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by

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THE SET FUNCTION $\mathcal{T}_a$

LEOBARDO FERNÁNDEZ

Abstract. In 1948, F. Burton Jones defined the set function $\mathcal{T}$ on the power set of a continuum in the following way: Given a subset A of a continuum X, $\mathcal{T}(A)$ is the set of points x in X such that x can not be separated from A by a continuum neighborhood, this is, x is in $\mathcal{T}(A)$ if for each subcontinuum W of X containing x in its interior, we have that $W \cap A \neq \emptyset$. In this paper, we define the set function $\mathcal{T}_a$ on the power set of a continuum X as follows: Given a subset A of X, $\mathcal{T}_a(A)$ is the set of points x in X such that x can not be separated from A by an arcwise connected continuum neighborhood. Properties of this set function are presented.

1. Introduction

In 1948, F. Burton Jones defined the set function $\mathcal{T}$ [9, Theorem 3] on a continuum. He defined this set function just for points and it was called L. Later, Robert P. Hunter changed its name to a set function $\mathcal{T}$ [8], and in [5] the set function $\mathcal{T}$ is generalized to the power set of a continuum. Since then, the set function $\mathcal{T}$ has been studied, it is possible to give characterizations of locally connected continua, indecomposable continua and aposyndetic continua using this function. Now we introduce the set function $\mathcal{T}_a$ in a similar way as Jones's set function $\mathcal{T}$ was defined, the main difference is that $\mathcal{T}_a$ requires that the continuum on which it is defined is arcwise connected. After preliminaries in section 2, in section 3 we define the set function $\mathcal{T}_a$ and give examples and basic properties. In section 4 we study the $\mathcal{T}_a$-additivity, $\mathcal{T}_a$-symmetry, and idempotency. In

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section 5 we focus on uniquely arcwise connected continua. In section 6 we study cyclically connected continua. In section 7 we review continuity of the set function $\mathcal{T}_a$ and in section 8 we combine set function $\mathcal{T}_a$ with continuous functions.

2. Preliminaries

A *continuum* is a nonempty compact connected metric space. A continuum X is *locally connected at a point* $p \in X$ if, for every open set U containing p, there exists an open connected set V such that $p \in V \subseteq U$ and the continuum X is *locally connected* if it is locally connected at each of its points. The continuum X is *aposyndetic at* p *with respect to* q if there exists a subcontinuum W of X such that $p \in Int(W)$ and $q \notin W$. We say that a continuum X is *aposyndetic* at a point $p \in X$ if it is aposyndetic at p with respect to any point in $X \setminus \{p\}$ and a continuum is *aposyndetic* if it is aposyndetic at each of its points. We say that a continuum X is *arcwise connected* if, for every pair of its points, there exists an arc joining them in X and it is *hereditarily arcwise connected* if every subcontinuum of X is arcwise connected. The *power set* of a topological space X is the set $\mathcal{P}(X) = \{A\colon A \text{ is a subset of } X\}$. Given a continuum X, we define $2^X = \{A \in \mathcal{P}(X)\colon A \text{ is nonempty and closed}\}$ and $C(X) = \{A \in 2^X\colon A$ is connected$\}$. The set 2^X is called the *hyperspace of closed nonempty subsets of* X and the set $C(X)$ is called the *hyperspace of subcontinua of* X. The topology on the hyperspaces 2^X and $C(X)$ is the Vietoris topology.

3. Definition and General Properties of the Set Function $\mathcal{T}_a$

We begin this section with the definition of the set function $\mathcal{T}_a$.

Definition 3.1. Let X be an arcwise connected continuum. We define the *set function* $\mathcal{T}_a : \mathcal{P}(\mathrm{X}) \to \mathcal{P}(\mathrm{X})$ as follows: for each $A \in \mathcal{P}(X)$, $\mathcal{T}_a(A) = \{x \in X\colon$ if $W \in C(X)$ is arcwise connected and $x \in Int(W)$, then $W \cap A \neq \emptyset\}$.

Sometimes it is better to work with complements, so the definition of the set function $\mathcal{T}_a$ is equivalent to $\mathcal{T}_a(A) = X \setminus \{x \in X\colon$ there is an arcwise connected continuum W such that $x \in Int(W) \subseteq W \subseteq X \setminus A\}$, for each $A \in \mathcal{P}(X)$.

Remark 3.2. By the definition of $\mathcal{T}_a$, as the continuum X is arcwise connected, we can see easily that $\mathcal{T}_a(\emptyset) = \emptyset$ and for each $A \in \mathcal{P}(X)$, $A \subseteq \mathcal{T}_a(A)$ and $\mathcal{T}_a(A) \in 2^X$. Also, by definition, it is easy to see that if $A, B \in \mathcal{P}(X)$ are such that $A \subseteq B$, then $\mathcal{T}_a(A) \subseteq \mathcal{T}_a(B)$.

Recall that if X is a continuum, the set function $\mathcal{T} : \mathcal{P}(X) \to \mathcal{P}(X)$ is given by $\mathcal{T}(A) = X \setminus \{x \in X\colon$ there exists a continuum W such that $x \in Int(W) \subseteq W \subseteq X \setminus A\}$.

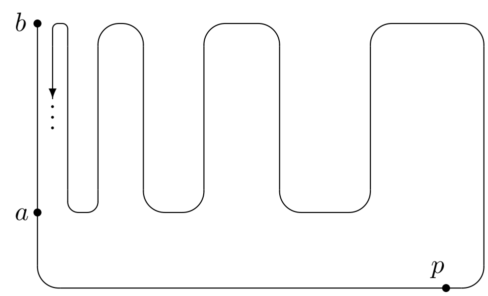
Remark 3.3. Note that, by the definitions of the set functions $\mathcal{T}$ and $\mathcal{T}_a$, we have that if X is an arcwise connected continuum, then $A \subseteq \mathcal{T}(A) \subseteq \mathcal{T}_a(A)$ for each $A \in \mathcal{P}(X)$. In general, the equality does not hold; see Example 3.5.

As a consequence of the definitions of $\mathcal{T}$ and $\mathcal{T}_a$, we have the following theorem.

Theorem 3.4. *Let X be an arcwise connected continuum. If X is hereditarily arcwise connected, then $\mathcal{T}(A) = \mathcal{T}_a(A)$ for each $A \in \mathcal{P}(X)$.*

It is known that for each $W \in C(X)$ we have $\mathcal{T}(W) \in C(X)$. Example 3.5 shows that $\mathcal{T}_a(W)$ is not necessarily connected, even for singletons. Example 3.5 also shows that, in general, $\mathcal{T}(A) \neq \mathcal{T}_a(A)$.

Example 3.5. Let X be the Warsaw circle with limit bar the segment ab. We can see that $\mathcal{T}(\{p\}) = \{p\}$; meanwhile, for every $x \in X$, $\mathcal{T}_a(\{x\}) = \{x\} \cup ab$.



Remark 3.6. A continuum X is indecomposable if and only if $\mathcal{T}(A) = X$ for every $A \in 2^X$ [10, Theorem 3.1.34]. A natural question is What happens for the set function $\mathcal{T}_a$; i.e., is there an arcwise connected continuum X such that $\mathcal{T}_a(A) = X$ for each $A \in 2^X$? In [1] an arcwise connected continuum is constructed such that every arcwise connected proper subcontinuum of X has empty interior. This continuum shows that there are continua for which the set function $\mathcal{T}_a$ is a constant map.

Definition 3.7. Let X be an arcwise connected continuum and let $p \in X$. We say that X is *arcwise connected im kleinen at p* if, for every open set U of X containing p, there exists an arcwise connected continuum neighborhood W of p such that $W \subseteq U$.

Theorem 3.8. *Let X be an arcwise connected continuum. Then X is arcwise connected im kleinen at each of its points if and only if X is locally connected.*

Proof. Suppose that X is arcwise connected im kleinen at each of its points. By definition, X is connected im kleinen at each of its points; thus, by [10, 1.7.12], X is locally connected.

Now suppose that X is locally connected. By [12, 0.65.4], we may assume that X has a convex metric. Let p be a point of X and let U be an open subset of X such that $p \in U$. Let $r > 0$ be such that $\mathcal{V}_r(p) \subseteq U$. As a consequence of [12, 0.65.3] and the fact that X has convex metric, $Cl(\mathcal{V}_{\frac{r}{2}}(p))$ is a subcontinuum of X such that $p \in \mathcal{V}_{\frac{r}{2}}(p) \subseteq Cl(\mathcal{V}_{\frac{r}{2}}(p)) \subseteq U$. Thus, X is arcwise connected im kleinen at p. Since p was arbitrary, X is arcwise connected im kleinen at each of its points. □

Theorem 3.9. *Let X be an arcwise connected continuum. Then X is locally connected if and only if $\mathcal{T}_a(A) = A$ for each $A \in 2^X$.*

Proof. Assume that X is locally connected and let $A \in 2^X$. Since X is locally connected we may also assume that X has a convex metric. By Remark 3.2, $A \subseteq \mathcal{T}_a(A)$. Let $x \in X \setminus A$. Since $X \setminus A$ is an open set and X is regular, there exists an open set U of X such that $x \in U \subseteq Cl(U) \subseteq X \setminus A$. By Theorem 3.8, X is arcwise connected im kleinen. Hence, there exists an arcwise connected neighborhood whose closure is an arcwise connected subcontinuum W (we are assuming that X has a convex metric) of X such that $x \in Int(W) \subseteq W \subseteq U$. Since $Cl(U) \cap A \subseteq (X \setminus A) \cap A = \emptyset$, then $W \subseteq X \setminus A$. Hence, $x \in X \setminus \mathcal{T}_a(A)$. Therefore, $\mathcal{T}_a(A) = A$.

Now suppose that $\mathcal{T}_a(A) = A$ for each $A \in 2^X$. Let $p \in X$ and let U be an open set of X containing p. Since $(X \setminus U) \in 2^X$, then $\mathcal{T}_a(X \setminus U) = X \setminus U$ and $p \notin \mathcal{T}_a(X \setminus U)$. Thus, there exists an arcwise connected subcontinuum W of X such that $p \in Int(W) \subseteq W \subseteq X \setminus (X \setminus U)$. This implies that $p \in Int(W) \subseteq W \subseteq U$. Hence, X is arcwise connected im kleinen at p. Since p was an arbitrary element of X, by Theorem 3.8, X is locally connected. □

Definition 3.10. Let X be an arcwise connected continuum and let $p, q \in X$. We say that X is *arcwise aposyndetic at p with respect to q* if there exists an arcwise connected subcontinuum W of X such that $p \in int(W) \subseteq W \subseteq X \setminus \{q\}$. We say that X is *arcwise aposyndetic at p* if X is arcwise aposyndetic at p with respect to each point of $X \setminus \{p\}$. Finally, we say that X is *arcwise aposyndetic* if it is arcwise aposyndetic at each of its points.

Recall that a *pseudo-arc* is a hereditarily indecomposable continuum such that, for every $\varepsilon > 0$, there exists a surjective map $f\colon X \to [0,1]$ such that $\operatorname{diam}(f^{-1}(t)) < \varepsilon$ for all $t \in [0,1]$.

Example 3.11. The cone over the pseudo-arc is an example of an arcwise connected continuum which is aposyndetic but it is not arcwise aposyndetic.

Theorem 3.12. *Let X be an arcwise connected continuum and let A be a subset of X. If $x \in X \setminus \mathcal{T}_a(A)$, then X is arcwise aposyndetic at x with respect to each point of A.*

Proof. Let $x \in X \setminus \mathcal{T}_a(A)$ and let $a \in A$. Then there exists an arcwise connected subcontinuum W of X such that $x \in Int(W) \subseteq W \subseteq X \setminus A$. Since $X \setminus A \subseteq X \setminus \{a\}$, W is a subcontinuum of X such that $x \in Int(W) \subseteq W \subseteq X \setminus \{a\}$. Therefore, X is arcwise aposyndetic at x with respect to a. □

Theorem 3.13. *Let X be an arcwise connected continuum. Then X is arcwise aposyndetic if and only if $\mathcal{T}_a(\{p\}) = \{p\}$ for each $p \in X$.*

Proof. Assume that X is arcwise aposyndetic. By Remark 3.2, we have that $\{p\} \subseteq \mathcal{T}_a(\{p\})$. Let $q \in X \setminus \{p\}$. Since X is arcwise aposyndetic, there exists an arcwise connected continuum W such that $q \in Int(W) \subseteq W \subseteq X \setminus \{p\}$. This implies that $q \in X \setminus \mathcal{T}_a(\{p\})$. Therefore, $\mathcal{T}_a(\{p\}) = \{p\}$.

The converse is a consequence of the Theorem 3.12. □

Definition 3.14. Let X be an arcwise connected continuum and let $p \in X$. We say that X is *strongly semilocally arcwise connected at* p if, for every open subset U of X containing p, there is an open subset V of X such that $p \in V \subseteq U$ and $X \setminus V$ is the union of a finite number of mutually disjoint arcwise connected continua. We say X is *strongly semilocally arcwise connected* if X is strongly semilocally arcwise connected at p for each $p \in X$.

Theorem 3.15. *Let X be an arcwise connected continuum and let $x \in X$. Then X is strongly semilocally arcwise connected at x if and only if $\mathcal{T}_a(\{x\}) = \{x\}$.*

Proof. Let X be an arcwise connected continuum and let $x \in X$. Suppose that X is strongly semilocally arcwise connected at x, and let $y \in X \setminus \{x\}$. Let U be an open set such that $x \in U$ and $y \notin Cl(U)$. Since X is strongly semilocally arcwise connected at x, there is an open set V such that $x \in V \subseteq U$ and $X \setminus V$ is the union of a finite number of arcwise connected continua. Since $y \in X \setminus V$, the component of y in $X \setminus V$ is an arcwise connected subcontinuum W of X such that $y \in Int(W) \subseteq W \subseteq X \setminus V \subseteq X \setminus \{x\}$. This implies that $\mathcal{T}_a(\{x\}) = \{x\}$.

Now suppose that $\mathcal{T}_a(\{x\}) = \{x\}$ for each $x \in X$. Let U be an open set containing x. For each $y \in X \setminus U$ there is an arcwise connected subcontinuum W_y such that $y \in Int(W_y) \subseteq W_y \subseteq X \setminus \{x\}$. The family $\{Int(W_y) \colon y \in X \setminus U\}$ is an open cover of $X \setminus U$. Since $X \setminus U$ is compact, there exist $y_1, y_2, ..., y_n \in X \setminus U$ such that $X \setminus U \subseteq \bigcup_{i=1}^{n} W_{y_i}$. Let $V = X \setminus \bigcup W_{i=1}^{n}$. Then $X \setminus V$ is the union of a finite number of arcwise connected subcontinua of X. Therefore, X is strongly semilocally arcwise connected in x. □

As a consequence of theorems 3.13 and 3.15 we have the following corollary.

Corollary 3.16. *X is strongly semilocally arcwise connected if and only if X is arcwise aposyndetic.*

Definition 3.17. Let X be a continuum and let $p \in X$. We say that X is *semilocally connected at p* provided that, for each open subset U of X such that $p \in U$, there exists an open subset V of X such that $p \in V \subseteq U$ and $X \setminus V$ has a finite number of components. We say X is *semilocally connected* if X is semilocally connected at each of its points.

Definition 3.18. Let X be an arcwise continuum and let $p \in X$. We say that X is *semilocally arcwise connected at p* provided that, for each open subset U of X such that $p \in U$, there exists an open subset V of X such that $p \in V \subseteq U$ and $X \setminus V$ has a finite number of arc components. We say X is *semilocally arcwise connected* if X is semilocally arcwise connected at each of its points.

Question 3.19. Example 3.5 shows that the concepts of semilocally arcwise connected at p and strongly semilocally arcwise connected at p are different. It follows from the definitions that every strongly semilocally arcwise connected continuum is semilocally arcwise connected. Is the other implication always true?

Definition 3.20. Let X be an arcwise connected continuum and let $p \in X$. We say that X is *colocally arcwise connected at p* if, for every open subset U of X containing p, there is an open subset V of X such that $p \in V \subseteq U$ and $X \setminus V$ is arcwise connected. We say X is *colocally arcwise connected* if X is colocally arcwise connected at p for each $p \in X$.

Remark 3.21. It follows from the definitions that if X is an arcwise connected, semilocally arcwise connected continuum, then X is semilocally connected.

Theorem 3.22. *If X is a strongly semilocally arcwise connected continuum without cut points, then X is colocally arcwise connected.*

Proof. Let X be an arcwise connected, strongly semilocally arcwise connected continuum without cut points. Since X is strongly semilocally arcwise connected without cut points, by Remark 3.21, X is semilocally connected without cut points. It follows from [14, Corollary 6.22] that, for each $x \in X$, there is an open set U of X such that $x \in U$ and $X \setminus U$ is connected. Since X is strongly semilocally arcwise connected, there is an open set V of X such that $x \in V \subseteq U$ and $X \setminus V$ is the union of a finite number of arcwise connected subcontinua. Let $K_1, K_2, ..., K_n$ be arcwise connected subcontinua of X such that $X \setminus V = \bigcup_{i=1}^{n} K_i$. Then $X \setminus U \subseteq K_j$ for some $j \in \{1, 2, ..., n\}$; otherwise, we would have that $(X \setminus U) \cap K_i \neq \emptyset$ and $(X \setminus U) \cap K_j \neq \emptyset$, where $i \neq j$. Hence, $X \setminus U = [(X \setminus U) \cap K_i] \bigcup [(X \setminus U) \cap \bigcup_{l \neq i} K_l]$. But this contradicts the fact that $X \setminus U$ is connected. Thus, $X \setminus U \subseteq K_j$ for some $j \in \{1, 2, ..., n\}$. Then $X \setminus K_j$ is an open set such that $x \in X \setminus K_j \subseteq U$ whose complement K_j is arcwise connected. □

As a consequence of theorems 3.15 and 3.22, we have the following corollary.

Corollary 3.23. *Let X be an arcwise connected continuum without cut points and let $x \in X$. Then X is colocally arcwise connected at x if and only if $\mathcal{T}_a(\{x\}) = \{x\}$.*

4. $\mathcal{T}_a$-Additivity and $\mathcal{T}_a$-Symmetry

Now we are going to define when a continuum is $\mathcal{T}_a$-additive and $\mathcal{T}_a$-symmetric.

Definition 4.1. Let X be an arcwise connected continuum. We say that X is *$\mathcal{T}_a$-additive* if, for every pair of closed subsets of X, A, and B, we have that $\mathcal{T}_a(A \cup B) = \mathcal{T}_a(A) \cup \mathcal{T}_a(B)$.

Theorem 4.2. *Let X be an arcwise connected continuum. If, for every point $x \in X$ and for every pair of arcwise connected subcontinua W_1 and W_2 of X such that $x \in Int(W_j)$, $j \in \{1, 2\}$, there is an arcwise connected subcontinuum W_3 of X such that $x \in Int(W_3)$ and $W_3 \subseteq W_1 \cap W_2$, then X is $\mathcal{T}_a$-additive.*

Proof. Let A_1 and A_2 be two closed subsets of X. Since $\mathcal{T}_a(A_1) \cup \mathcal{T}_a(A_2) \subseteq \mathcal{T}_a(A_1 \cup A_2)$, we need to show that $\mathcal{T}_a(A_1 \cup A_2) \subseteq \mathcal{T}_a(A_1) \cup \mathcal{T}_a(A_2)$. For this, let x be a point in $X \setminus (\mathcal{T}_a(A_1) \cup \mathcal{T}_a(A_2))$. Then $x \in (X \setminus \mathcal{T}_a(A_1)) \cap (X \setminus \mathcal{T}_a(A_2))$. It follows that there are two arcwise connected subcontinua W_1 and W_2 of X such that $x \in Int(W_j) \subseteq W_j \subseteq (X \setminus A_j)$, $j \in \{1, 2\}$. By hypothesis, there is an arcwise connected subcontinuum W_3 of X such that $x \in Int(W_3) \subseteq W_3 \subseteq (W_1 \cap W_2)$ and $(W_1 \cap W_2) \subseteq X \setminus (A_1 \cup A_2)$. Therefore, $x \in X \setminus (\mathcal{T}_a(A_1 \cup A_2))$. □

Definition 4.3. Let X be a continuum. A *filterbase* $\mathcal{F}$ in X is a family $\mathcal{F} = \{A_\omega\}_{\omega\in\Omega}$ of subsets of X such that for each $\omega \in \Omega$, $A_\omega \neq \emptyset$, and for each pair $\omega_1, \omega_2 \in \Omega$, there exists $\omega_3 \in \Omega$ such that $A_{\omega_3} \subseteq A_{\omega_1} \cap A_{\omega_2}$.

Lemma 4.4. *Let X be an arcwise connected continuum. If $\mathcal{F}$ is a filterbase of closed sets of X, then $\mathcal{T}_a(\bigcap\{G\colon G \in \mathcal{F}\}) = \bigcap\{\mathcal{T}_a(G)\colon G \in \mathcal{F}\}$.*

Proof. We know, by Remark 3.2, that $\mathcal{T}_a(\bigcap\{G\colon G \in \mathcal{F}\}) \subseteq \bigcap\{\mathcal{T}_a(G)\colon G \in \mathcal{F}\}$. Let x be a point of $X \setminus \mathcal{T}_a(\bigcap\{G\colon G \in \mathcal{F}\})$. Then there is an arcwise connected subcontinuum W of X such that $x \in Int(W) \subseteq W \subseteq X \setminus (\bigcap\{G\colon G \in \mathcal{F}\})$. Since W is compact, there exist $G_1, G_2, ..., G_n$ in $\mathcal{F}$ such that $W \subseteq X \setminus (\bigcap_{i=1}^n G_i)$. Then $W \cap (\bigcap_{i=1}^n G_i) = \emptyset$. Since $\mathcal{F}$ is a filterbase, there is a closed set G in $\mathcal{F}$ such that $G \subseteq \bigcap_{i=1}^n G_i$. This implies that $W \cap G = \emptyset$. Thus, $p \in X \setminus \mathcal{T}_a(G)$. Therefore, $p \in X \setminus \bigcap\{\mathcal{T}_a(G)\colon G \in \mathcal{F}\}$. □

Theorem 4.5. *Let X be an arcwise connected continuum. Then X is $\mathcal{T}_a$-additive if and only if, for each family Λ of closed subsets of X whose union is closed in X, we have that $\mathcal{T}_a(\bigcup\{L\colon L \in \Lambda\}) = \bigcup\{\mathcal{T}_a(L)\colon L \in \Lambda\}$.*

Proof. Suppose that X is $\mathcal{T}_a$-additive. By Remark 3.2, we know that $\bigcup\{\mathcal{T}_a(L)\colon L \in \Lambda\} \subseteq \mathcal{T}_a(\bigcup\{L\colon L \in \Lambda\})$. Let x be a point of $X \setminus (\bigcup\{\mathcal{T}_a(L)\colon L \in \Lambda\})$. For each $L \in \Lambda$, let $F(L) = \{A \subseteq X\colon A$ is closed and $L \subseteq Int(A)\}$.

Case 1: If $L = \emptyset$, then $\emptyset \in F(L)$. Since $\mathcal{T}_a(\emptyset) = \emptyset$, we have that $\emptyset = \mathcal{T}_a(L) = \bigcap\{\mathcal{T}_a(A)\colon A \in F(L)\}$.

Case 2: If $L \neq \emptyset$, then $F(L)$ is a filterbase of closed subsets of X. Since $\bigcap\{A\colon A \in F(L)\} = L$, then $\mathcal{T}_a(L) = \mathcal{T}_a(\bigcap\{A\colon A \in F(L)\})$. By Lemma 4.4, $\mathcal{T}_a(L) = \bigcap\{\mathcal{T}_a(A)\colon A \in F(L)\}$. Then, for each $L \in \Lambda$, $x \in X \setminus (\bigcap\{\mathcal{T}_a(A)\colon A \in F(L)\})$. Thus, for each $L \in \Lambda$, there exists $A_L \in F(L)$ such that $x \in X \setminus \mathcal{T}_a(A_L)$.

Now $\{Int(A_L)\colon L \in \Lambda\}$ is an open cover of $\bigcap\{L\colon L \in \Lambda\}$. Since $\bigcap\{L\colon L \in \Lambda\}$ is compact, there exist $L_1, L_2, ..., L_m \in \Lambda$ such that $\bigcap\{L\colon L \in \Lambda\} \subseteq \bigcup_{j=1}^m Int(A_{L_j})$. Then, by hypothesis and mathematical induction, we have $\mathcal{T}_a(\bigcup_{j=1}^m A_{L_j}) = \bigcup_{j=1}^m \mathcal{T}_a(A_{L_j})$. Hence, $\mathcal{T}_a(\bigcup\{L\colon L \in \Lambda\}) \subseteq \bigcup_{j=1}^m \mathcal{T}_a(A_{L_j})$. Since for every $j \in \{1, 2, ..., m\}$, $x \in X \setminus \mathcal{T}_a(A_{L_j})$, it follows that $x \in X \setminus \mathcal{T}_a(\bigcup\{L\colon L \in \Lambda\})$. Thus, we have that $\mathcal{T}_a(\bigcup\{L\colon L \in \Lambda\}) \subseteq \bigcup\{\mathcal{T}_a(L)\colon L \in \Lambda\}$.

The reverse implication is clear. □

Corollary 4.6. *Let X be an arcwise connected, $\mathcal{T}_a$-additive continuum. If K is a subcontinuum of X such that $\mathcal{T}_a(\{x\})$ is connected for each $x \in K$, then $\mathcal{T}_a(K)$ is a subcontinuum of X.*

Proof. Let X be an arcwise connected, $\mathcal{T}_a$-additive continuum, and let K be a subcontinuum of X such that $\mathcal{T}_a(\{x\})$ is connected for each $x \in K$. By hypothesis and Theorem 4.5, $\mathcal{T}_a(K) = \mathcal{T}_a(\bigcup_{x\in K}\{x\}) = \bigcup_{x\in K}\mathcal{T}_a(\{x\})$. Since $K \cup \mathcal{T}_a(\{x\})$ is connected for each $x \in K$, then $\bigcup_{x\in K}(K \cup \mathcal{T}_a(\{x\}))$ is connected. But then, since $K \subseteq \bigcup_{x\in K}\mathcal{T}_a(\{x\})$, we have $\bigcup_{x\in K}(K \cup \mathcal{T}_a(\{x\})) = K \cup (\bigcup_{x\in K}\mathcal{T}_a(\{x\})) = \bigcup_{x\in K}\mathcal{T}_a(\{x\}) = \mathcal{T}_a(K)$, which proves that $\mathcal{T}_a(K)$ is connected. □

Definition 4.7. Let X be an arcwise connected continuum. We say that X is $\mathcal{T}_a$-*symmetric* if for every pair of closed subsets A and B of X, such that $A \cap \mathcal{T}_a(B) = \emptyset$, then $\mathcal{T}_a(A) \cap B = \emptyset$. We say that X is *point* $\mathcal{T}_a$-*symmetric* if for every pair of points x and y, if $x \notin \mathcal{T}_a(\{y\})$, then $y \notin \mathcal{T}_a(\{x\})$.

Theorem 4.8. *Let X be an arcwise connected continuum. If X is $\mathcal{T}_a$-symmetric, then X is $\mathcal{T}_a$-additive.*

Proof. Let X be an arcwise connected, $\mathcal{T}_a$-symmetric continuum, and let A and B be closed subsets of X. Since $\mathcal{T}_a(A) \cup \mathcal{T}_a(B) \subseteq \mathcal{T}_a(A \cup B)$, we need to prove that $\mathcal{T}_a(A \cup B) \subseteq \mathcal{T}_a(A) \cup \mathcal{T}_a(B)$. Let $x \in \mathcal{T}_a(A \cup B)$. Then $\{x\} \cap \mathcal{T}_a(A \cup B) \neq \emptyset$. Since X is $\mathcal{T}_a$-symmetric, $\mathcal{T}_a(\{x\}) \cap (A \cup B) \neq \emptyset$. Thus, either $\mathcal{T}_a(\{x\}) \cap A \neq \emptyset$ or $\mathcal{T}_a(\{x\}) \cap B \neq \emptyset$. Without lost of generality, assume that $\mathcal{T}_a(\{x\}) \cap A \neq \emptyset$. Since X is $\mathcal{T}_a$-symmetric, $\{x\} \cap \mathcal{T}_a(A) \neq \emptyset$, which implies that $x \in \mathcal{T}_a(A)$. Hence, $x \in \mathcal{T}_a(A) \cup \mathcal{T}_a(B)$. Therefore, X is $\mathcal{T}_a$-additive. □

Theorem 4.9. *Let X be an arcwise connected continuum such that $\mathcal{T}_a(A) = A$ for each A in 2^X, then X is both $\mathcal{T}_a$-additive and $\mathcal{T}_a$-symmetric.*

Proof. Suppose that $\mathcal{T}_a(A) = A$ for each closed subset A of X. Then $\mathcal{T}_a(A \cup B) = A \cup B = \mathcal{T}_a(A) \cup \mathcal{T}_a(B)$. Thus, X is $\mathcal{T}_a$-additive. If $A \cap \mathcal{T}_a(B) = \emptyset$, then $A \cap B = \emptyset$ and $\mathcal{T}_a(A) \cap B = \emptyset$. Thus, X is $\mathcal{T}_a$-symmetric. □

Theorem 4.10. *Let X be an arcwise connected continuum such that $\mathcal{T}_a(A) = X$ for each A in 2^X, then X is both $\mathcal{T}_a$-additive and $\mathcal{T}_a$-symmetric.*

Proof. Suppose that $\mathcal{T}_a(A) = X$ for each closed subset A of X. Then $\mathcal{T}_a(A \cup B) = X = \mathcal{T}_a(A) \cup \mathcal{T}_a(B)$; thus, X is $\mathcal{T}_a$-additive. Finally, for each pair of closed subsets A and B, $A \cap \mathcal{T}_a(B) \neq \emptyset$ and $\mathcal{T}_a(A) \cap B \neq \emptyset$. Therefore, X is $\mathcal{T}_a$-symmetric. □

Theorem 4.11. *Let X be an arcwise connected continuum. Then X is $\mathcal{T}_a$-additive and point $\mathcal{T}_a$-symmetric if and only if X is $\mathcal{T}_a$-symmetric.*

Proof. Let X be an arcwise connected continuum, $\mathcal{T}_a$-additive, and point $\mathcal{T}_a$-symmetric. Let A and B be two closed subsets of X such that $A \cap \mathcal{T}_a(B) = \emptyset$. Then, for every $x \in A$ and for every $y \in B$, $x \notin \mathcal{T}_a(\{y\})$. Since

X is point $\mathcal{T}_a$-symmetric, we have that $y \notin \mathcal{T}_a(\{x\})$ for every $x \in A$ and for every $y \in B$. Now, since X is $\mathcal{T}_a$-additive, $\mathcal{T}_a(A) = \mathcal{T}_a(\bigcup_{x\in A}\{x\}) = \bigcup_{x\in A}\mathcal{T}_a(\{x\})$ (Theorem 4.5). Thus, $y \notin \mathcal{T}_a(A)$ for each $y \in B$, which implies that $B \cap \mathcal{T}_a(A) = \emptyset$. Therefore, X is $\mathcal{T}_a$-symmetric.

The other implication follows from Theorem 4.8. □

Definition 4.12. Let X be an arcwise connected continuum. We say that $\mathcal{T}_a$ is *idempotent on* X if $\mathcal{T}_a^2(A) = \mathcal{T}_a(A)$ for each subset A of X, where $\mathcal{T}_a^2(A) = \mathcal{T}_a(\mathcal{T}_a(A))$.

Theorem 4.13. *Let X be an arcwise connected continuum. Then $\mathcal{T}_a$ is idempotent on X if and only if, for each arcwise connected subcontinuum W of X and for each $x \in Int(W)$, there exists an arcwise connected subcontinuum M of X such that $x \in Int(M) \subseteq M \subseteq Int(W)$.*

Proof. Assume that $\mathcal{T}_a$ is idempotent on X. Let W be an arcwise connected subcontinuum of X and let $x \in Int(W)$. Then $x \in X \setminus \mathcal{T}_a(X \setminus W)$. Since $\mathcal{T}_a$ is idempotent on X, $x \in X \setminus \mathcal{T}_a^2(X \setminus W)$. This implies that there exists an arcwise connected subcontinuum M of X such that $x \in Int(M) \subseteq M \subseteq X \setminus \mathcal{T}_a(X \setminus W) \subseteq X \setminus (X \setminus W) = W$. Hence, $x \in Int(M) \subseteq M \subseteq Int(W)$.

Now suppose that for each arcwise connected subcontinuum W of X and for each $x \in Int(W)$ there exists an arcwise connected subcontinuum M of X such that $x \in Int(M) \subseteq M \subseteq Int(W)$. Let $B \in \mathcal{P}(X)$. By Remark 3.2, we have that $\mathcal{T}_a(B) \subseteq \mathcal{T}_a^2(B)$. Let $x \in X \setminus \mathcal{T}_a(B)$. Then there exists an arcwise connected subcontinuum W of X such that $x \in Int(W) \subseteq W \subseteq X \setminus B$. By hypothesis, there exists an arcwise connected subcontinuum M of X such that $x \in Int(M) \subseteq M \subseteq Int(W)$. Since $Int(W) \cap \mathcal{T}_a(B) = \emptyset$, $x \in Int(M) \subseteq M \subseteq X \setminus \mathcal{T}_a(B)$. Thus, $x \in X \setminus \mathcal{T}_a^2(B)$. Therefore, $\mathcal{T}_a$ is idempotent. □

5. Uniquely Arcwise Connected Continua

In this section we are going to prove that in dendroids which are not dendrites there always exists a point p for which $\mathcal{T}_a(\{p\}) \neq \{p\}$.

Definition 5.1. Let X be an arcwise connected continuum. We say that X is *uniquely arcwise connected* if it does not contain simple closed curves.

Theorem 5.2. *Let X be a uniquely arcwise connected continuum and let $x \in X$. If M and N are arcwise connected subcontinua of X such that $x \in Int(M) \cap Int(N)$, then $M \cap N$ is an arcwise connected subcontinuum of X such that $x \in Int(M \cap N)$.*

Proof. Let X be an arcwise connected continuum and let $x \in X$. Let M and N be arcwise connected subcontinua of X such that $x \in Int(M) \cap$

$Int(N)$. Clearly, $x \in Int(M \cap N)$. To see that $M \cap N$ is arcwise connected, let $x, y \in M \cap N$. Since M is arcwise connected, there is an arc α from x to y in M. Since N is arcwise connected, there is an arc β from x to y in N. By hypothesis, X is uniquely arcwise connected, which implies that $\alpha = \beta$. Thus, $M \cap N$ is arcwise connected. □

As a consequence of Theorem 4.2 and Theorem 5.2, we have the following corollary.

Corollary 5.3. *Let X be a uniquely arcwise connected continuum, then X is $\mathcal{T}_a$-additive.*

Remark 5.4. Using Corollary 5.3, we see that the converse of Theorem 4.8 is not true. Since the Warsaw circle (Example 3.5) is uniquely arcwise connected, it is $\mathcal{T}_a$-additive. If x is a point on the limit bar and y is a point which is not in the limit bar, it is easy to see that $x \in \mathcal{T}_a(\{y\})$, but $y \notin \mathcal{T}_a(\{x\})$.

As a consequence of Corollary 4.6 and Corollary 5.3, we obtain the following corollary.

Corollary 5.5. *Let X be a uniquely arcwise connected continuum and let K be a subcontinuum of X. If $\mathcal{T}_a(\{x\})$ is connected for every $x \in K$, then $\mathcal{T}_a(K)$ is a subcontinuum of X.*

As a particular case of uniquely arcwise connected continua, we have the dendroids.

Recall that a continuum X is *unicoherent* provided that whenever $X = A \cup B$, where A and B are closed and connected subsets of X, then $A \cap B$ is connected. The continuum X is *hereditarily unicoherent* if each subcontinuum of it is unicoherent. A *dendroid* is an arcwise connected, hereditarily unicoherent continuum.

Theorem 5.6. *Let X be a dendroid. If X is not locally connected, then there exists $p \in X$ such that $\mathcal{T}_a(\{p\}) \neq \{p\}$.*

Proof. Let X be a dendroid. Since X is not locally connected, there is a point p of X such that X is not connected im kleinen at p. Then there exists $\delta > 0$ such that if V is a neighborhood of p and $V \subseteq \mathcal{V}_\delta(p)$, V is not connected. Let V be a neighborhood of p such that $p \in V \subseteq Cl(V) \subseteq \mathcal{V}_\delta(p)$, and let B the component of $Cl(V)$ containing p; hence, by [15, p. 18, Theorem 12.1], we have that

(i) there is a sequence $\{A_n\}_{n=1}^{\infty}$ of components of $Cl(V)$, and there is a subcontinuum A of B containing p such that $\lim_{n\to\infty} A_n = A$, and if $j \neq k$, $A_j \cap A_k = \emptyset$,

(ii) $p \in A$,

(iii) if $x \in A$, then X is not connected im kleinen at x,
(iv) $A \subseteq \mathcal{V}_\delta(p)$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{V}_\delta(p)$ such that $x_n \in A_n$ and $\lim_{n\to\infty} x_n = p$. Let $q \in A \setminus \{p\}$, and let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{V}_\delta(p)$ such that $y_n \in A_n$ and $\lim_{n\to\infty} y_n = q$. Let α_i be the arc joining x_i to p. If there are infinitely many arcs α_i containing q, then $p \in \mathcal{T}_a(\{q\})$ because if $p \notin \mathcal{T}_a(\{q\})$, there exists an arcwise connected subcontinuum W of X such that $p \in Int(W)$ and $q \notin W$. Since W is arcwise connected, X is a dendroid, and $p \in Int(W)$, there is an $N \in \mathbb{N}$ such that the arc α_i is contained in W for each $i \geq N$. But, since $q \notin W$, this implies that there are just finitely many arcs α_i containing q, which is a contradiction. Hence, $p \in \mathcal{T}_a(\{q\})$, and thus $\mathcal{T}_a(\{q\}) \neq \{q\}$. If there are just finitely many arcs α_i containing q, then, since X is uniquely arcwise connected, there are infinitely many arcs β_i joining y_i to q containing p. Hence, $q \in \mathcal{T}_a(\{p\})$, and thus $\mathcal{T}_a(\{p\}) \neq \{p\}$. □

Remark 5.7. Using the fact that in dendroids (in general, in hereditarily arcwise connected continua) the set functions $\mathcal{T}$ and $\mathcal{T}_a$ coincide, Theorem 5.6 implies also that if X is a dendroid which is not locally connected, then there is a point $p \in X$ such that $\mathcal{T}(\{p\}) \neq \{p\}$.

6. Cyclicly Connected Continua

Theorem 6.2 and Example 6.3 show that in every cyclicly connected continuum X, $\mathcal{T}_a(\{x\})$ is connected for every point x in X, but not necessarily for subcontinua.

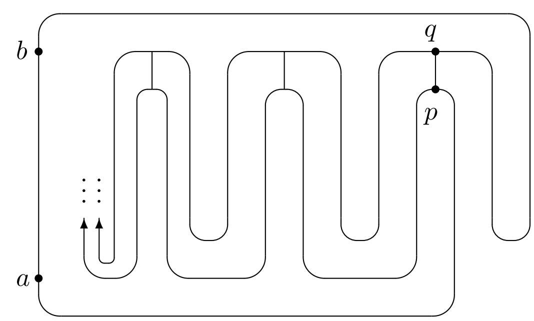
Definition 6.1. Let X be a continuum. We say that X is *cyclicly connected* if every pair of points of X is contained in a simple closed curve.

Theorem 6.2. *Let X be a cyclicly connected continuum. Then $\mathcal{T}_a(\{x\})$ is connected for each $x \in X$.*

Proof. Let X be a cyclicly connected continuum and let $x \in X$. Assume that $\mathcal{T}_a(\{x\})$ is not connected. Then there exist two nonempty disjoint closed subsets A and B of X such that $\mathcal{T}_a(\{x\}) = A \cup B$ and suppose that $x \in A$. Let U be an open subset of X such that $A \subseteq U$ and $Cl(U) \cap B = \emptyset$. Then $Bd(U) \cap \mathcal{T}_a(\{x\}) = \emptyset$, and thus, for each $z \in Bd(U)$, there exists an arcwise connected subcontinuum K_z of X such that $z \in Int(K_z) \subseteq K_z \subseteq X \setminus \{x\}$. Since $Bd(U)$ is compact, there exist $z_1, z_2, ..., z_n \in Bd(U)$ such that $Bd(U) \subseteq \bigcup_{i=1}^{n} Int(K_{z_i}) \subseteq \bigcup_{i=1}^{n} K_{z_i}$. Let $V = U \setminus (\bigcup_{i=1}^{n} K_{z_i})$ and let $Y = X \setminus V = (X \setminus U) \cup (\bigcup_{i=1}^{n} K_{z_i})$. By [13, Theorem 5.4], Y has a finite number of components. Note that $B \subseteq X \setminus Cl(U) \subseteq X \setminus U \subseteq X \setminus V = Y$, in other words, $B \subseteq Int(Y)$. Let $b \in B$ and let C be the component of Y containing b. Then C is a subcontinuum of X such that $b \in Int(C) \subseteq C \subseteq X \setminus \{x\}$. If C is arcwise connected, then $b \notin \mathcal{T}_a(\{x\})$, which is a contradiction. Suppose that C is not arcwise connected. We may assume that $K_{z_i} \cap C \neq \emptyset$ if $1 \leq i \leq l$ and $K_{z_i} \cap C = \emptyset$ if $i > l$. Let $D = C \cup \bigcup_{i=1}^{l} K_{z_i}$. Then D is a subcontinuum of X such that $b \in Int(D)$. Since X is cyclicly connected, $X \setminus \{x\}$ is arcwise connected. For each $i \in \{1, 2, ..., l\}$, let α_i be an arc in $X \setminus \{x\}$ from b to a point in K_{z_i}. Let $E = D \cup \bigcup_{i=1}^{l} \alpha_i$. Since X is arcwise connected, each arcwise component of C intersects some K_{z_i} for some $i \in \{1, 2, ..., l\}$. Then E is an arcwise connected subcontinuum of X such that $b \in Int(E) \subseteq E \subseteq X \setminus \{x\}$. But this implies that $b \notin \mathcal{T}_a(\{x\})$, which is a contradiction. Therefore, $\mathcal{T}_a(\{x\})$ is connected. □

The following example shows that if X is cyclicly connected and W is a subcontinuum of X, then $\mathcal{T}_a(W)$ is not necessarily connected.

Example 6.3. Consider the cyclicly connected continuum X as the continuum which appears in [4, Figure 1]. Let J be the limit bar ab and let W be the segment pq. We can see that $\mathcal{T}_a(W) = W \cup J$, which is not connected.



Since every arcwise connected homogeneous continuum is cyclicly connected [4], we have the following results.

Theorem 6.4. *Every arcwise connected homogeneous continuum is arcwise aposyndetic.*

Proof. Let X be an arcwise connected homogeneous continuum. By [3, p. 280, Theorem], we have that X is colocally arcwise connected. Since X is homogeneous, X has no cut points, and, by Corollary 3.23, we have that $\mathcal{T}_a(\{x\}) = \{x\}$ for each $x \in X$. □

Theorem 6.5. *Let X be an arcwise connected homogeneous continuum. Then X is $\mathcal{T}_a$-additive if and only if X is locally connected.*

Proof. If X is $\mathcal{T}_a$-additive and $A \in 2^X$, then $\mathcal{T}_a(A) = \mathcal{T}_a(\bigcup_{x\in A}\{x\}) = \bigcup_{x\in A}\mathcal{T}_a(\{x\})$. By Theorem 6.4, $\bigcup_{x\in A}\mathcal{T}_a(\{x\}) = \bigcup_{x\in A}\{x\} = A$. Thus, $\mathcal{T}_a(A) = A$. Then, by Theorem 3.9, X is locally connected.

If X is locally connected and $A, B \in 2^X$, then $A \cup B$ is closed. Again, by Theorem 3.9, $\mathcal{T}_a(A \cup B) = A \cup B = \mathcal{T}_a(A) \cup \mathcal{T}_a(B)$. Therefore, X is $\mathcal{T}_a$-additive. □

7. Continuity of $\mathcal{T}_a$

Let us recall the definitions of an upper semicontinuous and lower semicontinuous function. Let X be a continuum, let $F : \mathcal{P}(X) \to \mathcal{P}(X)$ be a set function, and let U be an open set of X. If the set $\{A \in 2^X : F(A) \subseteq U\}$ is open in 2^X, then we say that F is *upper semicontinuous*. If the set $\{A \in 2^X : F(A) \cap U \neq \emptyset\}$ is open in 2^X, then we say that F is *lower semicontinuous*. Finally, we say that the set function F is *continuous* if it is both upper and lower semicontinuous.

As an immediate consequence of Theorem 3.9, we have that $\mathcal{T}_a$ is continuous for every locally connected continuum.

Theorem 7.1. *The set function $\mathcal{T}_a$ is upper semicontinuous.*

Proof. Let U be an open set of X and let $\mathcal{U} = \{A \in 2^X : \mathcal{T}_a(A) \subseteq U\}$. By definition, we need to prove that $\mathcal{U}$ is open. Let $B \in Cl(2^X \setminus \mathcal{U})$. There is a sequence $\{B_n\}_{n=1}^{\infty} \subseteq 2^X \setminus \mathcal{U}$ such that $\lim_{n\to\infty} B_n = B$. For each $n \in \mathbb{N}$, $\mathcal{T}_a(B_n) \cap (X \setminus U) \neq \emptyset$. Let $x_n \in \mathcal{T}_a(B_n) \cap X \setminus U$. We assume, without loss of generality, that $\lim_{n\to\infty} x_n = x$ for $x \in X$. In fact, $x \in X \setminus U$. We claim that $x \in \mathcal{T}_a(B)$. Suppose $x \notin \mathcal{T}_a(B)$. Then there exists an arcwise connected subcontinuum W of X such that $x \in Int(W) \subseteq W \subseteq X \setminus B$. Since $\lim_{n\to\infty} B_n = B$ and $\lim_{n\to\infty} x_n = x$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, $B_n \subseteq X \setminus W$ and $x_n \in Int(W)$. Let $n \geq N$. Then $x_n \in Int(W) \subseteq W \subseteq X \setminus B_n$. This implies that $x_n \in X \setminus \mathcal{T}_a(B_n)$, which is a contradiction. Hence, $x \in \mathcal{T}_a(B) \cap (X \setminus U)$, and thus $B \in 2^X \setminus \mathcal{U}$. Therefore, $2^X \setminus \mathcal{U}$ is closed and $\mathcal{U}$ is open. □

Theorem 7.2. *Let X be an arcwise connected, $\mathcal{T}_a$-additive continuum. If $\mathcal{T}_a$ is continuous on $F_1(X)$, then $\mathcal{T}_a$ is continuous on 2^X.*

Proof. By Theorem 7.1, it suffices to prove that $\mathcal{T}_a$ is lower semicontinuous. Let U be an open set of X. We define $\mathcal{U} = \{A \in 2^X : \mathcal{T}_a(A) \cap U \neq \emptyset\}$. By definition, we need to prove that $\mathcal{U}$ is open in 2^X. Let $B \in Cl(2^X \setminus \mathcal{U})$, and let $\{B_n\}_{n=1}^{\infty} \subseteq 2^X \setminus \mathcal{U}$ be a sequence such that $\lim_{n\to\infty} B_n = B$. Then $\mathcal{T}_a(B_n) \cap U = \emptyset$ for each $n \in \mathbb{N}$. Suppose that $B \notin 2^X \setminus \mathcal{U}$, then $\mathcal{T}_a(B) \cap U \neq \emptyset$. Let $y \in \mathcal{T}_a(B) \cap U$. Since X is $\mathcal{T}_a$-additive, then $\mathcal{T}_a(B) = \bigcup_{x\in B}\mathcal{T}_a(\{x\})$ and $y \in \mathcal{T}_a(\{x\})$ for some $x \in B$. Since

$\lim_{n\to\infty} B_n = B$, there exists $x_n \in B_n$ such that $\lim_{n\to\infty} x_n = x$. By hypothesis, $\mathcal{T}_a$ is continuous on $F_1(X)$. Then $\lim_{n\to\infty} \mathcal{T}_a(\{x_n\}) = \mathcal{T}_a(\{x\})$. Hence, there exists $y_n \in \mathcal{T}_a(\{x_n\})$ such that $\lim_{n\to\infty} y_n = y$. Since $y \in U$, there exists $N \in \mathbb{N}$ such that $y_n \in U$ for each $n \geq N$. This implies that $\mathcal{T}_a(\{x_n\}) \cap U \neq \emptyset$, for $x_n \in B_n$ and $n \geq N$. Then $\mathcal{T}_a(B_n) \cap U \neq \emptyset$ for $n \geq N$ which contradicts the fact that $\mathcal{T}_a(B_n) \cap U = \emptyset$. Thus, $B \in 2^X \setminus \mathcal{U}$ and $\mathcal{U}$ is open in 2^X. Therefore, $\mathcal{T}_a$ is lower semicontinuous in X. □

Recall that a *generalized Warsaw circle* is a continuum which is a compactification of the ray $[0, \infty)$, having as a remainder a non-degenerate compact interval, say $[a, b]$, and joining with an arc the points 0 and a.

Theorem 7.3. *The set function $\mathcal{T}_a$ is continuous on the generalized Warsaw circle.*

Proof. Let X be the Warsaw Circle with limit bar J. We know that X is uniquely arcwise connected. Then, by Corollary 5.3, X is $\mathcal{T}_a$-additive. Thus, for each $A \in 2^X$, $\mathcal{T}_a(A) = \mathcal{T}_a\big(\bigcup_{x\in A}\{x\}\big)$. By Theorem 4.5, we have that $\mathcal{T}_a\big(\bigcup_{x\in A}\{x\}\big) = \bigcup_{x\in A}\mathcal{T}_a(\{x\})$. On the other hand, $\bigcup_{x\in A}\mathcal{T}_a(\{x\}) = \bigcup_{x\in A}(J \cup \{x\}) = J \cup \bigcup_{x\in A}\{x\} = J \cup A$. Thus, $\mathcal{T}_a(A) = J \cup A$.

Now let $\{A_n\}_{n=1}^\infty \subseteq 2^X$ be a sequence such that $\{A_n\}_{n=1}^\infty$ converges to A. Then

$$\lim_{n\to\infty} \mathcal{T}_a(A_n) = \lim_{n\to\infty}(J \cup A_n) = J \cup \left(\lim_{n\to\infty} A_n\right) = J \cup A = \mathcal{T}_a(A).$$

Therefore, $\mathcal{T}_a$ is continuous on X. □

8. $\mathcal{T}_a$ and Continuous Functions

We will give some relations between continuous functions and the set function $\mathcal{T}_a$. We will let ${}_X\mathcal{T}_a$ denote $\mathcal{T}_a\colon \mathcal{P}(X) \to \mathcal{P}(X)$ and ${}_Y\mathcal{T}_a$ denote $\mathcal{T}_a\colon \mathcal{P}(Y) \to \mathcal{P}(Y)$

The proof of the following theorem is similar to the proof of the same result for the set function $\mathcal{T}$ which appears in [10, Theorem 3.1.64] for which we omit its proof.

Theorem 8.1. *Let X and Y be arcwise connected continua and let $f : X \to Y$ be an onto map. Then ${}_Y\mathcal{T}_a(B) \subseteq f({}_X\mathcal{T}_a(f^{-1}(B)))$ for each $B \in 2^Y$.*

Theorem 8.2. *Let X be an arcwise connected continuum. If $h : X \to X$ is a homeomorphism, then $h(\mathcal{T}_a(A)) = \mathcal{T}_a(h(A))$ for each closed subset A of X.*

Proof. Let A be a closed subset of X and let $x \in X \setminus h(\mathcal{T}_a(A))$. Then $h^{-1}(x) \in X \setminus \mathcal{T}_a(A)$. Thus, there exists an arcwise connected subcontinuum W of X such that $h^{-1}(x) \in Int(W) \subseteq W \subseteq X \setminus A$. This implies that $x \in Int(h(W)) \subseteq h(W) \subseteq X \setminus h(A)$. Therefore, $x \in X \setminus \mathcal{T}_a(h(A))$.

Now let $x \in X \setminus \mathcal{T}_a(h(A))$. Then there exists an arcwise connected subcontinuum W of X such that $x \in Int(W) \subseteq W \subseteq (X \setminus h(A))$. Hence, $h^{-1}(x) \in h^{-1}(Int(W)) \subseteq h^{-1}(W) \subseteq h^{-1}(X \setminus h(A))$. This implies that $h^{-1}(x) \in Int(h^{-1}(W)) \subseteq h^{-1}(W) \subseteq (X \setminus A)$. Thus, $h^{-1}(x) \in X \setminus \mathcal{T}_a(A)$. Therefore, $x \in h(X \setminus \mathcal{T}_a(A))$. □

Theorem 8.3. *Let X and Y be arcwise connected continua and let $f : X \to Y$ be an onto map. If $f^{-1}(W)$ is arcwise connected for every arcwise connected subcontinuum W of Y, then $f({}_X\mathcal{T}_a(f^{-1}(B))) \subseteq {}_Y\mathcal{T}_a(B)$ for each $B \in 2^Y$.*

Proof. Let $B \in 2^X$ and let $y \in Y \setminus {}_Y\mathcal{T}_a(B)$. Then there exists an arcwise connected subcontinuum W of Y such that $y \in Int(W) \subseteq W \subseteq Y \setminus B$. Note that $f^{-1}(W) \cap f^{-1}(B) = \emptyset$. Since f is continuous and $f^{-1}(y)$ is compact, $f^{-1}(y) \subseteq Int(f^{-1}(W))$. Then, for every $x \in f^{-1}(y)$, $x \notin {}_X\mathcal{T}_a(f^{-1}(B))$. Thus, $f^{-1}(y) \cap {}_X\mathcal{T}_a(f^{-1}(B)) = \emptyset$. If $f(x) \in f({}_X\mathcal{T}_a(f^{-1}(B)))$, then $f^{-1}(y) \cap {}_X\mathcal{T}_a(f^{-1}(B)) \neq \emptyset$, which is a contradiction. Thus, $y \in Y \setminus f({}_X\mathcal{T}_a(f^{-1}(B)))$. Therefore, $f({}_X\mathcal{T}_a(f^{-1}(B))) \subseteq {}_Y\mathcal{T}_a(B)$. □

As a consequence of Theorem 8.1 and Theorem 8.3, we have the following corollaries.

Corollary 8.4. *Let X and Y be arcwise connected continua and let $f : X \to Y$ be an open onto map. If $f^{-1}(W)$ is arcwise connected for every arcwise connected subcontinuum W of Y, then $f({}_X\mathcal{T}_a(f^{-1}(B))) = {}_Y\mathcal{T}_a(B)$ for each $B \in 2^Y$.*

Corollary 8.5. *Let X and Y be arcwise connected continua and let $f : X \to Y$ be an onto map. If X is hereditarily arcwise connected, then $f({}_X\mathcal{T}_a(f^{-1}(B))) \subseteq {}_Y\mathcal{T}_a(B)$ for each $B \in 2^Y$.*

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