = 1 - \frac{m+n}{2} + \left(\frac{m \cdot n}{2}\right)\frac{1}{2} = \frac{(m-2)(n-2)}{4}.$$

So, again by Corollary 3.4.3,

$$\gamma(\Gamma_{m,n}) \geq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil. \tag{3.3}$$

Of course, the classical (non-ℓ^2-homological) methods give exact calculations for the genus of complete and complete bipartite graphs. That is, that

$$\gamma(\Gamma_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \text{and} \quad \gamma(\Gamma_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$$

([13], [12] and [8, p. 118–119]). While it is interesting that the above ℓ^2-homological methods imply (part) of the same, it is also clear that the strength of this method does not lie in these classical cases, but rather in its versatility. Indeed, given any graph, one can determine a Coxeter labeling and calculate $\chi^{\text{orb}}(\Sigma_\Gamma/W_\Gamma)$ to find a lower bound for the genus of the graph, as we will see below in Example 3.6.

Note that in equation (3.1), increasing any one edge label of Γ decreases χ^{orb}, so in case Γ contains no 3-cycles, it is clear that a labeling of 2's on each edge will give you the largest possible orbihedral Euler characteristic. It is also the case that uniform labeling by 3's can always produce a Coxeter labeling of a graph. But there are, in general, many labelings that result in metric flag graphs, and in case there are 3-cycles, it is possible that non-uniform labelings produce a larger Euler characteristic than a uniform labeling with 3.

Example 3.6. Let Γ be the graph pictured in Figure 4, a member of the Petersen family of graphs, where $V = 8$ and $E = 15$. So a labeling by 3's gives $\chi^{\text{orb}}(\Sigma_\Gamma/W_\Gamma) = -\frac{1}{2}$ and does not detect that Γ has genus ≥ 1. However, with the indicated Coxeter labeling,

$$\chi^{\text{orb}}(\Sigma_\Gamma/W_\Gamma) = \frac{1}{4}.$$

Hence, by Corollary 3.4.3, we can conclude that $\gamma(\Gamma) \geq 1$. Note: In Figure 4, we demonstrate an embedding for this graph in a genus 1 surface, so we do know that $\gamma(\Gamma) = 1$.

4. Furthering the Singer Program

Since the terminology and notation are developed, we now take a moment to further the program for Singer's Conjecture for Coxeter groups. We begin with the following, the main result of [17].

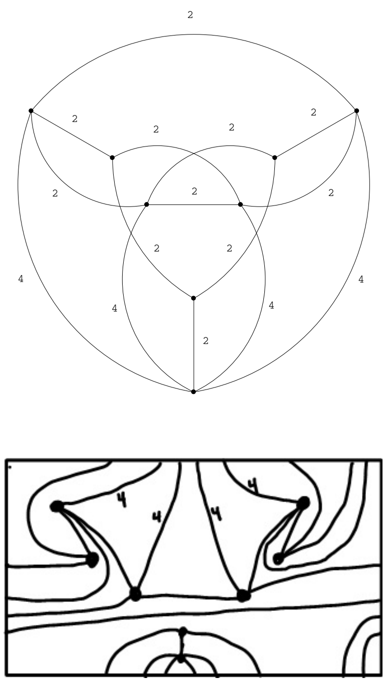

FIGURE 4. A Petersen graph with Coxeter labeling and embedded in genus 1 surface.

Theorem 4.1. *Let L be a metric flag triangulation of $\mathbb{S}^2$ and $A \subseteq L$ a full subcomplex with right-angled complement. Then*

$$\beta_i(A) = 0 \text{ for } i > 1.$$

Proof. Let B be a full subcomplex of L such that $A \subseteq B \subseteq L$. We induct on the number of vertices of $L - B$, the case $L = B$ cited here in 2.3.2. Assume $\mathfrak{h}_i(B) = 0$ for $i > 1$. Let v be a vertex of $B - A$ and set $B' = B - v$. Then $B = B' \cup CB_v$ where B_v and B' are full subcomplexes, the former by the first paragraph of §3.3. We have the following Mayer–Vietoris sequence:

$$\ldots \to \mathfrak{h}_i(B_v) \to \mathfrak{h}_i(B') \oplus \mathfrak{h}_i(CB_v) \to \mathfrak{h}_i(B) \to \ldots .$$

B_v is a full subcomplex of L_v, the link of v in L, a metric flag triangulation of $\mathbb{S}^1$. So Lemma 3.3.1 implies $\mathfrak{h}_i(B_v) = 0$, for $i > 1$. Thus, by equation (2.10), $\mathfrak{h}_i(CB_v) = 0$ for $i > 1$. It follows from exactness that $\mathfrak{h}_i(B') = 0$. $\square$

The statements in Lemma 3.3.1 and Theorem 4.1 are specific versions of the following statement, which is a variation of $\mathbf{V}(n)$ as found in [5, §8].

$\mathbf{V}'(n)$**.** *Suppose L is any metric flag triangulation of $\mathbb{S}^{n-1}$ and that A is a full subcomplex with a right-angled complement.*

- *If $n = 2k$ is even, then $\mathfrak{h}_i(L, A) = 0$ for all $i > k$.*
- *If $n = 2k + 1$ is odd, then $\mathfrak{h}_i(A) = 0$ for all $i > k$.*

Compare Theorem 4.2 with [5, Lemma 9.2.1]. As we point out, the proof given in [5] generalizes from the right-angled case to the case where the subcomplexes simply have a right-angled complement. Of course, with this hypothesis, we are unable to extend $\mathbf{V}'(2k)$ to the generic statement of Conjecture 1.1 as in [5], but we do include the following as furthering the program on Singer's Conjecture for Coxeter systems.

Theorem 4.2. *$V'(2k-1) \Longrightarrow V'(2k)$.*

Proof. Let (L, A) be as in $V'(2k)$ and let $A \subseteq B \subseteq S$. As in the proof of Theorem 4.1, we induct on the number of vertices of $L - B$, the case $L = B$ being trivial. Assume $\mathfrak{h}_i(L, B) = 0$ for $i > k$ and let v be a vertex of $B - A$ and set $B' = A - v$. Then $B = B' \cup CB_v$, and we consider the exact sequence of the triple (L, B, B'):

$$\ldots \to \mathfrak{h}_i(B, B') \to \mathfrak{h}_i(L, B') \to \mathfrak{h}_i(L, B) \to \ldots .$$

$\mathfrak{h}_(L, B) = 0$ by induction. By excision, $\beta_i(B, B') = \beta_i(CB_v, B_v)$, and by equation (2.12), $\beta_i(CB_v, B_v) = \frac{1}{2}\beta_{i-1}(B_v)$. B_v is a full subcomplex of L_v with a right-angled complement, and since $i - 1 > k - 1$, $\mathbf{V}'(2k-1)$ gives us that $\beta_{i-1}(B_v) = 0$. Thus, $\beta_i(B, B') = 0$ for $i > k$, and the proof follows from exactness. □

See [16] for more information on furthering the program to prove Singer's Conjecture for Coxeter groups (Conjecture 1.1).

5. Examples of the Generalized Singer Conjecture for Surfaces

Let Γ_n denote the complete graph on n vertices and let $\Gamma_{m,n}$ denote the complete bipartite graph on $m + n$ vertices. As mentioned above, it is known (independent of the given ℓ^2-homological methods) that

(5.1)
$$\gamma(\Gamma_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \text{and} \quad \gamma(\Gamma_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

Using these and ℓ^2-homological results noted in §§5.1 and 5.2, we demonstrate examples of triangulations of higher genus surfaces for which Conjecture 1.2 is true.

5.1. **Complete, bipartite graphs.**

Let P_k denote k disjoint points. Σ_{P_k} is a k-valent tree, and then, using the results from 2.3.1 and equation (2.8), we have that

$$\beta_i(L_m) = \begin{cases} \frac{m}{2} - 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases} \tag{5.2}$$

$K_{m,n}$ with each edge labeled 2 is metric flag and is the right-angled join $P_m *_2 P_n$. So by Künneth formula, equation (2.9), we have that

$$\beta_i(\Gamma_{m,n}) = \begin{cases} \left(\frac{m}{2} - 1\right)\left(\frac{n}{2} - 1\right) = \frac{(m-2)(n-2)}{4} & \text{if } i = 2 \\ 0 & \text{if } i \neq 2. \end{cases} \tag{5.3}$$

Note that the ℓ^2-Betti number calculations above are exactly the right-hand side of the "classical" genus calculations listed in equation (5.1).

5.2. **Complete graphs.**

Complete graphs are not themselves flag complexes, but labeling each edge with 3 does result in a metric-flag graph. Then, by equation (3.1), we have that $\chi^{\text{orb}}(\Sigma_{\Gamma_n}/W_{\Gamma_n}) = \frac{(n-4)(n-3)}{12}$. We also know that if Γ_n has at least two vertices, then $\beta_0(\Gamma_n) = 0$ (see §2.3.1). Thus, by equation (2.8), we know that $\chi^{\text{orb}}(\Sigma_{\Gamma_n}/W_{\Gamma_n}) \leq \beta_2(\Gamma_n)$. But Wiktor J. Mogilski [11] shows that, in fact, $\beta_1(\Gamma_n) = 0$ as well. That is, Mogilski shows

$$\beta_i(\Gamma_n) = \begin{cases} 1 - \frac{n}{2} + \frac{n(n-1)}{12} = \frac{(n-3)(n-4)}{12} & \text{if } i = 2 \\ 0 & \text{if } i \neq 2. \end{cases} \tag{5.4}$$

Note that the corresponding Davis complex is 2-dimensional, so it is automatic that $\beta_i(\Gamma_n) = 0$ for $i \geq 3$.

5.3. **Euclidean circuits.**

Let A denote a triangle, with each edge labeled 3 or a 4-gon, with each edge labeled with 2 and with opposite vertices not connected by an edge. In either case, we call A a Euclidean circuit (as in [15]), for $\Sigma_A \cong \mathbb{E}^2$ and we know that $\mathfrak{h}_i(A) = 0$ for all i [2]. These circuits will form the boundary of complementary regions in embeddings of Γ_n or $\Gamma_{m,n}$ in surfaces for specific m and n. As in the paragraph preceding Lemma 3.2.1, we will cone off these complementary regions, labeling each cone edge with 2. Then by equation (2.10), we know $\beta_i(CA) = 0$ for all i.

5.4. Embeddings of Γ_n and $\Gamma_{m,n}$.

We now consider examples in which the formulas in equation (5.1) above are exactly integers, for example, if $n = 7$ in the complete graph case or if $m = n = 4$ in the complete bipartite graph case. We will see that in these cases, the graphs contain all the ℓ^2-homological information and can be embedded in such a way as to generate examples of Conjecture 1.2.

First, suppose Γ_n is a complete graph for which $(n-3)(n-4) \equiv 0 \pmod{12}$. Then Γ_n embeds in a genus $\left(\frac{(n-3)(n-4)}{12}\right)$ surface. Γ_n has $(n(n-1))/2 = E$ edges and let F denote the number of complementary regions in the surface. In the embedding, each edge is part of the boundary of two of the F complementary regions or has, on each side, the same complementary region. So, for this embedding, we have the Euler characteristic calculation:

$$n - \frac{n(n-1)}{2} + F = 2 - \frac{(n-3)(n-4)}{6} \Longrightarrow F = \frac{n(n-1)}{3},$$

which implies that $3F = 2E$. So, we know that each complementary region is a triangle. Now, take a uniform labeling of 3 on every edge, add a vertex in each complementary region, and cone off on each triangle, labeling each cone edge with 2. We then have Γ_n embedded as a full subcomplex of a triangulation of a genus g surface, where $g = ((n-3)(n-4))/12$. For an example, see Figure 5.

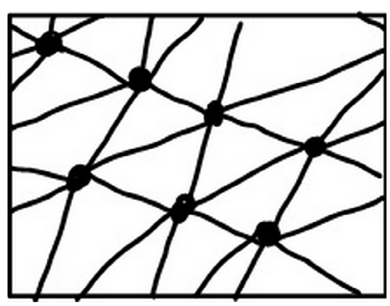

Figure 5. K_7 embedded in a genus 1 surface with 14 complementary regions, all 3-gons.

Similarly, consider a complete bipartite graph $\Gamma_{m,n}$ where either m and n are both even or $m \equiv 2 \bmod 4$ and n is odd. In either case, the calculation above gives that $K_{m,n}$ embeds in a surface of genus g where $g = \frac{(m-2)(n-2)}{4}$. Take such an embedding and let F denote the number of complementary regions and note that the number of edges E is (mn). We then have that

$$(m+n)-(mn)+F = 2-2\left(\frac{(m-2)(n-2)}{4}\right).$$

This implies that $4F = 2(mn)$. Then again, since each edge is used twice as a part of a boundary of one of the F complementary regions and each complementary region is at least a 4-gon, we know that each complementary region is a 4-gon. As above, add a vertex in each complementary region and cone off each 4-gon. Label each edge with 2 and we obtain a metric flag triangulation of a surface for which $\Gamma_{m,n}$ is a full subcomplex. For an example, see Figure 6.

We now consider such triangulations of genus g surfaces. That is, we consider triangulations L that contain either Γ_n, where $(n-3)(n-4) \equiv 0(\text{mod}12)$, or $\Gamma_{m,n}$, where m and n are both even or $m \equiv 2 \pmod 4$ and n is odd, as full subcomplexes and for which the complements of these graphs are collections of cones on Euclidean circuits. We have the following result.

Proposition 5.4.1. *Let L denote a metric flag triangulation of a genus g surface as described above. Then $\beta_i(L) = 0$ for $i \neq 2$ and $\beta_2(L) = g$.*

Proof. L contains, as a full subcomplex, a graph Γ where either Γ is a complete graph on n vertices for which $(n-3)(n-4)/12 = g$ or Γ is a complete bipartite graph on $m+n$ vertices for which $(m-2)(n-2)/4 = g$. Let R_k, $k = 1, \ldots, F$, denote the boundaries of the complementary regions of Γ in L. By the above, each R_k is either a 3-gon with edges labeled with 3 or a 4-gon, with edges labeled with 2, and we have that $L = \Gamma \cup CR_1 \cup CR_2 \cup \ldots \cup CR_F$. For $j = 0, 1, \ldots F$, let $L_i = \Gamma \cup CR_{i+1} \cup \ldots \cup CR_F$. Then $L_0 = L$ and $L_F = \Gamma$. By Mayer–Vietoris, we have

$$\ldots \to \mathfrak{h}_i(R_1) \to \mathfrak{h}_i(L_1) \oplus \mathfrak{h}_i(CR_1) \to \mathfrak{h}_i(L) \to \ldots.$$

Since $\mathfrak{h}_i(R_1) = 0$ and $\mathfrak{h}_i(CR_1) = 0$ for all i (see §5.3), we have that $\beta_i(L_1) = \beta_i(L)$. Using Mayer–Vietoris again with $L_1 = CR_2 \cup L_2$, we have $\beta_i(L_1) = \beta_i(L_2)$. Proceeding in this way, we get that $\beta_i(L) = \beta_i(L_1) = \ldots = \beta_i(L_F) = \beta_i(\Gamma)$ for all i. Finally, applying equation (5.3), in the case that Γ is a complete bipartite graph, or equation (5.4) in the case that Γ is a complete graph, we have that $\beta_i(\Gamma) = 0$ if $i \neq 2$ and $\beta_2(\Gamma) = g$. $\square$

5.5. The limitations of our examples.

It is interesting to note why the above Mayer–Vietoris method cannot be applied to any complete graph: It requires that the boundaries of the complementary regions are themselves full subcomplexes. Indeed, Figure

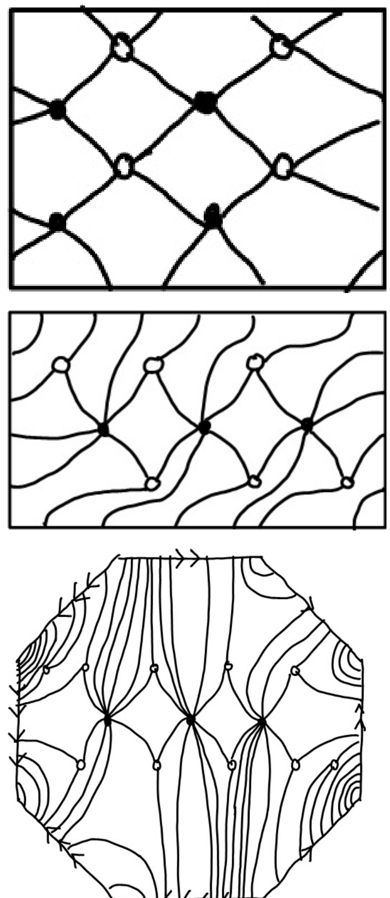

FIGURE 6. $K_{4,4}$, $K_{3,6}$, and $K_{3,10}$ embedded in surfaces with complementary regions all 4-gons.

7 contains an embedding of $\Gamma_{3,3}$ in a genus 1 surface. Labeling each edge with 2 defines a Coxeter labeling for $\Gamma_{3,3}$, and the method described in §3 does accurately bound the genus of $\Gamma_{3,3}$ above 0, and the given embedding shows the genus of $\Gamma_{3,3}$ is 1; but the methodology of §5 does not produce examples of Conjecture 1.2. Indeed, the boundaries of the complementary regions are two full 4-gons and one "10-gon." (Two edges are used twice and four vertices are used multiple times, but a complete circuit along the boundary of this region is a path of length 10.) Note that there are edges not in the boundary of this region that connect vertices. Thus, the

"10-gon" is not a full subcomplex, and the Mayer-Vietoris method does not provide proof of an example of Conjecture 1.2.

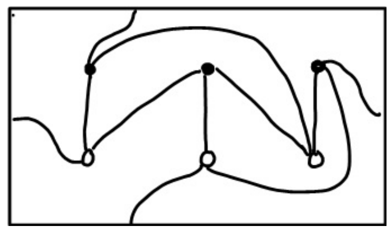

FIGURE 7. $K_{3,3}$ embedded in a genus one surface.

For another example, the embedding in Figure 4 does not fit the methodology of §5 because one complementary region is a 6-gon in which vertices are connected by an edge not contained in the 6-gon; thus, it is not a full subcomplex.

REFERENCES

[1] Michael W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983), no. 2, 293–324.

[2] ———, *The Geometry and Topology of Coxeter Groups*. London Mathematical Society Monographs Series, 32. Princeton, NJ: Princeton University Press, 2008.

[3] Michael W. Davis, Jan Dymara, Tadeusz Januszkiewicz, and Boris Okun, *Weighted L^2-cohomology of Coxeter groups*, Geom. Topol. **11** (2007), 47–138.

[4] Michael W. Davis and Gábor Moussong, *Notes on nonpositively curved polyhedra* in Low Dimensional Topology (Eger, 1996/Budapest, 1998). Ed. Károly Böröczky, Jr., Walter Neumann, and András Stipsicz. Bolyai Soc. Math. Stud., 8. Budapest: János Bolyai Math. Soc., 1999. 11–94. (Available at https://people.math.osu.edu/davis.12/notes.pdf).

[5] Michael W. Davis and Boris Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geom. Topol. **5** (2001), 7–74.

[6] Warren Dicks and Ian J. Leary, *On subgroups of Coxeter groups* in Geometry and Cohomology in Group Theory (Durham, 1994). Ed. Peter H. Kropholler, Graham A. Niblo, and Ralph Stöhr. London Math. Soc. Lecture Note Ser., 252. Cambridge: Cambridge Univ. Press, 1998. 124–160.

[7] Beno Eckmann, *Introduction to ℓ^2*-methods in topology: Reduced ℓ^2-homology, harmonic chains, ℓ^2-Betti numbers. Israel J. Math. **117** (2000), 183–219.

[8] Frank Harary, *Graph Theory*. Reading, Mass.: Addison-Wesley Publishing Co., 1969.

[9] James E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29. Cambridge: Cambridge University Press, 1990.

[10] C. R. F. Maunder, *Algebraic Topology.* London, New York: Van Nostrand Reinhold Co., 1970. C.R.F. Maunder, *Algebraic topology*, Van Nostrand Reinhold, 1970.

[11] Wiktor J. Mogilski, *The fattened Davis complex and weighted L2-(co)homology of Coxeter groups.* Unpublished. 2013.

[12] Gerhard Ringel, *Das geschlecht des vollständigen paaren graphen*, Abh. Math. Sem. Univ. Hamburg **28** (1965), 139–150.

[13] Gerhard Ringel and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 438–445.

[14] Timothy A. Schroeder, *The ℓ^2-homology of even Coxeter groups*, Algebr. Geom. Topol. **9** (2009), no. 2, 1089–1104.

[15] ———, *Geometrization of 3-dimensional Coxeter orbifolds and Singer's conjecture*, Geom. Dedicata **140** (2009), 163–174.

[16] ———, *Statements and dilemmas regarding the ℓ^2-homology of Coxeter groups.* Preprint. Available at arXiv:1212.4215 [math.GT].

[17] ———, *ℓ^2-homology and planar graphs*, Colloq. Math. **131** (2013), no. 1, 129–139.

Department of Mathematics and Statistics; Murray State University; Murray, KY 42071

E-mail address: tschroeder@murraystate.edu