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by

Jane Hawkins

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Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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TOPOLOGICAL DYNAMICS OF DIANALYTIC MAPS ON KLEIN SURFACES

JANE HAWKINS

Abstract. We study analytic maps of the sphere, the torus, and the punctured plane with extra symmetries; namely, they project to well-defined maps on corresponding nonorientable Klein surfaces. We define Julia and Fatou sets on Klein surfaces and discuss their topological dynamical properties; in particular, we construct new examples where the Julia set is the entire Klein surface. We characterize examples with Julia set the full surface for Klein bottles and construct the first such examples on the real punctured projective plane, as well as discuss new examples on the real projective plane. The examples show that topologically and measure theoretically complex behavior occurs on these surfaces.

1. Introduction

This paper combines topological and complex dynamics to produce new examples of maps of some nonorientable surfaces that are as close to analytic as is possible. This refers to both the surface structure and the map in the sense that the surfaces are doubly covered by connected orientable Riemann surfaces and each map has an analytic lifting to one on its Riemann surface double cover. Up to now there have been few examples given of Julia sets on nonorientable surfaces; the only examples we are aware of are on the real projective plane, denoted by $\mathbb{RP}^2$, were studied by the author with Sue Goodman in [9]. We recently learned there

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are also examples of Julia sets on $\mathbb{RP}^2$ in unpublished work in progress by Araceli Bonifant, Xavier Buff, and John Milnor [7] with topological properties different from the examples in this paper. In this paper we extend work in [9] to compute explicit examples of Julia sets on additional nonorientable surfaces; namely the Klein bottle and the punctured projective plane. We also provide new examples on $\mathbb{RP}^2$ as well. The maps we focus on are topologically exact and transitive, and have a dense set of repelling periodic points. On each surface of interest we specify a Riemannian metric, which in turn defines a probability measure; we denote the measure by m and discuss some measure theoretic properties of the maps with respect to m. The examples we construct in this paper show that imposing extra symmetry on analytic maps does not restrict topological or measure theoretical dynamical mixing behavior.

Given an open set $\Omega \subset \mathbb{C}$, a map $f : \Omega \to \mathbb{C}$ is called *dianalytic on* Ω if its restriction $f|V$ to any component V of Ω satisfies either $\partial(f|V) = 0$ or $\overline{\partial}(f|V) = 0$, i.e., f is either holomorphic or anti-holomorphic on each connected component of Ω. The domain for the maps considered in this paper is a Klein surface S, which is a surface with an atlas $\mathcal{A} = \{U_i, \varphi_i\}_{i\in\mathbb{N}}$ such that all overlap maps $\varphi_i \circ \varphi_j^{-1}$ are holomorphic or anti-holomorphic on their respective domains. We call this a *dianalytic structure* on S. This definition of Klein surface is from the book of Norman L. Alling and Newcomb Greenleaf [1], though others appear in the literature. All Riemann surfaces are Klein surfaces, but Klein surfaces are more general and include surfaces with boundary and nonorientable surfaces.

We say S is a *nonorientable Klein surface* if in addition to the dianalytic structure there exists a connected surface X which is an orientable double cover of S, and a fixed point free anti-holomorphic involution φ of X such that S is dianalytically conjugate to the quotient surface $X/[\varphi]$. The orientable double covering space X is uniquely determined up to a conformal mapping. The focus of this paper is to construct dynamically interesting maps on nonorientable Klein surfaces.

Notions of equicontinuity pass from a Riemann surface to a nonorientable Klein surface with an appropriate choice of Riemannian metric, so Fatou and Julia sets are well-defined concepts on Klein surfaces. The maps we focus on in this paper have Julia set the entire surface S because we are interested in maximizing the topological mixing behavior of the dynamics. Therefore, our attention is restricted to the genus 0 and 1 case for the double cover, by Montel's theorem; this was pointed out explicitly by Hans Rådström in 1953 [16]. Once we know that the set of normality is empty for each set of examples constructed, we obtain interesting topological dynamical information as a consequence of classical properties of Julia sets.

The paper is organized as follows. In section 1.1 we give the definitions of Julia and Fatou sets and the dynamical properties of interest. Section 2 begins with a review of work in [9] and adds to it with two results: namely, Proposition 2.4, where new examples are given, and Theorem 2.5, where limit sets are discussed. In section 3 we construct the first explicit examples of dianalytic maps of the punctured real projective plane with Julia set the entire surface; we show that there are many such examples and discuss their dynamical behavior. In section 4 we give the complete picture for the dianalytic maps on the Klein bottle and discuss the Julia sets and ergodicity. We also take the opportunity to add to statements in [4], where some of the maps appearing in this paper were discussed quite generally, but neither their dynamical properties nor their Julia sets were studied.

1.1. Definitions and notation.

Let $\mathbb{C}_\infty$ denote the Riemann sphere, and let $\Omega \subseteq \mathbb{C}_\infty$ be some domain (an open connected set). We consider analytic maps $H : \Omega \to \Omega$, and let $H^n = H \circ H \circ \cdots \circ H$ denote the n-fold composition. We put a metric d on Ω (usually the spherical metric), and consider the family of maps on Ω, $\mathcal{F} = \{H^n\}_{n\geq 1}$; we say $\mathcal{F}$ is *equicontinuous* at $z \in \Omega$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(\omega, z) < \delta \Rightarrow d(H^n(\omega), H^n(z)) < \epsilon \;\forall n \in \mathbb{N}.$$

Definition 1.1. We define the *Fatou set* of H, by

$$F(H) = \{z \in \Omega \,:\, \mathcal{F} \text{ is equicontinuous at } z\}.$$

The *Julia set* of H is its complement in Ω:

$$J(H) = \Omega \setminus F(H).$$

In this paper we consider only the following three cases for Ω (and H):

(1) $\Omega = \mathbb{C}_\infty$; in this case, then $H = p/q$, with p and q polynomials, a *rational function*. We assume $\deg(H) \geq 2$.
(2) If $\Omega = \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C}$, we use the Euclidean metric for d in this case and $H = az + b$, $a \in \mathbb{C}^*, b \in \mathbb{C}$, and assume $|a| > 1$.
(3) If $\Omega = \mathbb{C}_\infty \setminus \{0, \infty\}$, then $H(z) = z^n e^{f(z)+g(1/z)}$ with f and g entire functions on $\mathbb{C}$, often called a *Rådström function* (see e.g., [12]).

These properties of Julia sets are well established; most of them date back to Fatou. For a discussion of the history and for proofs of these results in this more general setting, see [8], [10], or [12].

Proposition 1.2. *Under the standing hypotheses above for $H : \Omega \to \Omega$, the following hold.*

(1) *$J(H)$ is a nonempty perfect set.*
(2) *$J(H^k) = J(H)$ for all $n \in \mathbb{N}$.*
(3) *$J(H)$ is completely invariant, i.e., $H^{-1}(J(H)) = J(H)$.*
(4) *$J(H)$ is the closure of the set of repelling points, i.e.,*

$$J(H) = cl\{z \in \Omega : H^p(z) = z \, and \, |(H^p)'(z)| > 1\},$$

where cl denotes the topological closure in Ω.
(5) *If $K \subset \Omega$ is compact, given any neighborhood U of a point $z \in J(H)$, there exists an integer n_0 such that $K \subset H^{n_0}(U)$.*

In the case where $\Omega = \mathbb{C}_\infty \setminus \{3 \text{ or more points}\}$, Ω is a hyperbolic surface and, using the Poincaré metric on Ω, it is a classical result that $J(H) = \emptyset$ always holds (see [16] or [15] for a statement and proof).

Proofs of the statements in Theorem 1.3 when S is compact can be found, for example, in [5] or [14]; they follow from Proposition 1.2 as well. Most were known already to Fatou on the sphere.

Theorem 1.3. *Suppose that $H : \Omega \to \Omega$ is analytic and Ω is of type (1)-(3) above. Assume that $J(H) = \Omega$, and that H induces a dianalytic map $\hat{H} : S \to S$ on a Klein surface S. Then $J(\hat{H}) = S$, and the following hold:*

(1) *$\hat{H}$ is* topologically exact *in the sense that if $U \subset S$ is a nonempty open set, then $\cup_{n\geq 0}\hat{H}^n(U) = \mathbb{C}^*$.*
(2) *$\hat{H}$ is* topologically transitive*: There exists some $z \in S$ such that $\mathcal{O}^+(z) = \cup_{k\geq 0}\hat{H}^k(z)$ is dense in S.*
(3) *Periodic points are dense in S.*
(4) *Every point $z \in S$ satisfies $\mathcal{O}^-(z) = \{w : \exists k \geq 0 : \hat{H}^k(w) = z\} = \cup_{k\geq 0}\hat{H}^{-k}(z)$ is dense in S.*

Proof. These properties follow from the corresponding properties on the orientable double cover Ω and Proposition 1.2. □

When S is compact, we define the *ω-limit set* of $z \in S$ for a continuous map $H : S \to S$ by

$$\omega(z) = \{y \in S \,|\, \exists n_i \to \infty \text{ with } H^{n_i}(z) \to y\}.$$

If S is not compact, it can occur (and does for some of our examples below) that $\omega(z)$ is empty for many $z \in S$.

2. The Real Projective Plane

In this section we add to the results known about dynamics of dianalytic maps of the real projective plane, which were studied by the author and Sue Goodman in [9] as well as by Milnor [14] and Ilie Barza and Dorin Ghisa [4]. A wide variety of dynamical behavior occurs and here we increase the understanding of what occurs in parameter space for some families of mappings. We obtain dianalytic maps of the real projective plane as follows. The Riemann sphere $\mathbb{C}_\infty$ is an orientable double cover of $\mathbb{RP}^2$, via the anti-holomorphic involution $\varphi(z) = -1/\overline{z}$, which has no fixed points. This induces an equivalence relation on $\mathbb{C}_\infty$ with each equivalence class containing exactly 2 points; we take the quotient by the relation $\sim_\varphi$, and this gives $\mathbb{RP}^2$ the structure of a nonorientable Klein surface [4]. We use $\mathfrak{p}$ to denote the quotient map from $\mathbb{C}_\infty$ to $\mathbb{RP}^2$ using the relation $\sim_\varphi$. To simplify notation we will write $\mathbb{RP}^2 \cong \mathbb{C}_\infty/[\varphi]$, with " $\cong$ " meaning dianalytically conjugate, to denote this relationship. Maps on $\mathbb{RP}^2$ that lift via $\mathfrak{p}$ to analytic maps of the double cover $\mathbb{C}_\infty$ are dianalytic. In [9, Theorem 3.1], the author and Goodman showed the following result.

Theorem 2.1. *Every rational map of $\mathbb{C}_\infty$ of (odd) degree $n \geq 3$ with exactly two distinct critical points which induces a dianalytic map of $\mathbb{RP}^2$, is conformally conjugate to exactly one of the form*

$$h_\alpha(z) = \frac{z^n + \alpha}{-\overline{\alpha} z^n + 1}, \quad \arg(\alpha) \in [0, \pi/(n-1)]. \tag{2.1}$$

We denote the induced dianalytic map on $\mathbb{RP}^2$ by $\hat{h}_\alpha$. In [9] it is shown that a wide variety of dynamics occurs as the parameter value α varies in the space described in the theorem; in Figure 1 (see page 345), we show the parameter space, with regions of solid color corresponding to areas of stability in which there is an attracting periodic orbit. Darker shades represent higher periods.

If $\mathbb{R}_\infty$ denotes the one point compactification of real line (i.e., the equator of $\mathbb{C}_\infty$), we have the following.

Proposition 2.2 ([9], Proposition 5.8)**.** *Let $h_\alpha(z) = \frac{z^3+\alpha}{-\alpha z^3+1}$, $\alpha \in \mathbb{R}$, and set $h \equiv h_\alpha|_{\mathbb{R}_\infty}$. If $\rho(h)$, the rotation number of h, is irrational, then*

1. *there exists a homeomorphism $\varphi : \mathbb{R}_\infty \to S^1$ conjugating h to $T_{\rho(h)}$, irrational rotation on S^1; and*
2. *$J(h_\alpha) = \mathbb{C}_\infty$ and $J(\hat{h}_\alpha) = \mathbb{RP}^2$.*

While these maps satisfy the hypotheses of Theorem 1.3, the measure theoretic properties of such a map are difficult to determine since the equator of the sphere is a forward invariant rotation set. This is discussed in Theorem 2.5 below. We now turn to a different type of map in this family with Julia set all of $\mathbb{RP}^2$.

In the next few results we prove that we can obtain maps of the form (2.1) that are ergodic, exact, and conservative.

Theorem 2.3 ([5], Theorem 4.3.1)**.** *If every critical point of h_α is preperiodic but not periodic, then $J(h_\alpha) = \mathbb{C}_\infty$.*

We endow the Riemann sphere with the usual Borel structure and consider the probability measure m on $\mathbb{C}_\infty$ which is normalized surface area measure on $S^2 \cong \mathbb{C}_\infty$ where the conjugacy is implemented via stereographic projection. We say $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is *ergodic* if, whenever $h^{-1}(A) = A$ for a measurable set, we have $m(A) = 0$ or $m(A) = 1$. We say $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is *exact* if, whenever $h^{-n} \circ h^n(A) = A$ for a measurable set, and for all $n \in \mathbb{N}$, we have $m(A) = 0$ or $m(A) = 1$. Exact maps are ergodic but the converse does not necessarily hold. The map h is called *conservative* if, for all sets A of positive measure, there exists a $k \in \mathbb{N}$ such that $m(h^{-k}(A) \cap A) > 0$.

Proposition 2.4. *If the unique critical point of $\hat{h}_\alpha$ is pre-periodic but not periodic, then $J(\hat{h}_\alpha) = \mathbb{RP}^2$ and $\hat{h}_\alpha$ is ergodic, exact, and conservative with respect to the $\hat{m}$ on $\mathbb{RP}^2$ where $\hat{m}$ is the pullback measure from m. Moreover, these maps exist with liftings of the form (2.1).*

Proof. It is was shown in [2] that a critically finite map of $\mathbb{C}_\infty$ is ergodic and exact. If either of these properties were to fail for $\hat{h}_\alpha$, then it would fail for the lifting h_α as well.

To show that parameters α with this property exist, namely that the critical points are pre-periodic but not periodic (and obtain their approximate values), we use the following algorithm:

- We consider only the critical point 0, since the other critical point ∞ exhibits antipodal behavior.
- Since $h_\alpha(0) = \alpha$, then α and $-1/\overline{\alpha}$ are the only critical values.
- We show that $h_\alpha(\alpha) = \alpha$ has only $\alpha = 0$ as a solution. To see this, if we set $h_\alpha(\alpha) = \alpha$ and assume $\alpha \neq 0$ the equation reduces easily to $\alpha = -1/\overline{\alpha}$, but the involution has no fixed points. Therefore, the critical value is never fixed.
- For $k \geq 2$, we set $h_\alpha^{k+1}(\alpha) = h_\alpha^k(\alpha)$ and solve for α.

The Fundamental Theorem of Algebra guarantees there are solutions. Each solution represents a parameter α corresponding to a map h_α with the property that $h_\alpha^k(\alpha)$ is a (necessarily) repelling fixed point and then Theorem 2.3 implies the result. □

We show a few of the parameters satisfying the hypotheses of Proposition 2.4 using red (dark) dots in Figure 1; finding precise values for the parameters is difficult.

In Table 1 we show the approximate values of some parameters α such that h_α satisfies the hypothesis of Proposition 2.4. While the family of maps h_α does not vary homomorphically in α due to the presence of $\overline{\alpha}$ in Equation (2.1), the dependence of h_α on α is real analytic, and the dependence of fixed points on the parameter is smooth but quite

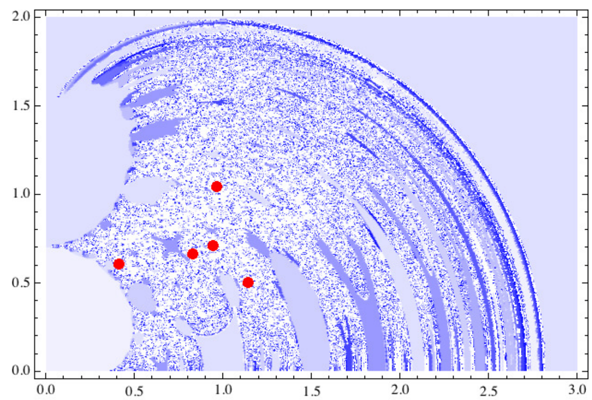

FIGURE 1. Each $\alpha = x + iy$ shown corresponds to a map of the form h_α; the solid regions are stable parts of parameter space. The red (dark) dots mark post critically finite parameters.

Value of k	Approximate α
2	$.94165 + .71130i$
3	$.96511 + 1.04128i$
4	$1.14318 + .49905i$
5	$.82911 + .66296i$
6	$.414056 + .60668i$

TABLE 1. The values of α with repelling fixed critical value $h_\alpha^k(\alpha) = h_\alpha^{k-1}(\alpha)$.

complicated. Therefore, it is likely that a result of Mary Rees [17], which was proved for analytic maps of the sphere, can be modified to show that there is a set of parameters of positive measure that correspond to maps that are ergodic and exact with respect to m (these sets would contain the parameters obtained in Proposition 2.4). However, we do not prove it here; rather, we conjecture that it is true.

2.1. The ω-limit sets.

For the maps h_α resulting from Proposition 2.2, a variety of ω-limit sets can occur. In particular, letting $\mathbb{R}_\infty$ denote the equator as above, for any $z \in B = \{\cup_{n\geq 1} h_\alpha^{-n}(\mathbb{R}_\infty)\}$, $\omega(z) = \mathbb{R}_\infty$; the set B projects to a dense set $\hat{B}$ in $\mathbb{RP}^2$, with the property that each point in $\hat{B}$ has a limit set a closed curve in $\mathbb{RP}^2$. We also have a dense set of periodic points whose ω-limit set is its finite forward orbit, and, in addition, there are points $z \in \mathbb{RP}^2$ such that $\omega(z) = \mathbb{RP}^2$. We set $\mathcal{E} = \mathfrak{P}(\mathbb{R}_\infty)$, the image of the equator under the antipodal map.

For the maps h_α resulting from Proposition 2.4, the last two cases of ω-limit sets occur. We summarize the possibilities in the next result.

Theorem 2.5. *For maps of the form* $h_\alpha(z) = \frac{z^3+\alpha}{-\bar{\alpha}z^3+1}$,

(1) *if h_α satisfies the hypotheses of Proposition 2.2, then for the corresponding map $\hat{h}_\alpha$; either*
   (a) $\omega(z) = \mathcal{E}$ *for m-a.e. point $z \in \mathbb{RP}^2$, or*
   (b) $\omega(z) = \mathbb{RP}^2$ *for m-a.e. point $z \in \mathbb{RP}^2$;*

(2) *if h_α satisfies the hypotheses of Proposition 2.4, then for the corresponding map $\hat{h}_\alpha$, $\omega(z) = \mathbb{RP}^2$ for m-a.e. point $z \in \mathbb{RP}^2$;*

(3) *for any map h_α, there is a dense set of points in $\mathbb{RP}^2$ such that for the map $\hat{h}_\alpha$, $\omega(z)$ is finite.*

Proof. If h_α satisfies the hypotheses of Proposition 2.2, then both critical points are real and recurrent and have a dense forward orbit in $\mathbb{R}_\infty$. We also have that $J(h_\alpha) = \mathbb{C}_\infty$. If we apply a result from ([6], Theorem 2.1), then either m-a.e. point is attracted to $\mathbb{R}_\infty$, or m-a.e. $z \in \mathbb{C}_\infty$ satisfies $\omega(z) = \mathbb{C}_\infty$. We project via the antipodal map to $\mathbb{RP}^2$ to obtain the result in (1).

If h_α is post-critically finite, then the result on the sphere is standard and appears in [2] and [6]. We project to $\mathbb{RP}^2$ to obtain the result in (2). Statement (3) follows from the fact that periodic points are dense in $J(\hat{h}_\alpha)$. □

3. The Punctured Projective Plane ${\mathbb{RP}^2}_*$

The dynamics of iterated analytic functions of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ have been studied by several authors, for example [8], [10], [11], [12], [13], and [16]. In this section we study a nonorientable Klein surface covered by $\mathbb{C}^*$, namely the punctured real projective plane which we denote by ${\mathbb{RP}^2}_*$. Properties of Julia sets for analytic maps on $\mathbb{C}^*$ have been considered in the papers listed above, but no examples have been computed or studied on the punctured real projective plane. The surface ${\mathbb{RP}^2}_*$ is the projective plane with a point removed; it has the structure of a nonorientable Klein surface with orientable double cover $\mathbb{C}^*$. We consider only analytic maps $H : \mathbb{C}^* \to \mathbb{C}^*$ with essential singularities at 0 and ∞ (using a Mobius map to place the essential singularities at 0 and ∞); otherwise we could just consider H as being rational or entire. By the commuting diagram, each such map lifts via the exponential map to an entire map of $\mathbb{C}$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{H}} & \mathbb{C} \\ \exp\downarrow & & \downarrow\exp \\ \mathbb{C}^* & \xrightarrow{H} & \mathbb{C}^* \end{array}$$

Therefore, $H(z)$ has the following form:

$$H(z) = z^n e^{f(z)+g(1/z)} \tag{3.1}$$

for n an integer, with f and g entire functions.

It is discussed in [4] and easy to check that if we regard $\mathbb{C}^* \cong \mathbb{C}_\infty \setminus \{0, \infty\}$, a dianalytic map on $\mathbb{RP}^2{}_*$ is induced by an analytic map $H : \mathbb{C}^* \to \mathbb{C}^*$ that commutes with the involution: $\varphi(z) = -1/\overline{z}$ as above.

Maps of $\mathbb{C}^*$ that commute with φ appear in [4]; here, we add to the discussion with the following result, proving some dynamical properties about some basic examples.

Proposition 3.1. (1) *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ and $q(z) = -\overline{a_0} + \overline{a_1} z - \overline{a_2} z^2 + \cdots + (-1)^{m+1} \overline{a_m} z^m$; then if $n \in \mathbb{Z}$ is odd,*

$$H(z) = z^n e^{p(z)+q(1/z)} \tag{3.2}$$

commutes with φ on $\mathbb{C}^*$.

(2) *If $n \in \mathbb{Z}$ is odd and $m \in \mathbb{N}$ is odd, $H_{n,m}(z) = z^n e^{z^m + z^{-m}}$ commutes with φ on $\mathbb{C}^*$.*

(3) *If $n \in \mathbb{Z}$ is odd and $m \in \mathbb{N}$ is even, $K_{n,m}(z) = z^n e^{z^m - z^{-m}}$ commutes with φ on $\mathbb{C}^*$.*

Proof. To show (1), we have that

$$H \circ \varphi(z) = (-1)^n \overline{z}^{-n} e^{p(-1/\overline{z})+q(-\overline{z})},$$

and

$$\varphi \circ H(z) = -\overline{z}^{-n} e^{-(\overline{p(z)}+\overline{q(1/z)})} = -\overline{z}^{-n} e^{-\overline{p(z)}-\overline{q(1/z)}}.$$

By our hypotheses, n is odd, so $(-1)^n \overline{z}^{-n} = -\overline{z}^{-n}$; and by our choice of q, $q(-\overline{z}) = -\overline{p(z)}$. It then follows, replacing z by $-1/\overline{z}$, that $p(-1/\overline{z}) = -\overline{q(-1/z)}$, from which (1) follows.

Both (2) and (3) are special cases of (1), with $p(z) = z^m$.

This proves the result. $\square$

Remark 3.2. (1) Consider functions of the form (3.2). The proof of Proposition 3.1 shows that n cannot be an even integer if H induces a well-defined dianalytic map on $\mathbb{RP}^2{}_*$. The integer n is sometimes called the degree of H.

(2) In Proposition 3.1(1), the form of the coefficients given for p and q in (3.2) extends to infinite series for entire functions (see [4]). In particular, all that is needed is that p and q be entire with $q(-\overline{z}) = -\overline{p(z)}$.

(3) If $a_0 = x + it$, where $x, t \in \mathbb{R}$, then $a_0 - \overline{a}_0 = 2it$, so any function of the form

$$H(z) = z^n e^{it} e^{p(z)+q(1/z)}, \tag{3.3}$$

with p and q entire, $p(0) = 0$ and $q(-\overline{z}) = -\overline{p(z)}$, using any $t \in \mathbb{R}$, will induce a dianalytic map on $\mathbb{RP}^2{}_*$.

Corollary 3.3. *Under the hypotheses of Proposition 3.1, all maps of the form (3.3) induce dianalytic maps of* $\mathbb{RP}^2{}_*$.

We describe the Julia sets and dynamics of the simplest examples: $H_{n,1}(z) = z^n e^{(z+1/z)}$.

Proposition 3.4. *If* $H_{n,1}(z) = z^n e^{(z+1/z)}$, *for* n *an odd integer, then* $J(\hat{H}) = \mathbb{RP}^2{}_*$.

Proof. It is enough to show that $J(H) = \mathbb{C}^*$ since the normality with respect to the spherical metric persists under the quotient map. Fix any odd integer (positive or negative) so that by Proposition 3.1 the map $g \equiv H_{n,1}$ induces a well-defined dianalytic map on $\mathbb{RP}^2{}_*$. We easily calculate:

- $g'(z) = e^{z+1/z} z^{n-2}(z^2 + nz - 1)$;
- there are two critical points in $\mathbb{C}^*$, both real, one negative and one positive, of the form $c_i = \dfrac{-n \pm \sqrt{n^2+4}}{2}$, $i = 1, 2$ with $c_1 = -1/c_2$;
- the calculus of g restricted to the real line shows that if c_1 is labelled to be the negative real critical point, then
$$\lim_{k\to\infty} g^k(c_1) = 0 \quad \text{and} \quad \lim_{k\to\infty} g^k(c_2) = \infty.$$

By the symmetry imposed on the maps, once we establish that one critical point iterates under g to 0, this forces the other critical point to iterate to ∞; in particular, the map $\hat{g} : \mathbb{RP}^2{}_* \to \mathbb{RP}^2{}_*$ is unicritical with the critical point c satisfying $\lim_{k\to\infty} \hat{g}^k(c) = [\infty]$. It was shown by J. Kotus ([12], Theorem 2) and independently by Linda Keen [10] and P. M. Makienko [13] that there are no Fatou components for maps of the punctured plane of the form (3.1) other than those occurring for rational maps. In particular, there are no wandering domains, and as is the case for rational maps, each of the periodic cycles of Fatou components require a critical point associated to it (as shown by [16]). Therefore, $J(g)$, and hence $J(\hat{g})$ is the entire surface. $\square$

In Figure 2 we show an approximation to the Julia set of a map on $\mathbb{RP}^2{}_*$, colored as follows. We color points yellow (light) if they're heading to 0 under iteration; the points colored blue (dark) iterate to ∞. The Julia set occurs at the interface between the two colors (see Theorem 3.5 below); under successive iterations a point will head towards 0 say under some number of iterations, then near 0, by the Big Picard Theorem, it gets "thrown" to an arbitrary value in the plane. After that it might head to ∞ (or 0 again), and the behavior at ∞ is identical. So the large blocks of color break up randomly near the omitted values.

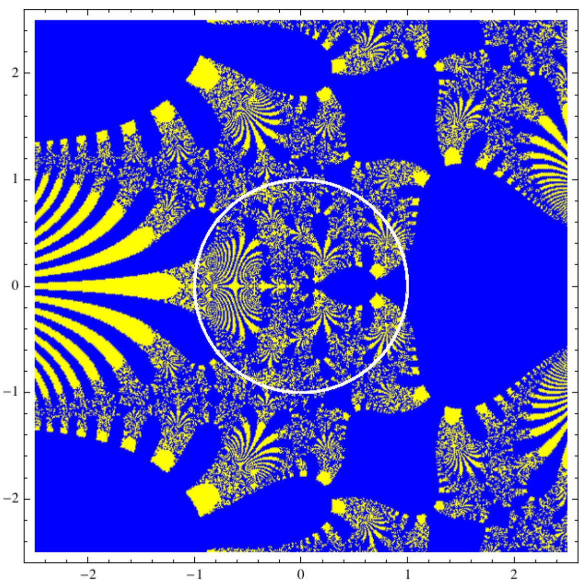

FIGURE 2. An approximation showing the structure of $J(g)$ for $g(z) = \frac{1}{z}e^{z+1/z}$, with the unit circle in white.

In [8] the following result was shown.

Theorem 3.5 ([8], Theorem 2)**.** *For any map of the form (3.2),*

$$J(g) = \partial\{z \in \mathbb{C}^* \mid \lim_{n\to\infty} g^n(z) = 0\} = \partial\{z \in \mathbb{C}^* \mid \lim_{n\to\infty} g^n(z) = \infty\}.$$

This leads to the following corollary.

Corollary 3.6. *If $H_{n,1}(z) = z^n e^{(z+1/z)}$, for n an odd integer, then there is a dense set of points in $\mathbb{RP}^2{}_*$ with an empty ω-limit set.*

Proof. The corollary holds since a dense set on $\mathbb{C}^*$, consisting of the preimages of the critical points, iterates to either 0 or ∞, and these project to a single point under the quotient map. Since this point is not in $\mathbb{RP}^2{}_*$, for any z in that dense set we have $\omega(z) = \emptyset$. $\square$

There are other rich dynamics in this family of maps obtained by varying the degrees and multiplying the standard form by a rotation (i.e., choosing $a_0 \neq 0$). In Figure 3 we show a Julia set for a map with one attracting period two orbit whose attracting basins are shown in white. The remaining colors show points iterating to 0 or ∞, or unresolved. When projected down onto $\mathbb{RP}^2{}_*$, the periodic orbit collapses to an attracting fixed point.

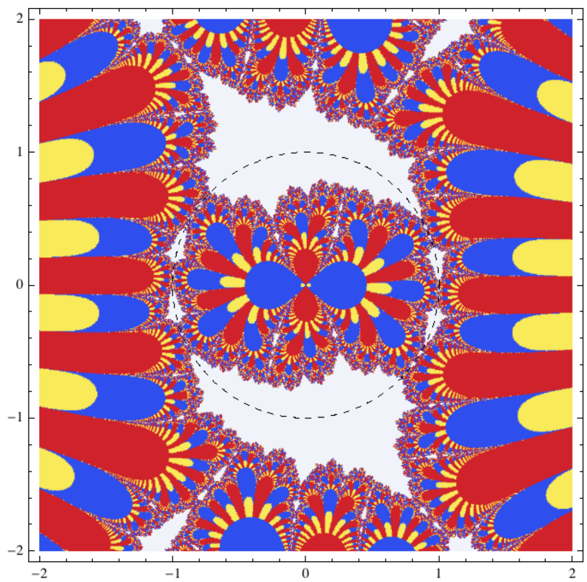

FIGURE 3. An approximation showing the structure of $J(f)$ for $f(z) = e^{.4i}z^3(e^{z^2-1/z^2})$, showing the attracting basin of a period two cycle in white. The antipodal symmetry with respect to the dotted black circle is evident.

4. The Klein Bottle

The case of the Klein bottle is different and simpler since the Klein bottle is a Klein surface which is double covered by the torus. We show that only a limited number of maps on tori give rise to dianalytic maps of the Klein bottle, but each is uniformly expanding and hence has the Julia set the entire surface. Due to the uniform expansion, ergodic properties are well understood in this setting.

We start with an analytic automorphism of $\mathbb{C}$, which is therefore of the form $F(z) = az + b$, with $a \neq 0$ and $b \in \mathbb{C}$. Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. We define a lattice of points in the complex plane by $\Lambda = [\lambda_1, \lambda_2]\colon = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\}$. Two different sets of vectors can generate the same lattice Λ; if $\Lambda = [\lambda_1, \lambda_2]$, then all other generators λ_3 and λ_4 of Λ are obtained by multiplying the vector (λ_1, λ_2) by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{ij} \in \mathbb{Z}$ and $a_{11}a_{22} - a_{12}a_{21} = 1$. We view Λ as a group acting on $\mathbb{C}$ by translation, each $\lambda \in \Lambda$ inducing a transformation of $\mathbb{C}$.

We consider here $\Lambda = [1, i\beta] = \{n + mi\beta \,:\, n, m \in \mathbb{Z},\, \beta > 1\}$, and we consider the map F as an induced map on $\mathbb{C}/\Lambda$. For F to be well defined, we need $F(\Lambda) \subseteq \Lambda$. Therefore, we assume that $a \in \mathbb{N}$, though in this generality it is not necessary, but sufficient (see e.g., [15, Problem 6-a]). We are interested in maps on the Klein bottle $\mathbb{K}$, which we view as a Klein surface so we need the following condition: Given the maps $T(z) = z + i\beta$ and $C(z) = \overline{z} + 1/2$, we need F to commute with both maps on $\mathbb{T} = \mathbb{C}/\Lambda$. We note that $C \circ C(z) = z + 1$, which is an involution on $\mathbb{C}/\Lambda$.

If $F \circ T(z) = az + b + ai\beta \bmod \Lambda$ equals $T \circ F(z) = az + b + i\beta \bmod \Lambda$, $a \neq 0$, this imposes the condition that a be an integer. If $b \in \mathbb{R}$, then forcing them to commute gives

$$F \circ C(z) = a\overline{z} + \frac{a}{2} + b \mod \Lambda = a\overline{z} + b + 1/2 \mod \Lambda = C(F(z)),$$

so a must be an odd integer. Moreover, if b has any imaginary component, say $b = x + iy$, $y \neq 0$, then

$$F \circ C(z) = a\overline{z} + x + \frac{a}{2} + iy \bmod \Lambda = a\overline{z} + x - iy + \frac{1}{2} \bmod \Lambda = C(F(z)),$$

which only holds if $y = \frac{n\beta}{2}$ for an integer n. This also implies a is an odd integer.

This discussion leads to the following result, which is a sharpening of a statement in [4].

Theorem 4.1. *An analytic map of $F : \mathbb{C} \to \mathbb{C}$ induces a dianalytic map $\hat{f}$ of $\mathbb{K} \cong \mathbb{T}/[C]$ if and only if is of the form:*

$$F(z) = az + \alpha + \frac{n_0 i\beta}{2}, \tag{4.1}$$

with $a \in \mathbb{Z}$ odd, $n_0 \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, and $\Lambda = [1, i\beta]$. Moreover, writing $c = \alpha + \frac{n_0 i\beta}{2}$, if $a \neq 1$, and $n_0 = m_0(a-1)$, $m_0 \in \mathbb{Z}$ the map $\hat{f}$ is dianalytically *conjugate to* $\hat{f}_a(z) = a\,z$ *via the conjugating map* $h(z) = z + \frac{c}{1-a}$*. That is,* $\hat{f} = h \circ \hat{f}_a \circ h^{-1}$.

Theorem 4.2. *For any integer* a*,* $|a| \geq 3$ *any map of the form (4.1) induces a dianalytic map on the Klein bottle:* $\hat{f} : \mathbb{K} \to \mathbb{K}$ *that is uniformly expanding with respect to the flat metric. Moreover,* $J(\hat{f}) = \mathbb{K}$*, and* $\hat{f}$ *is ergodic, measure-preserving, and exact with respect to* m*.*

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Department of Mathematics; University of North Carolina at Chapel Hill; CB #3250; Chapel Hill, North Carolina 27599-3250

E-mail address: jmh@math.unc.edu