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TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 277–290

http://topology.nipissingu.ca/tp/

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Electronically published on July 31, 2014

Topology Proceedings

Web: http://topology.auburn.edu/tp/
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Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

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Volume 46 (2015)
Pages 277-290

E-Published on July 31, 2014

EXTENDING GROUP ACTIONS

JAMES KEESLING, JAMES MAISSEN, AND DAVID C. WILSON

Abstract. The Hilbert-Smith conjecture proposes that every effective compact group action on a compact manifold is a Lie group. The conjecture is equivalent to saying that there is no effective action of a p-adic group on a compact manifold.

The goal of this paper is to present a different approach to the Hilbert-Smith conjecture by looking internally at the space under the action of the group. The approach is in contrast with the one most have taken in the past by studying the quotient space of the action.

We show that free p-adic actions on the space of irrationals are unique up to conjugation. We also show that should a counter-example to the Hilbert-Smith conjecture exist, the counter-example would be an extension of this unique free p-adic group action on the space of irrationals.

Our search for a solution to the Hilbert-Smith conjecture motivates our investigation of compact extensions of compact metric group actions on separable metric spaces. We give sufficient conditions to guarantee an extension of a group action to a given compactification and present an example which illustrates the need for those conditions.

1. Introduction

The Hilbert-Smith conjecture asks whether or not a p-adic group can act effectively on a manifold. We present another approach to attempt to solve this question by looking internally rather than through the quotient space of the action. The latter has been a popular approach ([1], [11], [13]) and has given some partial solutions to the conjecture ([7], [8], [12]). The

2010 *Mathematics Subject Classification.* Primary: 57S10, 57M60, 20F34, 57S05, 57N10. Secondary: 54H15, 55M35, 57S17.

Key words and phrases. Hilbert's fifth problem, Hilbert-Smith conjecture, manifolds, transformation groups.

most recent partial solution [9] is that the conjecture holds for manifolds of dimension 3. (It had long been known to hold for 2-manifolds ([2], [6]).) The solution did not make use of the quotient space, and our focus is likewise internal to the action, rather than dimension theoretic. We will show that for a given 0-dimensional compact group G, there is only one free G-action, up to conjugation by a homeomorphism, on the space of irrationals. Moreover, this G-action will be a sub-action of any effective G-action on a complete metric space where the periodic points contain no open set and the action has no isolated orbits. These results arose from discussions on a possible attack on the age-old Hilbert-Smith conjecture, since any potential counter-example to the conjecture would contain this unique free action on the space of irrationals as a sub-action. We turn to investigate how group actions on non-compact spaces can be extended, as group actions, to compactifications of those spaces. Requiring that each group element extends to a homeomorphism on the compactification is generally not sufficient to ensure that the group action extends to the compactification. We have examples and theorems to show exactly what happens.

2. Terms and Notation

We will let $\mathbb{R}$ denote the real numbers, $\mathbb{Q} \subset \mathbb{R}$ the rational numbers, $\mathbb{N} \subset \mathbb{Q}$ the natural numbers, and $\mathcal{I} := \mathbb{R} \setminus \mathbb{Q}$ the space of irrationals. We will use $\mathbb{Z}_n$ to denote the cyclic group of integers modulo n. For a given metric d, let $\operatorname{diam}(A) := \sup\{d(a,b) | a, b \in A\}$ and let $B_r(x)$ denote the open ball of radius r about the point x. Let $\bar{A}$ denote the closure of a given set A.

During the course of this paper, when we write $G = \varprojlim\{G_i, \phi_i^{i+1}\}$, we will have the following understanding:

(1) each G_i is a finite group with identity e_i,
(2) $G_0 = \{e_0\}$, and
(3) for each $i \geq 0$, $\phi_i^{i+1} : G_{i+1} \to G_i$ is an onto homomorphism.

3. Actions on the Space of Irrational Numbers

The main goal of this section is to prove Theorem 3.5. We begin by showing that the quotient space of any compact 0-dimensional group action on $\mathcal{I}$ is homeomorphic to $\mathcal{I}$. This result will follow from Theorem 3.2, where we show that the image of any perfect open map on $\mathcal{I}$ is homeomorphic to $\mathcal{I}$.

Lemma 3.1. *If (X,d) is a complete metric space and the mapping $p : X \to Y$ is a perfect open surjection, then the space Y is topologically complete.*

Proof. Recall a mapping $p : X \to Y$ is a perfect surjection if it is a closed surjection such that, for every $y \in Y$, we have $p^{-1}(y)$ is compact. Now we can assume that d is a bounded metric (replacing it with an equivalent $\frac{d}{1+d}$ to obtain a bounded metric from an unbounded one if needed). We define a new metric ρ for Y induced by the Hausdorff distance between pre-images of points under p using the complete metric d. In other words, for $\alpha, \beta \in Y$, let $A := p^{-1}(\alpha)$ and $B := p^{-1}(\beta)$, and define

$$\rho(\alpha, \beta) := \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

First, we show that the topology on (Y, ρ) is equivalent to the original topology on Y. Let $y \in Y$ and $U \subset Y$ be an open neighborhood of Y under the original topology. Since p is perfect, the pre-image $p^{-1}(y) \subset X$ is a compact set. Since p is continuous, we have $p^{-1}(U) \subset X$ is an open neighborhood of $p^{-1}(y)$. Since $p^{-1}(y)$ is compact, we can pick $\delta > 0$ such that the δ neighborhood of $p^{-1}(y)$ is contained within $p^{-1}(U)$. Let $\mathcal{U} := \{B_\delta(x) | x \in p^{-1}(y)\}$ be an open cover of $p^{-1}(y)$. Since $p^{-1}(y)$ is compact, take a finite sub-cover of $\mathcal{U}$ and denote it by $\{U_i | 1 \le i \le n\}$. For each $1 \le i \le n$, we have $U_i \cap p^{-1}(y) \ne \emptyset$, and since p is an open map, the set $p(U_i) \subset U$ is an open neighborhood of y. So the finite intersection $V := \bigcap_{i=1}^n p(U_i)$ is an open neighborhood of y. By construction, the set V is also an open neighborhood with respect to the metric ρ, and thus the two metrics induce the same topology on Y.

We shall now show that (Y, ρ) is complete. Let $\{\alpha_n\}_{n=1}^\infty \subset Y$ be a Cauchy sequence with respect to the metric ρ. For $\epsilon = 1/2^1$, there is an $N_1 > 0$ such that $n, m \ge N_1$ implies that $\rho(\alpha_n, \alpha_m) < 1/2^1$. Pick $b_1 \in p^{-1}(\alpha_{N_1})$ and observe that the open ball $B_1 := B_{\frac{1}{2^1}}(b_1)$ intersects $p^{-1}(\alpha_m)$ for every $m > N_1$. Inductively, pick b_{n+1} such that $N_{n+1} > N_n$ corresponds to a choice of $\epsilon = 1/2^{n+1}$ and $b_{n+1} \in p^{-1}(\alpha_{N_{n+1}}) \cap B_n$, then observe that the open ball $B_{n+1} := B_{\frac{1}{2^{n+1}}}(b_{n+1})$ intersects $p^{-1}(\alpha_m)$ for every $m > N_{n+1}$. Let $N_0 := 0$ and $B_0 := X$. Choose a sequence $\{a_n\}_{n=1}^\infty$ by picking $a_i := b_k$ if $i = N_k$ for some $k \ge 1$ and by choosing $a_i \in p^{-1}(\alpha_i) \cap B_k$ when $N_k < i < N_{k+1}$ for some $k \ge 0$. By construction, the sequence $\{a_n\}_{n=1}^\infty$ is Cauchy with respect to d. Since (X, d) is a complete metric space, this latter sequence has a limit, $a := \lim a_n$. Let $\alpha := p(a)$ and observe that $\alpha = \lim \alpha_n$, and thus the metric ρ is a complete metric for Y. Since there exists a complete metric on it, the space Y is topologically complete. □

Theorem 3.2. *If the mapping $p : \mathcal{I} \to Y$ is a perfect open surjection, then the space $Y \cong \mathcal{I}$, the space of irrationals.*

Proof. F. Hausdorff [4] characterized the irrationals as the 0-dimensional, nowhere locally compact, separable metric space that is topologically complete.

Since p is both an open and closed mapping, any subset $\mathcal{O} \subset \mathcal{I}$ that is both open and closed will have the property that $p(\mathcal{O})$ is both open and closed. Since p is a surjection, the open-closed basis of $\mathcal{I}$ is mapped to an open-closed basis of Y; hence, Y is a 0-dimensional separable metric space.

Further, since p is a perfect and light surjection and the space $\mathcal{I}$ is nowhere locally compact, the image space $p(\mathcal{I}) = Y$ is as well.

Since the space $\mathcal{I}$ is topologically complete (say viewed as $\mathbb{N}^{\mathbb{N}}$), there is a metric d for which $\mathcal{I}$ is complete. By Lemma 3.1, the space Y is topologically complete.

Hence, by Hausdorff's characterization of the irrationals, the space $Y \cong \mathcal{I} = \mathbb{R} \setminus \mathbb{Q}$, as stated. □

Now we apply this result to 0-dimensional compact metric group actions on the space of irrationals. L. S. Pontryagin [10] showed that every 0-dimensional compact group is the inverse limit of finite groups. It is easy to see that, for a given 0-dimensional compact metric group G, either G is finite or G has the topology of a Cantor set. If G is not finite, we call G a *Cantor group*. In either case, we can write $G = \varprojlim\{G_i, \phi_i^{i+1}\}$ with the conventions established at the start of the paper. The 0-dimensional compact group of p-adic integers is a Cantor group of the form $\varprojlim\{\mathbb{Z}_{p^i}, \phi_i^{i+1}\}$; hence, the results below will hold for them as well.

Corollary 3.3. *Let $G = \varprojlim\{G_i, \phi_i^{i+1}\}$ be a 0-dimensional compact group. If the group G acts freely on the space of irrationals $\mathcal{I}$, then the quotient space $\mathcal{I}/G \cong \mathcal{I}$.*

Proof. Let the map $p : \mathcal{I} \to \mathcal{I}/G$ denote the quotient map induced by the free action. The mapping p is a perfect open surjection, so by Theorem 3.2, the quotient space $\mathcal{I}/G \cong \mathcal{I}$. □

We will use this result to show that for any given 0-dimensional compact group, G, there is only one free G action on the space of irrationals, as all others are conjugate to it by homeomorphisms.

Theorem 3.4. *Let $G = \varprojlim\{G_i, \phi_i^{i+1}\}$ be a 0-dimensional compact group. If the mapping $A : G \times \mathcal{I} \to \mathcal{I}$ is a free topological group action, then there is a homeomorphism $h : \mathcal{I} \to \mathcal{I} \times G$ such that*

$$\begin{array}{ccc} G \times \mathcal{I} & \xrightarrow{A} & \mathcal{I} \\ {\scriptstyle 1_g \times h}\downarrow & & \downarrow{\scriptstyle h} \\ G \times (\mathcal{I} \times G) & \xrightarrow{B} & \mathcal{I} \times G \end{array}$$

commutes, where $B : G \times (\mathcal{I} \times G) \to \mathcal{I} \times G$ *is by* $B(j,(w,g)) := (w, j \circ g)$ *(i.e., left translation on the second factor).*

Proof. Let $G = \varprojlim\{G_i, \phi_i^{i+1}\}$. For each $k \in \mathbb{N}$, let $\pi^k : G \to G_k$ be the projection map, and define $G^k := \ker \pi^k \trianglelefteq G$. Let $\mathcal{B} := \{B_n\}_{n\in\mathbb{N}}$ be a countable open-closed basis of $\mathcal{I}$.

Claim. For a given $i \geq 1$ and $w \in \mathcal{I}$ there is a set $B_{n^{w,i}} \in \mathcal{B}$ such that $w \in B_{n^{w,i}}$ and $G^i(B_{n^{w,i}}) \cap (G \setminus G^i)(B_{n^{w,i}}) = \emptyset$.

Since G acts freely on $\mathcal{I}$, the set $G^i(w) \cap (G \setminus G^i)(w) = \emptyset$. Since $G \setminus G^i$ and G^i are both open, compact subsets of G and $\mathcal{B}$ is an open-closed basis of $\mathcal{I}$, there is an open-closed set $B \in \mathcal{B}$ such that $w \in B$ but $(G \setminus G^i)(w) \cap B = \emptyset$. Since $w \notin (G \setminus G^i)(B)$ which is a closed set, there is a $B_{n^{w,i}} \in \mathcal{B}$ with $w \in B_{n^{w,i}} \subseteq B$ and $(G \setminus G^i)(B) \cap B_{n^{w,i}} = \emptyset$. Suppose for sake of contradiction that there exists an $x \in G^i(B_{n^{w,i}}) \cap (G \setminus G^i)(B_{n^{w,i}})$. Thus there are $g_1 \in G^i, g_2 \in G \setminus G^i$, and $b_1, b_2 \in B_{n^{w,i}}$ with $x = g_1 b_1 = g_2 b_2$. The point $b_1 = (g_1^{-1} g_2) b_2$ would be in both $B_{n^{w,i}}$ and $(G \setminus G^i)(B)$ which contradicts their intersection being empty, and the claim is proven.

Since the point $w \in \mathcal{I}$ was arbitrary, for a given $i \geq 1$ there is a map $t_i : \mathcal{I} \to \mathbb{N}$ by $t_i(w) = n^{w,i} \in \mathbb{N}$ from the claim above. For $i = 1$, let $N_1 := (n_j)_{j=1}^{\omega}$ be the strictly increasing (possibly infinite) sequence of natural numbers such that we have $m \in N_1$ if and only if $m \in t_1(\mathcal{I})$.

Now, using the order on this subset of the naturals and the open-closed nature of the sets B_{n_k}, we will define a continuous map $s_1 : \mathcal{I} \to G_1$. For $k \geq 1$ and for all $w \in G(B_{n_k} \setminus G(\bigcup_{m<k} B_{n_m}))$, let $s_1(w) = \eta \in G_1$ if and only if $w \in g_\eta(B_{n_k})$ for some $g_\eta \in G$ such that $\pi^1(g_\eta) = \eta$. Since all of the sets B_{n_m} are both open and closed, so are sets of the form $G(B_{n_k} \setminus G(\bigcup_{m<k} B_{n_m}))$ and the mapping s_1 is continuous.

Assume for some $n \geq 1$, that we have for all $1 \leq m \leq n$,

(1) $s_m : \mathcal{I} \to G_m$ is continuous, and
(2) the following diagram commutes

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{s_m} & G_m \\ & {\scriptstyle s_{m-1}} \searrow & \downarrow{\scriptstyle \phi_{m-1}^m} \\ & & G_{m-1}\,. \end{array}$$

Let $N_n := (n_j)_{j=1}^{\omega}$ be the strictly increasing (possibly infinite) sequence of natural numbers such that we have $m \in N_n$ if and only if $m \in t_n(\mathcal{I})$. Again using the order on N_n as a subset of the naturals and the open-closed nature of the associated sets B_{n_k}, we will define a continuous map $s_n : \mathcal{I} \to G_n$. For $n \geq 1$ and for all points $w \in G([B_{n_k} \cap s_{n-1}^{-1}\{e_{n-1}\}] \setminus G(\bigcup_{m<k} B_{n_m}))$, let $s_n(w) = \eta \in G_n$ if and only if, for some $g_\eta \in G$ with $\pi^n(g_\eta) = \eta$, the point $w \in g_\eta(B_{n_k} \cap s_{n-1}^{-1}\{e_{n-1}\})$. Since all of the sets B_{n_m} are both open and closed, so are sets of the form $G([B_{n_k} \cap s_{n-1}^{-1}\{e_{n-1}\}] \setminus G(\bigcup_{m<k} B_{n_m}))$ and the mapping s_n is continuous.

The diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{s_n} & G_n \\ & \searrow^{s_{n-1}} & \downarrow \phi_{n-1}^{n} \\ & & G_{m-1} \end{array}$$

commutes from the definitions of s_* and the commutativity of

$$\begin{array}{ccc} G & \xrightarrow{\pi^m} & G_m \\ & \searrow^{\pi^{m-1}} & \downarrow \phi_{m-1}^{m} \\ & & G_{m-1}\,. \end{array}$$

By strong induction, we obtain a sequence of continuous maps $\{s_k\}_{k\in\mathbb{N}}$ and can define a continuous map $s : \mathcal{I} \to G$ using them. By Corollary 3.3, we have ${}^{\mathcal{I}}/_{G} \cong \mathcal{I}$. We can then define $q : \mathcal{I} \to \mathcal{I}$ to be the quotient map of the free G action. And, finally, we define a continuous bijection $h : \mathcal{I} \to \mathcal{I} \times G$ by $h(w) := (q(w), s(w))$. To see that h is a homeomorphism, merely observe that h is an open mapping. Let $U \subset \mathcal{I}$ and $w \in U$ be an arbitrary point of U. Now there exists an $n \in \mathbb{N}$ such that $G^n(w) \subset U$ and there exists a $\delta > 0$ such that $G^n(B_\delta(w)) \subseteq U$. Finally, $h(w) \in q[G^n(B_\delta(w))] \times G^n \subset h(U)$ and we see that w is an interior point of $h(U)$. Since w was arbitrary, it follows that $h(U)$ is open and h is a homeomorphism as desired. $\square$

Let G be a 0-dimensional compact group. Not every effective G-action on a space X will contain, as a sub-action, the free G-action on the space of irrationals. A trivial example would be when X is finite. Slightly less trivial would be letting G be a Cantor group and the space $X = G$, with G acting on X by left (or right) translation. In this second example, $\mathcal{I} \subset X$, but as ${}^{X}/_{G}$ is a singleton, obviously $\mathcal{I} \not\subseteq {}^{X}/_{G}$. To avoid this type of problem, we restrict our attention to Cantor groups and impose restrictions on the action and the space.

Theorem 3.5. *Let $G = \varprojlim\{G_i, \phi_i^{i+1}\}$ be a Cantor group. If X is a complete metric space upon which G acts effectively such that the set of* *periodic points contains no open set and the action has no isolated orbits, then there is a dense subspace $Y \subset X$, with $Y \cong \mathcal{I}$, such that G acts freely on Y.*

Proof. Let $\pi : X \to {}^{X}\!/_{G}$ be the quotient map induced by the effective action on X by G.

Since X is a complete metric space, the quotient space ${}^{X}\!/_{G}$ is as well. Denote by $P_k \subset X$ the collection of periodic points of X of period k under the G action. Let $P = \bigcup P_k$.

Now the space $W := {}^{X}\!/_{G} \setminus \pi(P)$ is a dense G_δ set in ${}^{X}\!/_{G}$, so it is a complete metric space. Since X has no isolated orbits, the space W has no isolated points.

Let Q and Q' be disjoint, countable, dense subsets of W. Since Q is 0-dimensional, it is contained in a 0-dimensional G_δ subset Z' of ${}^{X}\!/_{G}$. Moreover, since $W \setminus Q'$ is also a G_δ subset of ${}^{X}\!/_{G}$, so is the intersection $Z := Z' \cap [W \setminus Q']$. Since both Z and $W \setminus Z$ are dense in W, it follows that Z is nowhere locally compact.

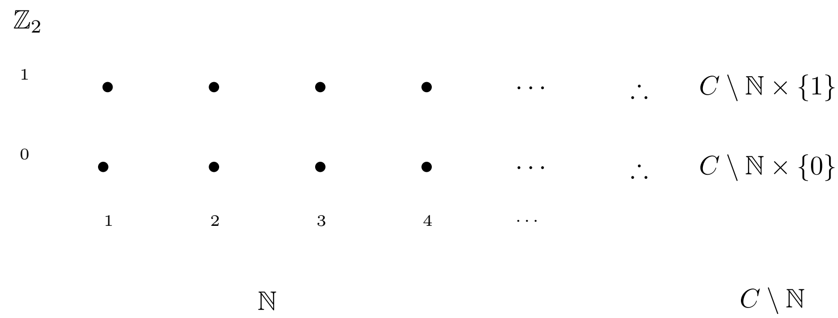
Since π is an open mapping, the set $Y := \pi^{-1}(Z)$ is a dense G_δ set of the complete metric space X. Since π is a light open mapping, Y is nowhere locally compact and 0-dimensional. Since Y has no isolated points, by Hausdorff's characterization of the irrationals [4], we have $Y \cong \mathcal{I}$. Finally, the group G acts freely upon Y, since $Y \subset X \setminus P$ and $Y = \pi^{-1}(Z)$ was the full pre-image of a set in ${}^{X}\!/_{G}$. □

If there is a counterexample to the Hilbert-Smith conjecture, then it will contain this unique free p-adic action on the space of irrationals. Moreover, this counterexample will be the extension of that p-adic group action to a manifold compactification of the space of irrationals. This observation is why we turn to examine equivariantly extending group actions and understand the potential obstacles that might be found therein.

4. Motivating Examples

One might think that simply requiring every element of a group action upon a space to extend would be sufficient to force the group action to extend. However, we will impose on our subsequent Theorem 5.1 two additional criteria: the group be complete and the compactification be metric. We will first give an example of a simple space, provide effective group actions on that space, and demonstrate that they will not extend when one of the criteria from the theorem is not met.

Consider the space $X := \mathbb{N} \times \mathbb{Z}_2$, where $\mathbb{N}$ denotes the natural numbers and $\mathbb{Z}_2$ the group of two elements. Let C be a compactification of $\mathbb{N}$ and let $Y := C \times \mathbb{Z}_2$ be the corresponding compactification of X.



We will first show that the condition that the group be complete is not a spurious choice. We will define an effective group action on X by a non-complete group and show that this group action does not extend to any compactification Y, defined above. Define, for each $i \in \mathbb{N}$, the homeomorphism $f_i : X \to X$ given by

$$f_i(n, z) := \begin{Bmatrix} (n, z+1) & n = i \\ (n, z) & n \neq i \end{Bmatrix}.$$

Let $e : X \to X$ be the identity and let $G_w := \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$, the group generated by $\{f_i\}_{i=1}^{\infty}$ (in other words, the countable weak product of $\mathbb{Z}_2$ actions). For any compact set K, let $N_K := \max\{n \in \mathbb{N} | (n, z) \in K\}$, then, for $i > N_K$, we have $f_{i|_K} = e$. We have that $f_i \to e$ with the topology generated by convergence on compact sets.

For each $i \in \mathbb{N}$, the homeomorphism f_i extends to a $\hat{f}_i : Y \to Y$ by

$$\hat{f}_i(\alpha) := \begin{Bmatrix} f_i(\alpha) & \alpha \in X \\ \alpha & \alpha \in Y \setminus X \end{Bmatrix}.$$

Let $\hat{e} : Y \to Y$ be the identity on Y. The extension $\hat{f}_i$ is also a homeomorphism since the function f_i is a homeomorphism, the composition $\hat{f}_i \circ \hat{f}_i = \hat{e}$, and, for any neighborhood N_y of $Y \setminus X$ where $N_y \subset Y \setminus \{(n, z) | n \leq i, z \in \mathbb{Z}_2\}$, we have $\hat{f}_{i|_{N_y}} = \hat{e}$, and thus $\hat{f}_i$ is continuous. Since every $g \in G_w$ is the finite composition of elements from $\{f_i\}_{i=1}^{\infty}$, the map $g : X \to X$ extends to a homeomorphism $\hat{g} : Y \to Y$ as well.

Yet, in the sequence $\{\hat{f}_i\}_{i=1}^{\infty}$, we do not have $\hat{f}_i \to \hat{e}$. As proof, consider $\alpha \in Y \setminus X$ and any neighborhood $N_\alpha \subset Y$ of α. For every $N \in \mathbb{N}$, there is an $n_N > N$ such that $(n_N, z) \in N_\alpha$ for some $z \in \mathbb{Z}_2$. Now $\hat{f}_{n_N}(n_N, z) \neq \hat{e}(n_N, z) = (n_N, z)$, so $\lim \hat{f}_i \neq \hat{e}$.

Although every element of the group G_w extends to a homeomorphism from Y to itself and the group structure on G_w is maintained, the topology of the group action is not. It is possible to define extensions of G_w to some compactifications of X, but not those compactifications in the form that we defined for Y. Given Y as defined above, define $Z := {}^{Y}/{\sim}$ by $(c, z) \sim (c', z')$ if and only if $c = c' \in C \setminus \mathbb{N}$. The group action G_w extends as a group action on Z, where, for every $f \in G_w$, we have $f_{|Z\setminus X} \equiv 1_Z$, the identity on the space Z.

Next, we will justify our decision to require that the compactification be a metric compactification. Let us consider a larger group, a compact group that was mentioned earlier, namely the Cantor group $G_s \cong \Pi_{i=1}^{\infty}\mathbb{Z}_2$. For every $\gamma = (\gamma_i) \in \Pi_{i=1}^{\infty}\mathbb{Z}_2$, define $f_\gamma : X \to X$ by $f_\gamma(n, z) := (n, z+\gamma_n)$. These functions yield an effective G_s group action on X. If, for every $\gamma \in G_s$, the homeomorphism f_γ extends to a homeomorphism $\hat{f}_\gamma : Y \to Y$, then we will show that $C = \beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$.

For each $\gamma = (\gamma_i) \in \Pi_{i=1}^{\infty}\mathbb{Z}_2$, define a function $F_\gamma : \mathbb{N} \to \{0, 1\}$ by $F_\gamma(n) = \gamma_n$. Since $C \times \{0\}$ is open in Y and since f_γ extends to a homeomorphism $\hat{f}_\gamma$, there is a continuous extension $\hat{F}_\gamma : C \to \{0, 1\}$ given by

$$\hat{F}_\gamma(\alpha) = \left\{\begin{matrix} F_\gamma(\alpha) & \alpha \in \mathbb{N} \\ 0 & \alpha \in C \setminus \mathbb{N}, \hat{f}_\gamma(\alpha, 0) \in C \times \{0\} \\ 1 & \alpha \in C \setminus \mathbb{N}, \hat{f}_\gamma(\alpha, 0) \in C \times \{1\} \end{matrix}\right\}.$$

Since every function $F : \mathbb{N} \to \{0, 1\}$ extends continuously to a function $\hat{F} : C \to \{0, 1\}$ by definition of the Stone-Čech compactification, we have that the compactification $C = \beta\mathbb{N}$. Since $Y = C \times \mathbb{Z}_2$, we have $Y \cong \beta\mathbb{N}$. Since the only compact sets of $\beta\mathbb{N}$ that are metrizable are finite sets, the orbits of a compact group action on it would be perforce trivial. Consider the extension $\hat{f}_\gamma : Y \to Y$ of f_γ, where $\gamma = (1)_{i=1}^{\infty}$, which would perforce have to swap $(C \setminus \mathbb{N}) \times \{0\}$ with $(C \setminus \mathbb{N}) \times \{1\}$. Thus, G_s does not act on Y as a group action.

Again, as was the case with G_w acting on X, there are extensions of the group action G_s to compactifications of X, but we cannot mandate that those compactifications be of the form given to Y. To extend the group action G_s to a metric compactification, the compactification would need to be of the form $Z := {}^{Y}/{\sim}$ that was defined by $(c, z) \sim (c', z')$ if and only if $c = c' \in C \setminus \mathbb{N}$ which we described earlier in the section. For every $\gamma \in G_s$, we extend the homeomorphism $f_\gamma : X \to X$ to the homeomorphism $\hat{f}_\gamma : Z \to Z$ by $\hat{f}_\gamma(\alpha) := \alpha$ for every $\alpha \in Z \setminus X$.

Moreover, the use of $\mathbb{Z}_2$ as a factor of X in our example could have been replaced by a more interesting compact group, such as the p-adic integers or some other Cantor group, without difficulty. Likewise, given a 0-dimensional compact group action on a separable metric space, it is not hard to see an example of this kind occurring within it as a sub-action. Thus, the obstacles that this simple example highlights are, in some sense, fundamental.

5. Extending Group Actions

The theorem below will show that the conditions discussed in the previous section are sufficient to guarantee the extension of the group action as a group action.

Theorem 5.1. *Let X be a metric space, G be a complete metric group, and $A : G \times X \to X$ be a topological group action. If (C, d_c) is a metric compactification of X such that for all $g \in G$ the map $A(g, \cdot) = g(x) : X \to X$ extends continuously to $\hat{A}(g, \cdot) = \hat{g}(x) : C \to C$, then $\hat{A} : G \times C \to C$ is a continuous group action.*

Proof. Let ρ be a complete metric on the group G. For ease of notation, define for each $g \in G$ the map $g : C \to C$ by $g(x) = \hat{A}(g, x)$.

By way of contradiction, suppose that $\hat{A}$ is not continuous. With this assumption we will show that there is an element $h \in G$ such that the function $h := \hat{A}(h, \cdot) : C \to C$ is not continuous.

Since $\hat{A}$ is not continuous, C is metric, and X is dense in C, there is a sequence $\{(g_i, x_i)\}_{i=1}^{\infty} \subset G \times X$ such that $(g_i, x_i) \to (g, x) \in G \times C$ but $g_i(x_i) \nrightarrow g(x) \in C$.

Claim 1. Without loss of generality, we may assume $g = e \in G$.

If the sequence $g^{-1}g_i(x_i) \to x \in C$, then $gg^{-1}g_i(x_i) \to g(x)$ since $g \in G$ implies g is continuous. But then $g_i(x_i) \to g(x)$ since $gg^{-1}g_i(x_i) = g_i(x_i)$ as $x_i \in X$ and A is a group action on X. Thus, the sequence $g^{-1}g_i(x_i) \nrightarrow x \in C$ and we may assume that $g = e \in G$.

Claim 2. Without loss of generality, we may assume that $g_i(x_i)$ converges to $y \neq x$.

Since $g_i(x_i) \nrightarrow x \in C$ and C is a compact metric space, the sequence $\{g_i(x_i)\}_{i=1}^{\infty}$ has a convergent subsequence such that $g_{i_k}(x_{i_k}) \to y \in C \setminus \{x\}$. So we can assume that $g_i(x_i) \to y \neq x$ as claimed.

Let $0 < r < d_c(x, y) \neq 0$, let $U := B_{\frac{r}{2}}(x)$, and let $V := B_{\frac{r}{2}}(y)$ so that $\bar{U} \cap \bar{V} = \emptyset$ (which we will use in the following two claims).

Claim 3. There is a sequence of group elements $\{h_j \in G\}_{j=0}^{\infty}$, sets $\{W_j \subseteq C\}_{j=1}^{\infty}$, and points $\{u_j \in X\}_{j=1}^{\infty}$ and $\{v_j \in X\}_{j=1}^{\infty}$ such that

(1) $\operatorname{diam}(W_j) \to 0$,
(2) $u_j \in W_j \cap X$ and $v_j \in W_j \cap X$, and
(3) $h_j(u_i) \in U$ and $h_j(v_i) \in V$ for all $i \leq j$.

For sake of induction, we define $h_0 := e \in G$ and $W_1 := B_{\frac{r}{2}}(h_0^{-1}(x)) = U$ and let $n_0 = 0$.

Since $x_i \to x$ and $g_i(x_i) \to y$ in C, there exists a number $M_0 > 0$ such that whenever $n > M_0$ both $x_n \in U$ and $g_n(x_n) \in V$. Let $m_1 := M_0 + 1$ and let $u_1 := h_0^{-1}(x_{m_1}) = x_{m_1} \in U = W_1$.

Since $g_i \to e$ in G, for the compact (finite) set $\{x_{m_1}\}$, there is a number $N_1 > m_1$ such that whenever $n > N_1$ the point $g_n(x_{m_1}) \in U$ and $\rho(g_n, e) < 1/2^1$. Let $n_1 := N_1 + 1$ and let $v_1 := h_0^{-1}(x_{n_1}) = x_{n_1} \in U = W_1$.

Let $h_1 := g_{n_1} \circ h_0 = g_{n_1} \in G$, then $h_1(u_1) = g_{n_1}(x_{m_1}) \in U$, $h_1(v_1) = g_{n_1}(x_{n_1}) \in V$, and $\rho(h_1, h_0) < 1/2^1$. Suppose for a number $k \geq 1$ and all $1 \leq j \leq k$ that

(1) $h_j \in G$,
(2) $W_j = B_{\frac{r}{j+1}}(h_{j-1}^{-1}(x))$,
(3) $u_j \in W_j \cap X$ and $v_j \in W_j \cap X$,
(4) $n_j > m_j > n_{j-1}$,
(5) $h_{j-1}(u_j) = x_{m_j}$ and $h_{j-1}(v_j) = x_{n_j}$, and
(6) $h_j(u_i) \in U$ and $h_j(v_i) \in V$ for all $1 \leq i \leq j$.

Let $W_{k+1} := B_{\frac{r}{k+2}}(h_k^{-1}(x))$.

Since $g_i \to e$ in G, for the compact (finite) set $\{h_k(u_i), h_k(v_i) : 1 \leq i \leq k\}$ there is a number $M_k > n_k$ such that whenever $n > M_k$, the point $h_k^{-1}(x_n) \in W_{k+1}$, the points $g_n(h_k(u_i)) \in U$ for all $1 \leq i \leq k$, and the points $g_n(h_k(v_i)) \in V$ for all $1 \leq i \leq k$. Let $m_{k+1} := M_k + 1$ and let $u_{k+1} := h_k^{-1}(x_{m_{k+1}}) \in W_{k+1}$.

Since $g_i \to e$ in G, for the compact (finite) set $\{x_{m_{k+1}}\}$ there is a number $N_{k+1} > m_{k+1}$ such that whenever $n > N_{k+1}$, the point $g_n(x_{m_{k+1}}) = g_n(h_k(u_{k+1})) \in U$ and $\rho(g_n, e) < 1/2^{k+1}$. Let $n_{k+1} := N_{k+1} + 1$ and let $v_{k+1} := h_k^{-1}(x_{n_{k+1}})$.

Let $h_{k+1} := g_{n_{k+1}} \circ h_k \in G$, then $h_{k+1}(u_i) \in U$ and $h_{k+1}(v_i) \in V$ for $1 \leq i \leq k+1$. Moreover, $\rho(h_{k+1}, h_k) < 1/2^{k+1}$.

Then, by strong induction, we have the following for all $n \in \mathbb{N}$ and all $k \leq n$:

(1) $h_n \in G$,
(2) $W_n = B_{\frac{r}{n+1}}(h_{n-1}^{-1}(x)) \subseteq X$, thus $\operatorname{diam}(W_n) \to 0$,
(3) $u_n \in W_n \cap X$ and $v_n \in W_n \cap X$,
(4) $h_n(u_k) \in U$ and $h_n(v_k) \in V$, and
(5) $\rho(h_n, h_{n-1}) < 1/2^n$.

Claim 4. There is an $h \in G$ such that $h(x) := \hat{A}(h, \cdot) : C \to C$ is not continuous which contradicts our assumption that $\hat{A}$ is continuous.

Since C is a compact metric space, there is a subsequence of $\{u_k\}_{k=1}^{\infty}$ which converges to a single point in C. Without loss of generality, ignore subindices. Let $z \in C$ such that $u_k \to z$ in C. Since $\text{diam}(W_n) \to 0$, we also have $v_k \to z$ in C as well.

Since G is a complete metric group, the constructed Cauchy sequence $\{h_k\}_{k=1}^{\infty}$ converges to an element $h \in G$.

For a fixed $i \in \mathbb{N}$ and the corresponding compact (finite, two point) set $\{u_i, v_i\}$, it follows from $h_k \to h$ in G that $h_k(u_i) \to h(u_i)$ and $h_k(v_i) \to h(v_i)$. For all $k > i$, the points $h_k(u_i) \in U$ and $h_k(v_i) \in V$; thus, $h(u_i) \in \bar{U}$ and $h(v_i) \in \bar{V}$.

Since $h \in G$, we should have $h : C \to C$ being continuous, so $h(u_i) \to h(z) \in \bar{U}$ and $h(v_i) \to h(z) \in \bar{V}$. Thus, we have $h(z) \in \bar{U} \cap \bar{V} = \emptyset$ which means that $h : C \to C$ is discontinuous as was claimed.

We have reached a contradiction and thus it must be the case that $\hat{A}$ is continuous as desired. □

6. Some Applications

In this section we give some examples that illustrate Theorem 5.1.

Theorem 6.1. *Let M^n be any connected or compact manifold with $n \in \{2, 3\}$. If Z is any 0-dimensional subset of M^n, then for any p-adic group Δ_p, there is no effective action of Δ_p on $M^n \setminus Z$.*

Proof. We first note that under the assumptions in the theorem, the Freudenthal compactification of $M^n \setminus Z$ is M^n because $M^n \setminus Z$ is peripherally compact and locally connected. Also note that every homeomorphism $h : M^n \setminus Z \to M^n \setminus Z$ extends to a unique homeomorphism of M^n. Assuming that there is an effective action of Δ_p on $M^n \setminus Z$, then each homeomorphism defined by an element in the group would extend uniquely to M^n. By Theorem 5.1, we have that the action of Δ_p on M^n, defined by the extensions of the individual homeomorphisms, is continuous. However, there are no effective actions of Δ_p on any n-manifold for $n = 2$ or 3. □

In particular, there is no effective action of Δ_p on the irrational points of the Sierpiński carpet, because the irrational points of the Sierpiński carpet is homeomorphic to $S^2 \setminus Z$ where Z is a countable dense subset of S^2. If there were such an effective action, there would be an effective action on S^2.

Now let $B_0 = I^3$. Let B_1 be obtained from B_0 in the following way. Divide each coordinate in B_0 into three equal intervals. In this way, we divide B_0 into twenty-seven subcubes. Remove the central cube from B_0 to get B_1. Repeat this process on the remaining twenty-six cubes in B_1

to get B_2. Continuing in this way we get $B_0 \supset B_1 \supset B_2 \supset \cdots$. Let

$$M_2^3 = \bigcap_{i=0}^{\infty} B_i.$$

Let Z be the union of the boundaries of the removed subcubes in M_2^3. Then $M_2^3 \setminus Z$ is homeomorphic to $S^3 \setminus \{x_i\}_{i=1}^{\infty}$ where $\{x_i\}_{i=1}^{\infty}$ is a countable dense set of points in S^3. So there is no effective action of Δ_p on $M_2^3 \setminus \{x_i\}_{i=1}^{\infty}$ by the argument given for the previous example.

Theorem 6.2. *Let G be any connected, nontrivial, complete separable metric group. If M^n is a manifold with $n \geq 2$ and Z is a dense 0-dimensional subset of M^n, then there is no nontrivial action of G on $M^n \setminus Z$.*

Proof. By way of contradiction, suppose that there were such an action $A : G \times M^n \setminus Z \to M^n \setminus Z$. Then for each $g \in G$, the associated homeomorphism $g(x) = A(g, x) : M^n \setminus Z \to M^n \setminus Z$ extends to a homeomorphism on M^n. By Theorem 5.1, we have a continuous non-trivial action $\bar{A} : G \times M^n \to M^n$. Under this action, Z is invariant. However, since G is connected, the action on Z must be the identity. Since Z is dense in M^n, the action must be the identity on M^n. We have reached a contradiction with the assumption that we began with a non-trivial action of G on $M^n \setminus Z$. □

Corollary 6.3. *We can conclude that there are no non-trivial flows on $M^n \setminus Z$ whenever Z is a dense 0-dimensional subset of M^n.*

Acknowledgments. The authors wish to thank Sasha Dranishnikov for many helpful conversations about the subjects in this paper. They also wish to thank the referee of the paper for a very thorough analysis of the paper, some valuable corrections, and many helpful improvements.

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