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YINHE PENG

Abstract. We will first construct L spaces with the following properties: an L space whose square contains an uncountable closed discrete subset (in particular, it is not Lindelöf); an L space whose square is the closure of a countable union of closed discrete subsets. Then we will prove that Moore's original L space has a non-Lindelöf square, answering a problem of Boaz Tsaban and Lyubomur Zdomskyy.

Justin Tatch Moore in [3] constructed an L space and showed that its square is not hereditarily Lindelöf. Then a natural question is whether it is Lindelöf or even whether there exists an L space with non-Lindelöf square. It was first appeared in [6] where Marion Scheepers and Franklin D. Tall constructed an example by assuming the non-Lindelöf square property. Then Boaz Tsaban and Lyubomur Zdomskyy [9] proved that some finite power of Moore's L space is non-Lindelöf which would guarantee the existence of the example constructed in [6]. They also asked whether the square of Moore's L space is non-Lindelöf. Section 2 will give a positive answer. Also, the proof of Theorem 2.6 answers some questions in [5].

1. An L Space with Non-Lindelöf Square

Two theorems of this section are both built on the technology of minimal walk (see [8]). But instead of going into details on minimal walk, we will just introduce some combinatorial facts generated from it.

Theorem 1.1. *There is an L space such that every uncountable subspace has a non-Lindelöf square.*

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The L space we need here is the same as the one defined in [3] except for a slight change. First, let us fix $\{z_\alpha : \alpha < \omega_1\} \subset \mathbb{T}$ [1] that are rationally independent and introduce some notation.

Definition 1.2. (1) A set $\{w_\beta \in \mathbb{T}^{\omega_1} : \beta < \omega_1\}$ is said to be induced from a map $a : [\omega_1]^2 \to \omega$ if $w_\beta(\alpha) = z_\alpha^{a(\alpha,\beta)+1}$, for $\alpha < \beta$, $w_\beta(\beta) = z_\beta^{1/2}$, and $w_\beta(\alpha) = 1$ for $\alpha > \beta$.[2]

(2) For any space $\mathscr{L} = \{w_\beta \in \mathbb{T}^{\omega_1} : \beta < \omega_1\} \subset \mathbb{T}^{\omega_1}$ and any subset $X \subset \omega_1$, the restriction subspace is $\mathscr{L} \restriction_X = \{w_\beta \in \mathscr{L} : \beta \in X\}$.

For $a = (\alpha, \beta) \in {\omega_1}^2$, $w_a = (w_\alpha, w_\beta)$, and for $D \subset X^2$, $w_D = \{w_a : a \in D\}$.

(3) For two sets of ordinals a and b, say $a < b$ if any ordinal in a is less than any ordinal in b.

We recall some facts from [3, Corollary 7.11, Proposition 7.13] and [4, Lemma 2].

Proposition 1.3 ([3], [4]). *There is an oscillation map $osc : [\omega_1]^2 \to \omega$ with the following properties:*

- (i) *$T(osc) = \{osc(\cdot, \beta) \restriction_\alpha \in \omega^{<\omega_1} : \alpha \le \beta < \omega_1\}$ is an Aronszajn tree (where the tree order is the function extension);*
- (ii) *the space $\mathscr{L} = \{w_\beta \in \mathbb{T}^{\omega_1} : \beta < \omega_1\}$ induced from osc is an L space. In particular, $\mathscr{L} \restriction_X$ is hereditarily Lindelöf (and hence an L space);*
- (iii) *for every $\alpha \le \beta$, there is an integer $n(\alpha, \beta)$ such that $|osc(\xi, \alpha) - osc(\xi, \beta)| < n(\alpha, \beta)$ for any $\xi < \alpha$.*

Now we fix $\mathscr{L} = \{w_\beta \in \mathbb{T}^{\omega_1} : \beta < \omega_1\}$ to be the space induced from the map *osc* guaranteed by Proposition 1.3. Then we fix an uncountable $X \subset \omega_1$ and want to show that $\mathscr{L}^2 \restriction_{X^2} = \mathscr{L} \restriction_X \times \mathscr{L} \restriction_X$ is not Lindelöf. Actually, we are going to show that $\mathscr{L}^2 \restriction_{X^2}$ contains an uncountable closed discrete set.

First, let us pick a $D = \{a_\alpha = (\mu_\alpha, \nu_\alpha) \in X^2 : \mu_\alpha < \nu_\alpha \text{ and } \alpha < \omega_1\}$ such that for any $\alpha < \beta < \omega_1$, $\nu_\alpha < \mu_\beta$. Fix a natural number $n(\mu_\alpha, \nu_\alpha)$ for each $a_\alpha \in D$ guaranteed by property (iii) of Proposition 1.3. Let $D_n = \{a_\alpha \in D : n(\mu_\alpha, \nu_\alpha) = n\}$, then $D = \bigcup_{n<\omega} D_n$.

Lemma 1.4. *For each $n < \omega$ and for any $a \in X^2 \setminus \{(\alpha, \alpha) : \alpha \in X\}$, there is a neighborhood of w_a that is disjoint from $w_{D_n} \setminus \{w_a\}$. In particular, w_{D_n} is discrete.*

[1] Here $\mathbb{T}$ is the unit circle in $\mathbb{C}$ – complex numbers.

[2] Note here the definition of $w_\beta(\beta)$ is different from that in [3], and this difference is important to achieve our goal.

Proof. Fix $n < \omega$ and $a \in X^2 \setminus \{(\alpha, \alpha) : \alpha \in X\}$ and assume $\eta = min(a) < \tau = max(a)$. As there are at most finitely many $w_{a_\alpha} \in w_{D_n} \setminus \{w_a\}$ such that η or τ is a member of a_α, we can find a neighborhood O_1 of w_a disjoint from these w_{a_α}'s. Also,

$$O_2 = \{(x, y) : |x(\eta) - y(\eta)| \neq |z_\eta^i - 1| \text{ for any natural number } i < n\}$$

is an open neighborhood of w_a. Now define

$$O = O_1 \cap O_2 \cap \{(x, y) : x(\eta) \neq 1\}$$

and we will show this neighborhood of w_a works. Now pick any $w_{a_\alpha} \in w_{D_n} \setminus \{w_a\}$.

Case 1: η or τ is a member of a_α.

Then $w_{a_\alpha} \notin O_1 \supset O$.

Case 2: $\eta > \mu_\alpha$.

$w_{\mu_\alpha}(\eta) = 1$. So $w_{a_\alpha} \notin O$.

Case 3:: $\eta < \mu_\alpha$.

$|w_{\mu_\alpha}(\eta) - w_{\nu_\alpha}(\eta)| = |z_\eta^{osc(\eta,\mu_\alpha)-osc(\eta,\nu_\alpha)} - 1|$ where $|osc(\eta, \mu_\alpha) - osc(\eta, \nu_\alpha)| < n(\mu_\alpha, \nu_\alpha) = n$. So $w_{a_\alpha} \notin O_2 \supset O$.

This completes the proof of the lemma. □

We have proved that $\mathscr{L}^2 \restriction_{X^2} \setminus (w_{D_n} \cup \Delta)$ [3] is disjoint from $\overline{w_{D_n}}$ – the closure of w_{D_n}. So if Δ is disjoint from $\overline{w_{D_n}}$, then w_{D_n} is closed (and discrete by Lemma 1.4). From now on we shall additionally assume that $D = \{a_\alpha \in X^2 : \alpha < \omega_1\}$ has the following property:

(∗) There are an ordinal ξ and natural numbers $i \neq j$ such that for any $a_\alpha \in D$, $\xi < \mu_\alpha$, $osc(\xi, \mu_\alpha) = i$ and $osc(\xi, \nu_\alpha) = j$.

This can be done since $T(osc)$ is an Aronszajn tree (see property (i) of Proposition 1.3).

Proof of Theorem 1.1. Fix X, pick $D = \{a_\alpha = (\mu_\alpha, \nu_\alpha) : \alpha < \omega\}$ as before but with additional property (∗), and pick n such that D_n is uncountable. $O = \{(x, y) : x(\xi) \neq z_\xi^{i+1} \text{ or } y(\xi) \neq z_\xi^{j+1}\}$ is an open set containing Δ but disjoint from w_D, and therefore w_{D_n}. This, combined with Lemma 1.4, suffices to prove that w_{D_n} is closed and discrete in $\mathscr{L}^2 \restriction_{X^2}$. So $\mathscr{L}^2 \restriction_{X^2}$ is not Lindelöf. □

An easy corollary follows from the proof.

Corollary 1.5. *For any uncountable $X \subset \omega_1$, $\mathscr{L}^2 \restriction_{X^2}$ contains an uncountable closed discrete subset.*

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[3] $\Delta = \{(w_\alpha, w_\alpha) : \alpha \in X\}$.

We have proved that D is a countable union of closed discrete subsets. But this D is not dense by property $(*)$. Now we turn to find a dense one.

Theorem 1.6. *There is an L space such that its square is the closure of a countable union of closed discrete subsets.*

Before we start, let us introduce some definitions.

Definition 1.7. For any $\alpha < \omega_1$ and $r \in \mathbb{R}$, $w^r_\alpha(\xi) = z_\xi^{osc(\xi,\alpha)+r}$ [4] for $\xi < \alpha$, $w^r_\alpha(\alpha) = z^\pi_\alpha$,[5] and $w^r_\alpha(\xi) = 1$ for $\xi > \alpha$.

Now we will use Kronecker's Theorem to construct the required L space.[6]

Kronecker's Theorem ([2]). *Let A be a real $m \times n$ matrix and assume that $\{\boldsymbol{z} \in \mathbb{Q}^m : A^T\boldsymbol{z} \in \mathbb{Q}^n\} = \{\boldsymbol{0}\}$. Then for any $\epsilon > 0$ and for any $b_0, ..., b_{m-1} \in \mathbb{R}$, there exist $p_0, ..., p_{m-1} \in \mathbb{Z}$ and $\boldsymbol{q} \in \mathbb{Z}^n$ such that $|A_i\boldsymbol{q} - p_i - b_i| < \epsilon$ for all $i < m$ where A_i is the ith row of A.*

Now let $\{O_\alpha \times U_\alpha : \alpha < \omega_1\}$ be a base for $\mathbb{T}^{\omega_1} \times \mathbb{T}^{\omega_1}$. Use Kronecker's Theorem to inductively construct an increasing sequence $(\mu_\alpha, \nu_\alpha) \in \omega_1^2$ and $(q_{\mu_\alpha}, q_{\nu_\alpha}) \in \mathbb{Q}^2$ such that $(w^{q_{\mu_\alpha}}_{\mu_\alpha}, w^{q_{\nu_\alpha}}_{\nu_\alpha}) \in O_\alpha \times U_\alpha$.

At stage α, assume $O_\alpha \supset \{x \in \mathbb{T}^{\omega_1} : |x(\xi_i) - b_i| < \epsilon \text{ for all } i < m\}$ for some $\epsilon > 0$, $m < \omega$, $\xi_i \in \omega_1$, and $b_i \in \mathbb{T}$ $(i < m)$. Fix a large enough μ_α (greater than ξ_i and ν_β constructed before) and choose a $q_{\mu_\alpha} \in \mathbb{Q}$ guaranteed by Kronecker's Theorem such that $w^{q_{\mu_\alpha}}_{\mu_\alpha} \in O_\alpha$. Then choose $\nu_\alpha > \mu_\alpha$ and q_{ν_α} similarly.

Moreover, if we consider $w^q_{\mu_\alpha}$ (or $w^q_{\nu_\alpha}$) as a function from $\mathbb{Q}$ to $\mathbb{T}^{\omega_1}$ with variable q, then it is continuous. So we have enough choice and can assume additionally that $q_{\mu_\alpha} - q_{\nu_\alpha} \notin \mathbb{Z}$.

Let $X = \{\mu_\alpha, \nu_\alpha : \alpha \in \omega_1\}$ and $\mathscr{L}\restriction_X = \{w^{q_\beta}_\beta : \beta \in X\}$. Denote $D = \{(\mu_\alpha, \nu_\alpha) : \alpha \in \omega_1\}$ and for each $n < \omega$, $q \in \mathbb{Q} \setminus \mathbb{Z}$ and $D_{n,q} = \{(\mu_\alpha, \nu_\alpha) : n(\mu_\alpha, \nu_\alpha) = n \text{ and } q_{\mu_\alpha} - q_{\nu_\alpha} = q\}$.

Lemma 1.8. *$\mathscr{L}\restriction_X$ is an L space.*

Proof. Nonseparable is trivial. If $\mathscr{L}\restriction_X$ is not hereditarily Lindelöf, then there are an uncountable $Y \subset X$ and a rational q such that $\mathscr{L}\restriction_Y$ is not Lindelöf and $q_\beta = q$ for every $\beta \in Y$. The proof of Corollary 7.11 in [3] also shows that $\mathscr{L}\restriction_Y$ is Lindelöf. A contradiction. □

Since $w_D = \{(w^{q_{\mu_\alpha}}_{\mu_\alpha}, w^{q_{\nu_\alpha}}_{\nu_\alpha}) : \alpha \in \omega_1\}$ is dense in $\mathbb{T}^{\omega_1} \times \mathbb{T}^{\omega_1}$, Theorem 1.6 follows from the following lemma.

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[4] Here we assume $(e^{i\theta})^r = e^{i\theta r}$.

[5] π can be replaced by any irrational number.

[6] A proof can be found in [1].

Lemma 1.9. *$w_{D_{n,q}}$ is closed discrete in $\mathscr{L}^2 \restriction_{X^2}$ for each $n \in \omega$, $q \in \mathbb{Q} \setminus \mathbb{Z}$.*

Proof. Fix such n and q. It suffices to prove that for any $a \in X^2$, there is a neighborhood O of $w_a^{q_a} = (w_{a(0)}^{q_{a(0)}}, w_{a(1)}^{q_{a(1)}})$ disjoint from $w_{D_{n,q}} \setminus \{w_a^{q_a}\}$. So fix $a \in X^2$ and assume $\eta = min(a)$ and $\tau = max(a)$. As there are at most finitely many $(w_{\mu_\alpha}^{q_{\mu_\alpha}}, w_{\nu_\alpha}^{q_{\nu_\alpha}}) \in w_{D_{n,q}} \setminus \{w_a^{q_a}\}$ such that η or τ is in $\{\mu_\alpha, \nu_\alpha\}$, we can find a neighborhood O_1 of $w_a^{q_a}$ disjoint from them. Recall that $q \in \mathbb{Q} \setminus \mathbb{Z}$. So

$O_2 = \{(x, y) : |x(\tau) - y(\tau)| \neq |z_\tau^{i+q} - 1|$ for any integer i such that $|i| < n\}$

is a neighborhood of $w_a^{q_a}$. Now define

$$O = O_1 \cap O_2 \cap \{(x, y) : \ x(\tau) \neq 1 \text{ or } y(\tau) \neq 1\}.$$

We will show this neighborhood of $w_a^{q_a}$ works. Now pick any $(w_{\mu_\alpha}^{q_{\mu_\alpha}}, w_{\nu_\alpha}^{q_{\nu_\alpha}}) \in w_{D_{n,q}} \setminus \{w_a^{q_a}\}$. By definitions of X and D, there is no $\xi \in X$ such that $\mu_\alpha < \xi < \nu_\alpha$. So $\tau < \nu_\alpha$ if and only if $\tau \leq \mu_\alpha$.

Case 1: Either η or τ is a member of $\{\mu_\alpha, \nu_\alpha\}$.
Then $(w_{\mu_\alpha}^{q_{\mu_\alpha}}, w_{\nu_\alpha}^{q_{\nu_\alpha}}) \notin O_1$.

Case 2: $\nu_\alpha < \tau$.
$w_{\mu_\alpha}^{q_{\mu_\alpha}}(\tau) = w_{\nu_\alpha}^{q_{\nu_\alpha}}(\tau) = 1$, so $(w_{\mu_\alpha}^{q_{\mu_\alpha}}, w_{\nu_\alpha}^{q_{\nu_\alpha}}) \notin O$.

Case 3: $\nu_\alpha > \tau$ and $\tau \neq \mu_\alpha$ (so $\tau < \mu_\alpha$).
$|w_{\mu_\alpha}^{q_{\mu_\alpha}}(\tau) - w_{\nu_\alpha}^{q_{\nu_\alpha}}(\tau)| = |z_\tau^{osc(\tau,\mu_\alpha) - osc(\tau,\nu_\alpha) + q} - 1|$. Since $|osc(\tau, \mu_\alpha) - osc(\tau, \nu_\alpha)| < n(\mu_\alpha, \nu_\alpha) = n$, $(w_{\mu_\alpha}^{q_{\mu_\alpha}}, w_{\nu_\alpha}^{q_{\nu_\alpha}}) \notin O_2$.

So O is a neighborhood of $w_a^{q_a}$ disjoint from $w_{D_{n,q}} \setminus \{w_a^{q_a}\}$. □

2. Back to Moore's Original L Space

Now we need to go back to the definition of *osc* to get more information. To introduce *osc*, we need a few definitions and facts about minimal walk (see [8] and [3] for more information).

Definition 2.1. (1) A *C-sequence* is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that $C_{\alpha+1} = \{\alpha\}$ and C_α is a cofinal subset of α of order type ω for limit α's.

(2) For a C-sequence, the *maximal weight* of the walk is the function $\rho_1 : [\omega_1]^2 \to \omega$, defined recursively by $\rho_1(\alpha, \beta) = max\{|C_\beta \cap \alpha|, \rho_1(\alpha, min(C_\beta \setminus \alpha))\}$ with boundary value $\rho_1(\alpha, \alpha) = 0$.

(3) For any $\beta < \omega_1$, $e_\beta : \beta \to \omega$ is canonically induced from ρ_1: $e_\beta(\alpha) = \rho_1(\alpha, \beta)$ for any $\alpha < \beta$.

(4) For any C-sequence, the *lower trace* $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$ is recursively defined for any $\alpha \leq \beta < \omega_1$ as follows:
(i) $L(\alpha, \alpha) = 0$;
(ii) $L(\alpha, \beta) = (L(\alpha, min(C_\beta \setminus \alpha)) \cup \{max(C_\beta \cap \alpha)\}) \setminus max(C_\beta \cap \alpha)$.

(5) Suppose that s and t are two functions defined on a common finite set of ordinals F. $Osc(s,t;F)$ is the set of all ξ in $F \setminus \{minF\}$ such that $s(\xi^-) \leq t(\xi^-)$ and $s(\xi) > t(\xi)$ where ξ^- is the greatest element of F less than ξ.

(6) For $\alpha < \beta < \omega_1$, $Osc(\alpha,\beta)$ denotes $Osc(e_\alpha, e_\beta; L(\alpha,\beta))$ and $osc(\alpha,\beta) = |Osc(\alpha,\beta)|$ denotes the cardinality of $Osc(\alpha,\beta)$.

Here are some basic facts.

Fact 2.2. *Suppose that s and t are two functions defined on a common finite set of ordinals $F_0 \cup F_1$ where $F_0 < F_1$. Then* $|Osc(s,t;F_0)| + |Osc(s,t;F_1)| \leq |Osc(s,t;F_0 \cup F_1)| \leq |Osc(s,t;F_0)| + |Osc(s,t;F_1)| + 1.$

Proof. Note by definition

$$Osc(s,t;F_0) \cup Osc(s,t;F_1) \subset Osc(s,t;F_0 \cup F_1) \subset Osc(s,t;F_0) \cup Osc(s,t;F_1) \cup \{min(F_1)\}.$$ □

Fact 2.3 ([7])**.** *ρ_1 is coherent; i.e., for any $\alpha < \beta$, $\{\xi < \alpha : \rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)\}$ is finite.*

Fact 2.4 ([8], [3])**.** (1) *If $\alpha \leq \beta \leq \gamma$ and $L(\alpha,\beta) > L(\beta,\gamma)$, then $L(\alpha,\gamma) = L(\beta,\gamma) \cup L(\alpha,\beta)$.*

(2) *If $\beta > 0$ is a limit ordinal, then $\lim_{\alpha\to\beta} min\ L(\alpha,\beta) = \beta$.*

The following lemma will be needed.

Lemma 2.5 ([3, Lemma 4.4])**.** *Let $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ be uncountable and pairwise disjoint. There is a closed and unbounded set of $\delta < \omega_1$ such that if a is in $\mathcal{A} \setminus \delta$,[7] b is in $\mathcal{B} \setminus \delta$, and R is in $\{=, >\}$, then there are a^+ in $\mathcal{A} \setminus \delta$, and b^+ in $\mathcal{B} \setminus \delta$ such that for all $i < k$ and $j < l$*

(i) *$maxL(\delta, b(j))$ is less than both $\Delta(a(i), a^+(i))$ and $\Delta(b(j), b^+(j))$;* [8]

(ii) *$L(\delta, b(j))$ is a proper initial part of $L(\delta, b^+(j))$;*

(iii) *if ξ is in $L^+ = L(\delta, b^+(j)) \setminus L(\delta, b(j))$, then $e_{a^+(i)}(\xi)\ R\ e_{b^+(j)}(\xi)$.*

Now let us fix $\{z_\alpha : \alpha < \omega_1\} \subset \mathbb{T}$ that are rationally independent. Moore's original L space is $\mathscr{L} = \{w_\beta : \beta < \omega_1\}$ where

$$w_\beta(\alpha) = \begin{cases} z_\alpha^{osc(\alpha,\beta)+1} & : \alpha < \beta \\ 1 & : \alpha \geq \beta. \end{cases}$$

For $X \subset \omega_1$, $\mathscr{L}_X = \{w_\beta \restriction_X : \beta \in X\}$ is a subspace of $\mathbb{T}^X$. Throughout this section, we will write w_β for $w_\beta \restriction_X$ when referring to elements of $\mathscr{L}_X$.

[7] $\mathcal{A} \setminus \delta = \{a \in \mathcal{A} : a > \delta\}$.

[8] $\Delta(\alpha,\beta) = min(\{\xi < \alpha, \beta : e_\alpha(\xi) \neq e_\beta(\xi)\} \cup \{\alpha,\beta\})$ for $\alpha, \beta < \omega_1$.

Theorem 2.6. *For any uncountable $X \subset \omega_1$, $\mathscr{L}_X^2$ is not Lindelöf.*

Proof. Fix $X \in [\omega_1]^{\omega_1}$. Then fix a countable $M \prec H(\omega_2)$ containing all relevant objects and $\delta = M \cap \omega_1$. Note δ has the property mentioned in Lemma 2.5 for $\mathcal{A} = \mathcal{B} = X$ since there is a closed unbounded set of ordinals with this property in M. Using Lemma 2.5 four times, we can get μ_δ and ν_δ in $X \setminus \delta$ such that $|Osc(e_{\mu_\delta}, e_{\nu_\delta}; L(\delta, \nu_\delta))| > 1$.[9]

Now fix a $\gamma < \delta$ greater than $L(\delta, \mu_\delta) \cup L(\delta, \nu_\delta)$ and such that

(1) $e_{\mu_\delta} \restriction_{[\gamma,\delta)} = e_{\nu_\delta} \restriction_{[\gamma,\delta)}$;

(2) for any $\tau \in (\gamma, \delta)$ and for $\xi \in \{\mu_\delta, \nu_\delta\}$, $L(\tau, \xi) = L(\delta, \xi) \cup L(\tau, \delta)$ and $L(\delta, \xi) < \gamma \leq L(\tau, \delta)$.

Pick a sufficiently large $\gamma \in C_\delta$. $\gamma \in C_\delta$ will guarantee $\gamma \leq L(\tau, \delta)$ for $\tau \in (\gamma, \delta)$ and the rest is guaranteed by Fact 2.3 and Fact 2.4. Let $n(\alpha, \beta)$ be a natural number guaranteed by Proposition 1.3(iii) for any $\alpha, \beta < \omega_1$.

Denote $a = e_{\mu_\delta} \restriction_\gamma$, $b = e_{\nu_\delta} \restriction_\gamma$, $L' = L(\delta, \mu_\delta)$, $L'' = L(\delta, \nu_\delta)$, and $n = n(\mu_\delta, \nu_\delta)$. Note a, b, L', and L'' are all in M by definition and Fact 2.3. Consider the set

$$\begin{aligned} A = \{(\alpha, \mu_\alpha, \nu_\alpha) \in \omega_1 \times X \times X : {} & \mu_\alpha, \nu_\alpha \geq \alpha, |Osc(e_{\mu_\alpha}, e_{\nu_\alpha}; L(\alpha, \nu_\alpha))| \\ & > 1, \ n(\mu_\alpha, \nu_\alpha) = n, \ e_{\mu_\alpha} \restriction_\gamma = a, \ e_{\nu_\alpha} \restriction_\gamma = b, \ L(\alpha, \mu_\alpha) = L', \\ & L(\alpha, \nu_\alpha) = L'', \text{ and properties (1) and (2) above hold while} \\ & \text{replacing } (\delta, \mu_\delta, \nu_\delta) \text{ by } (\alpha, \mu_\alpha, \nu_\alpha)\}. \end{aligned}$$

Note A is in M and $(\delta, \mu_\delta, \nu_\delta) \in A$. So A is uncountable and the set of its first coordinates is uncountable as well.

Let A_i collect the ith coordinates of elements of A for $i < 3$. For each $\mu_\alpha \in A_1$, let

$$O_{\mu_\alpha} = \{(x, y) \in \mathscr{L}_X^2 : |x(\mu_\alpha) - y(\mu_\alpha)| \neq |z_{\mu_\alpha}^j - 1| \text{ for any } j \in \{1, 2, ..., n\}\}.$$

It is easy to see that O_{μ_α} is open and contains everything below μ_α, i.e., for $\eta, \tau < \mu_\alpha$, $(w_\eta, w_\tau) \in O_{\mu_\alpha}$.

Then we claim that $\mathcal{C} = \{O_{\mu_\alpha} : \mu_\alpha \in A_1\}$ is an open cover of $\mathscr{L}_X^2$ without countable subcover.

First, it is easy to see that $\mathcal{C}$ is an open cover since A_1 is unbounded in ω_1. The following claim will show that it has no countable subcover.

Claim. For $(\alpha, \mu_\alpha, \nu_\alpha)$, $(\beta, \mu_\beta, \nu_\beta) \in A$, if $\mu_\alpha < \beta$, then $osc(\mu_\alpha, \mu_\beta) < osc(\mu_\alpha, \nu_\beta)$.

Proof of Claim. Note by definition of A, $L(\mu_\alpha, \xi) = L(\beta, \xi) \cup L(\mu_\alpha, \beta)$ and $L(\beta, \xi) < \gamma \leq L(\mu_\alpha, \beta)$ for $\xi \in \{\mu_\beta, \nu_\beta\}$.

$$\begin{aligned} osc(\mu_\alpha, \mu_\beta) &= |Osc(e_{\mu_\alpha}, e_{\mu_\beta}; L(\beta, \mu_\beta) \cup L(\mu_\alpha, \beta))| \\ &\leq |Osc(e_{\mu_\alpha}, e_{\mu_\beta}; L(\beta, \mu_\beta))| + |Osc(e_{\mu_\alpha}, e_{\mu_\beta}; L(\mu_\alpha, \beta))| + 1 \end{aligned}$$

[9] Actually, we can make the size of the set arbitrarily large, but greater than 1 is sufficient here. See also the proof of Theorem 4.3 in [3].

$$= |Osc(a, a; L')| + |Osc(e_{\mu_\alpha}, e_{\mu_\beta}; L(\mu_\alpha, \beta))| + 1^{10}$$
$$= |Osc(e_{\mu_\alpha}, e_{\mu_\beta}; L(\mu_\alpha, \beta))| + 1.$$
$$osc(\mu_\alpha, \nu_\beta) = |Osc(e_{\mu_\alpha}, e_{\nu_\beta}; L(\beta, \nu_\beta) \cup L(\mu_\alpha, \beta))|$$
$$\geq |Osc(e_{\mu_\alpha}, e_{\nu_\beta}; L(\beta, \nu_\beta))| + |Osc(e_{\mu_\alpha}, e_{\nu_\beta}; L(\mu_\alpha, \beta))|$$
$$= |Osc(a, b; L'')| + |Osc(e_{\mu_\alpha}, e_{\nu_\beta}; L(\mu_\alpha, \beta))|$$
$$> 1 + |Osc(e_{\mu_\alpha}, e_{\nu_\beta}; L(\mu_\alpha, \beta))|.$$

Then $osc(\mu_\alpha, \mu_\beta) < osc(\mu_\alpha, \nu_\beta)$ since e_{μ_β} agrees with e_{ν_β} on $[\gamma, \beta)$.

This finishes the proof of the claim.

It follows from the claim that for any $(\alpha, \mu_\alpha, \nu_\alpha), (\beta, \mu_\beta, \nu_\beta) \in A$, if $\mu_\alpha < \beta$, then $osc(\mu_\alpha, \nu_\beta) - osc(\mu_\alpha, \mu_\beta) \in \{1, 2, ..., n\}$, and hence $(w_{\mu_\beta}, w_{\nu_\beta}) \notin O_{\mu_\alpha}$. Then $\mathcal{C}$ has no countable subcover since A_0 is uncountable. □

Remark 2.7. The above proof shows that $\{U_{\mu_\alpha} = \{x \in \mathbb{T}^X : x(\mu_\alpha) \neq z^i_{\mu_\alpha}$ for any $i \in \{1, 2, ..., n\}\} : \mu_\alpha \in A_1\}$ is an open cover of the group generated by $\mathscr{L}_X$ without countable subcover since $w_{\nu_\beta} w^{-1}_{\mu_\beta} \notin U_{\mu_\alpha}$ for $\beta \in A_0 \setminus (\mu_\alpha + 1)$.

We can also find an uncountable closed discrete subset of $\mathscr{L}^2_X$.

Corollary 2.8. *For any uncountable $X \subset \omega_1$, $\mathscr{L}^2_X$ contains an uncountable closed discrete subset.*

Proof. Let M, δ, A, and n be as in Theorem 2.6. We need one more property of δ.

Claim. For any $\xi < \delta$, the range of $osc(\cdot, \delta) \restriction_{[\xi,\delta)\cap X}$ is unbounded.

Proof of Claim. Otherwise, there are $\xi < \delta$ and $m < \omega$ such that $osc(\xi', \delta) < m$ for any $\xi' \in [\xi, \delta) \cap X$. Let $E = \{\alpha > \xi :$ the range of $osc(\cdot, \alpha) \restriction_{[\xi,\alpha)\cap X}$ is bounded by $m\}$. Then $\delta \in E$ and $E \in M$, and hence E is uncountable. Using Lemma 2.5 $2m$ times (or [3, Theorem 4.3]) for $\mathcal{A} = X \setminus \xi$ and $\mathcal{B} = E$, we can get some $\xi' \in X \setminus \xi$ and $\alpha \in E$ such that $osc(\xi', \alpha) \geq m$. A contradiction. This finishes the proof of the claim.

Without loss of generality, assume $\mu_\delta < \nu_\delta$ (actually ν_δ can be chosen arbitrarily large). Now define $B = \{(\alpha, \mu_\alpha, \nu_\alpha) \in A : \alpha < \mu_\alpha < \nu_\alpha$ and α has the property mentioned in the above claim when replacing δ by $\alpha.\}$. Note $B \in M$ and $(\delta, \mu_\delta, \nu_\delta) \in B$. So B and the set of its first coordinates are both uncountable.

Pick an uncountable $D' \subset \{(\mu_\alpha, \nu_\alpha) : (\alpha, \mu_\alpha, \nu_\alpha) \in B$ for some $\alpha\}$ such that for $(\mu_\alpha, \nu_\alpha) \neq (\mu_\beta, \nu_\beta)$ in D', $\nu_\alpha < \beta$ or $\nu_\beta < \alpha$. Let $(\mu_{\alpha_0}, \nu_{\alpha_0})$ be the least pair in D'. Let $D = D' \setminus \{(\mu_{\alpha_0}, \nu_{\alpha_0})\}$.

As in §1, it suffices to find a neighborhood for each $w_a \in \mathscr{L}^2_X$ disjoint from $w_D \setminus \{w_a\}$. Assume $\eta = min(a)$ and $\tau = max(a)$. We want to find

[10]Recall $L(\beta, \mu_\beta) = L' < \gamma$.

a neighborhood O disjoint from each $(w_{\mu_\alpha}, w_{\nu_\alpha}) \in w_D \setminus \{w_a\}$. Let us discuss case by case. Fix $(\alpha, \mu_\alpha, \nu_\alpha) \in B$ such that $(\mu_\alpha, \nu_\alpha) \in D$.

Case 1: η or τ is in interval $[\alpha, \nu_\alpha]$.

As there are at most finitely many $(w_{\mu_\beta}, w_{\nu_\beta})$ in w_D such that η or τ is in interval $[\beta, \nu_\beta]$, let O_1 be a neighborhood of w_a disjoint from all of them.

Case 2: $\nu_\alpha < \eta$.

Define $O_2 = \{(x, y) : |x(\mu_{\alpha_0}) - y(\mu_{\alpha_0})| \neq |z^i_{\mu_{\alpha_0}} - 1|$ for any $i \in \{1, 2, ..., n\}\}$ if $\eta = \tau$ and $O_2 = \{(x, y) : x(\eta) \neq 1$ or $y(\eta) \neq 1\}$ if $\eta < \tau$.

Case 3: $\alpha > \tau$.

Let ζ be the least such that $(w_{\mu_\zeta}, w_{\nu_\zeta}) \in w_D$ and $\tau < \zeta$. Pick a neighborhood O_3 of w_a such that $O_3 \subset \{(x, y) : |x(\mu_\zeta) - y(\mu_\zeta)| \neq |z^i_{\mu_\zeta} - 1|$ for any $i \in \{1, 2, ..., n\}\}$ and $(w_{\mu_\zeta}, w_{\nu_\zeta}) \notin O_3$.

Case 4: $\eta < \alpha < \nu_\alpha < \tau$.

Let β be the least such that $(w_{\mu_\beta}, w_{\nu_\beta}) \in w_D$ and $\eta < \beta$. Let $n(\beta, \tau)$ be the natural number guaranteed by Proposition 1.3(iii). Since β has the property mentioned in the above claim, we can find $\xi \in [\eta, \beta) \cap X$ such that $osc(\xi, \beta) > n + n(\beta, \tau)$. Then $osc(\xi, \tau) > n$. Define $O_4 = \{(x, y) : |x(\xi) - y(\xi)| \neq |z^i_\xi - 1|$ for any $i \in \{0, 1, 2, ..., n\}\}$.

Let $O = O_1 \cap O_2 \cap O_3 \cap O_4$. It is easy to see that O is uniquely determined by w_a and D. Then this O is a neighborhood of w_a disjoint from $w_D \setminus \{w_a\}$. □

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