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TOPOLOGY PROCEEDINGS

Volume 47, 2016

Pages 39–57

http://topology.nipissingu.ca/tp/

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Electronically published on February 13, 2015

**Topology Proceedings**

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| **ISSN:** | 0146-4124 |

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

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Volume 47 (2016)
Pages 39-57

E-Published on February 13, 2015

# FUNCTION SPACES AND $L$-PREORDERED SETS

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AND STEPHEN E. RODABAUGH

Abstract. In classical domain theory, the function-space constructor is the most interesting domain constructor. In this paper, we review some results involving the classical function-space constructor and the Scott topology, and then we begin to consider how these results could be extended if we replace preordered and partially ordered sets with $L$-preordered sets and $L$-partially ordered sets for a frame $L$. In this paper, we focus on $L$-preordered sets.

## 1. Introduction

One of the interesting issues in domain theory is how to define function-space domains so that they behave nicely with respect to the function-space constructor. Since the function-space constructor involves spaces of functions, cardinality issues may be problematic. A specific goal of this current study is to begin to examine conditions which could be applied in lattice-valued settings so that the function-space constructor would behave nicely.

There is a relatively rich literature on $L$-fuzzy preorders and $L$-fuzzy partial orders, where the $L$ may be a lattice structure different from a frame; see, for example, [1, 6, 11, 12]. In this paper, when compared to Lai and Zhang [6], we focus on $L$-preorders instead of $L$-partial orders, though we do compare our $L$-antisymmetry condition to theirs. When compared to Yao and Shi [11], we work with traditional topologies instead of many-valued topologies even when beginning with many-valued orders. This may be considered an intermediate step as one transitions

2010 *Mathematics Subject Classification.* 54B99, 54C35, 06A06, 68N15.

*Key words and phrases.* Partially ordered and preordered sets, $L$-partially ordered and $L$-preordered sets, domain theory, Scott topology, function-space constructor.

from the world of two-valued logic to a fully many-valued world. However, it is noteworthy that this situation of moving from many-valued relations to traditional topologies also occurs in the work of U. Höhle. Höhle uses many-valued equivalence relations to build stalk spaces equipped with traditional topologies [4, 5]. In fact, the moving from many-valued relations to traditional topologies is consistent with the "defuzzification" processes commonly employed in applications of many-valued logic.

Continuing with the idea of transitioning from $L$-preorders and $L$-partial orders though traditional topologies in developing a Scott-like domain theory, in future work the authors do plan to complete the transition though traditional topologies to many-valued topologies as they attempt to help develop a many-valued, Scott-like domain theory.

In this paper, we do address the question of how far Dana Scott's theory of continuous functions can be extended, i.e., what is a "maximal" extension of Scott's theory, if the equivalence of order continuity and topological continuity of functions is to be preserved. For example, may partially ordered sets be replaced by preordered sets, and does one have to work with "complete orders"?

## 2. Function Spaces in Classical Domain Theory

There are several good introductions to domain theory. For this paper, the book *The Formal Semantics of Programming Languages An Introduction* by Glynn Winskel [10] is used.

As stated in [10], "Domain theory is the mathematical foundation of denotational semantics," which is a formal and mathematical method of assigning semantics to computer programs. Domains are usually built from partially ordered sets. In this paper, we try to use preordered sets, which we call presets, as much as possible. As we are working in a more general than usual setting, we include proofs of some classical-like results.

**Definition 2.1.** Let $X$ be a set. A *partial order* $\leq$ on $X$ is a relation on $X$, i.e., $\leq\ \subset X \times X$ with the following properties: the relation $\leq$ is

(1) reflexive if for each $x \in X$, $x \leq x$;
(2) antisymmetric if for all $x, y \in X$, whenever $x \leq y$ and $y \leq x$, then $x = y$; and
(3) transitive if for all $x, y, z \in X$, whenever $x \leq y$ and $y \leq z$, then $x \leq z$.

When $\leq$ is a partial order on $X$, $(X, \leq)$ or just $X$ is called a *partially ordered set* or a *poset*.

If $\leq$ is reflexive and transitive but not necessarily antisymmetric, then $\leq$ is a *preorder*, and $(X, \leq)$ or simply $X$ is a *preordered set* or a *preset*. •

Domains are the data types for programming. Data types may be explicitly given. For example, the data type of Boolean values, denoted by

$$\mathcal{B} = (\{\mathbf{true}, \mathbf{false}\}, \leq_{\mathcal{B}}),$$

where **false** $\leq_{\mathcal{B}}$ **true**, is usually given as is the data type of natural numbers denoted by $(\mathcal{N}, =)$ where $\mathcal{N} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$. In programming languages, most data types are constructed using data type constructors. The most interesting standard constructor is the function space constructor.

The function space constructor is interesting in part because it can cause fatal problems as shown by an example which is given in [9] and paraphrased below.

A program execution may be thought of as a sequence of states where a state is modeled as values stored in memory locations. The first state in a program execution sequence has program inputs stored in memory locations. As the program executes, the stored values change as input values are modified and as temporary values are created and manipulated until the program computes its (final) output values. Each change in the stored values may be thought of as creating a new state, and thus, a program execution may be thought of as a finite sequence of states. Once all the output values have been calculated, the program may/should stop execution with output values stored in the final state of the program execution sequence.

To simplify matters, a program execution is often thought of as simply two states, an initial state and a final state where the initial state is the first one in the sequence mentioned above and the final state is the last one in that sequence, and a program itself is the set of all its program executions. Thus, a program is a function which takes (initial) states to (final) states.

These comments lead to Definition 2.2 and Example 2.3.

**Definition 2.2.** A *state* is a function from the set of memory locations to the set of possible values. •

We are being casual in our use of the terms such as "memory location" and "possible value". These terms may, of course, be defined more rigorously. However, doing so in this paper would contribute very little. The intuitive meanings for these terms suffice for our purposes.

Many programming languages allow programs themselves to be included in the values which may be stored in a memory location of a state. As pointed out in [9], these considerations lead to the following contradiction.

**Example 2.3.** We let **location** be the set of memory locations, **value** the set of values which may be stored in the memory locations, and **state** the set of functions from **location** to **value**, i.e.,

$$\mathbf{state} = [\mathbf{location} \to \mathbf{value}].$$

Further, if **program** is the set of programs, then by thinking of a program as a function from (initial) states to (final) states, we have

$$\mathbf{program} = [\mathbf{state} \to \mathbf{state}],$$

and, further, since programs may be values, then

$$\mathbf{program} \subset \mathbf{value}.$$

Hence, if the cardinality of the set of possible states is $\alpha$, then we have the following cardinality results.

- $$|\mathbf{state}| = \alpha$$
- $$|\mathbf{program}| = \alpha^\alpha$$
- $$|\mathbf{value}| \geq \alpha^\alpha$$
- $$|\mathbf{state}| \geq \alpha^\alpha$$
- $$\alpha = |\mathbf{state}| \geq \alpha^\alpha \quad \bullet$$

The above contradiction, when $\alpha > 1$, is created because the function space constructor for sets grows exponentially. The solution to this problem in domain theory is to restrict the set of functions, i.e., the set of programs, which may be included in a function space domain. Part of this restricting is accomplished by using the Scott topology. Dana Scott defined this topology which is named after him [8].

The Scott topology is developed by working with preordered or partially ordered sets which have additional properties.

**Definition 2.4.** Let $(D, \leq)$ be a preset. $(D, \leq)$ or just $D$ is a *directed set* if every finite subset of $D$ has an *upper bound*, i.e., if $E$ is a finite subset of $D$, then there exists $d \in D$ such that for each $e \in E$, $e \leq d$. Since $E$ may be empty, then $D$ must be non-empty.

If $(X, \leq)$ is a preset, then $D \subset X$ is a *directed subset* of $X$ if $(D, \leq_D)$ is a directed set where $\leq_D$ is the restriction of $\leq$ to $D$. $\bullet$

**Definition 2.5.** Let $(X, \leq)$ be a preset. $(X, \leq)$ or $X$ is a *directed complete preset* or a *dcpro* if every directed subset $D$ of $X$ has a least upper bound or supremum $\bigsqcup D$ in $X$. •

Directed completeness is usually defined for partially ordered sets, and a directed complete partially ordered set is abbreviated dcpo.

**Definition 2.6.** A preset $(X, \leq)$ is a *complete preset* or a *cpro* if it is directed complete and if it has a bottom element, i.e., if it has an element $\perp$ such that for each $x \in X$, $\perp \leq x$. •

As with dcpros, completeness is usually defined for partially ordered sets, and a complete partially ordered set is abbreviated cpo.

Definitions in domain theory are not fully standardized. For example, sometimes a bottom element is required for a cpo and sometimes not. We define Scott continuous functions between presets [7], and others may define them between partially ordered sets or cpos.

**Definition 2.7.** Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be presets, and let $f : X \to Y$ be a function. The function $f$ is *order-preserving* if whenever $a, b \in X$ with $a \leq b$, then $f(a) \sqsubseteq f(b)$. •

Scott continuous functions are usually defined as functions which preserve least upper bounds of directed sets. Often this means working with dcpros or dcpos, and requiring that a Scott continuous function map the least upper bound of a directed set to the least upper bound of the image of that directed set. As in [7], this setting can be generalized. When one considers generalizations, obvious questions include how can this setting be generalized and how much can it be generalized. In [7], the generalization is to require that when a directed set has a least upper bound, then the image of that least upper bound be the least upper bound of the image of the directed set. From the assumed conditions in this more general setting, it does follow that the image of a directed set with a least upper bound also has a least upper bound, and thus, the function is order-preserving.

An important result of Scott's work is the equivalence of Scott continuous functions in the preordered or partially ordered setting and continuous functions in the (Scott) topological setting. We want to preserve this equivalence in our generalization. Thus, our generalization questions include how and how much can we generalize our setting and our conditions and still preserve the equivalence of continuous function in the ordered setting and the topological setting.

There are, of course, multiple ways of generalizing the classical setting in which one works with dcpos or cpos. In the classical setting, directed sets always have least upper bounds or limits, and the functions are order-preserving. In Remark 2.9, we suggest a generalization which is more general than [7]. According to this proposed suggestion in Remark 2.9, it would not be required that the image of a directed set with a limit also have a limit. This seems like a natural generalization. However, as seen in Example 2.10, satisfying the order condition of Remark 2.9 does not ensure topological continuity, and in fact, the order condition of Remark 2.9 does not imply that the function is order-preserving. However, requiring that limits of directed sets are mapped to limits of the images of directed sets does imply order-preservation; see Lemma 2.11.

**Notation 2.8.** Let $f : X \to Y$ be a function; let $W \subset X$; and let $Z \subset Y$. We use $f^{\to}(W)$ for the subset of $Y$ containing the images of all points in $W$, i.e.,

$$f^{\to}(W) = \{f(w) \mid w \in W\} \subset Y.$$

Similarly, we use $f^{\leftarrow}(Z)$ for the subset of $X$ containing all the preimages of points in $Z$, i.e.,

$$f^{\leftarrow}(Z) = \{x \in X \mid f(x) \in Z\} \subset X. \ \bullet$$

**Remark 2.9.** It would seem that a possible generalization of the classical definition of a Scott continuous function could be: Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be presets, and let $f : X \to Y$ be a function. Then the function $f$ would be *Scott continuous* if it would preserve the suprema of directed sets when the image sets have suprema. That is, if $D$ is a directed subset of $X$, if $\bigsqcup D$ exists in $X$, and if $\bigsqcup f^{\to}(D)$ exists in $Y$, then $f(\bigsqcup D)$ is a supremum of $f^{\to}(D)$. However, as we see in Example 2.10, this generalization does not work because this "order" condition can be satisfied without having topological continuity. Thus, it seems that the generalization in [7], which is the generalization which we use in this paper, is maximal. It is also worth noting that the order condition in this remark does not ensure that the function is order-preserving. $\bullet$

**Example 2.10.** Let $X = Y = \{a, b\}$ where $a$ and $b$ are distinctive elements. Define a reflexive relation $\leq$ on $X$ by $a \leq b$, and define a reflexive relation $\sqsubseteq$ on $Y$ such that $a$ and $b$ are not related. Also, define $f : X \to Y$ to be the identity function. In this example, the condition "if $D$ is a directed subset of $X$, if $\bigsqcup D$ exists in $X$, and if $\bigsqcup f^{\to}(D)$ exists in $Y$, then $f(\bigsqcup D)$ is a supremum of $f^{\to}(D)$" holds, but we have a "problem" with topological continuity.

The Scott topology on $X$ is $\{\emptyset, \{b\}, X\}$, and the Scott topology on $Y$ is the powerset of $Y$. The function $f : X \to Y$ is the identity function, which is not topologically continuous because $\{a\}$ is open in $Y$ but $\{a\} = f^{\leftarrow}(\{a\})$ is not open in $X$. Thus, the order-theoretic condition of this remark holds, but the function is not continuous with respect to the Scott topologies.

Further note that the function $f$ is not order-preserving. We have that $a \leq b$, but $a = f(a) \not\sqsubseteq f(b) = b$.

It seems as though requiring our functions to be order-preserving is an important condition. $\bullet$

The requirement that our functions be order-preserving is in keeping with Scott's information ordering which assumes when given two order related inputs to a program or "computation function", that the output from the input with the more information will contain at least as much information as the output from the input with the lesser information. Thus, we are working in the category **Preset** of presets and order-preserving functions.

We want our functions to be order-preserving, but as in the classical setting, we do not need to explicitly require that our functions are order-preserving.

**Lemma 2.11.** *Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be presets, and let $f : X \to Y$ be a function. The function $f$ is order-preserving if whenever $D$ is a directed subset of $X$ and $D$ has a least upper bound $\bigsqcup D$, then $f(\bigsqcup D)$ is a least upper bound of $f^{\to}(D)$ in $Y$.* $\bullet$

*Proof.* Let $a, b \in X$ with $a \leq b$. Then $D = \{a, b\}$ is a directed subset of $X$ with least upper bound $b$. Since $f$ preserves this least upper bound, then $f(a) \sqsubseteq f(b)$. $\square$

**Definition 2.12.** Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be presets. The function $f$ is *Scott continuous* if it preserves the suprema of directed sets. That is, if $D$ is a directed subset of $X$ and if $\bigsqcup D$ exists in $X$, then $f(\bigsqcup D)$ is a supremum of $f^{\to}(D)$. $\bullet$

As we will see below, calling these functions continuous is, in fact, justifiable from a topological perspective.

When working with presets in place of partially ordered sets, least upper bounds need not be unique. However, for a distinct directed subset, all least upper bounds are related, in that each one is less than or equal to every least upper bound.

This lack of uniqueness of least upper bounds in presets is why the last sentence in Definition 2.12 reads "then $f(\bigsqcup D)$ is a supremum of $f^{\to}(D)$" instead of reading "then $f(\bigsqcup D) = \bigsqcup f^{\to}(D)$".

**Definition 2.13.** Let $(X, \leq)$ be a preset, and let $V \subset X$. The *up-closure* of $V$ in $X$ is $\{x \in X \mid \exists v \in V \,.\, v \leq x\}$. The up-closure of $V$ is denoted by $\uparrow V$. $V$ is said to be *up-closed* if $V = \uparrow V$. $\bullet$

**Definition 2.14.** Let $(X, \leq)$ be a preset. A subset $U \subset X$ is *Scott open* if $U$ is up-closed and if it non-trivially intersects every directed set whose least upper bound or limit it contains. Thus, $U$ is Scott open if $U = \uparrow U$ and whenever $D$ is a directed subset of $X$ with $\bigsqcup D$ existing and in $U$, i.e., $\bigsqcup D \in U$, then $U \cap D \neq \emptyset$. •

**Proposition 2.15.** *Let $(X, \leq)$ be a preset; $\tau = \{U \subset X | U \text{ is Scott open}\}$ is a topology on $X$.* •

**Definition 2.16.** In Proposition 2.15, $\tau$ is the *Scott topology* on $(X, \leq)$ or $X$, and $(X, \tau)$ is a Scott topological space. •

We have taken a quick and direct approach to get the Scott topology. There are interesting connections between a given preorder and related topologies, of which one is the Scott topology. For each topological space $(X, \sigma)$, the topology $\sigma$ generates a preorder on $X$. This preorder is called the specialization preorder, and it is defined such that for $x, y \in X$, $x \leq_\sigma y$ if and only if whenever $x \in U \in \sigma$, then $y \in U$, i.e., if and only if $\text{cl}(\{x\}) \subset \text{cl}(\{y\})$, where for $A \subset X$, $\text{cl}(A)$ is the closure of $A$, i.e., the smallest closed set containing $A$.

Multiple topologies on a set $X$ may generate the same preorder. In fact, there is a complete lattice of topologies which generate the same preorder. Each of these topologies is called an *order-consistent topology* [3] with respect to the generated preorder. This complete lattice of topologies is ordered by subset inclusion of the topologies. The finest topology in each of these complete lattices is the Alexandroff topology which is the topology of up-closed subsets. The coarsest or weakest topology is the topology generated by the collection of sets of the form $(X - \downarrow a)$ when $a \in X$ and when $\downarrow a = \{x \in X \,|\, x \leq a\}$. The Scott topology is the topology whose open sets are the up-closed sets which non-trivially intersect the directed sets whose limits they contain. More about the interrelationships between preorders and topologies may be found in [3, 7].

Lemma 2.17 may be proven directly. It also follows from the preceding discussion because the Scott topology for a given preorder, i.e., the order-consistent Scott topology for a given preorder, is a subset of the order-consistent Alexandroff topology and a superset of the order-consistent weakest topology for the same preorder.

**Lemma 2.17.** *Let $(X, \leq)$ be a preset, and let $a \in X$. Then $U = (X - \downarrow a)$ is a Scott open set.* •

**Theorem 2.18.** *Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be presets; let $(X, \tau_X)$ and $(Y, \tau_Y)$ be the corresponding Scott topological spaces; and let $f : X \to Y$ be a function. $f : (X, \leq) \to (Y, \sqsubseteq)$ is Scott continuous in the order theoretic sense if and only if $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous in the topological sense.* •

*Proof.* Suppose $f : (X, \leq) \to (Y, \sqsubseteq)$ is Scott continuous in the order theoretic sense. Let $U$ be a Scott open subset of $Y$. Suppose that $a \in f^{\leftarrow}(U)$ and that $a \leq b$ for $b \in X$. Since $f$ is order-preserving and since $f(a) \in U$, then $f(b) \in U$. It follows that $b \in f^{\leftarrow}(U)$, and therefore, $f^{\leftarrow}(U)$ is up-closed.

Let $D$ be a directed subset of $X$ such that $\bigsqcup D$ exists and $\bigsqcup D \in f^{\leftarrow}(U)$. Since $D$ is directed and since $f$ is order-preserving, then $f^{\rightarrow}(D)$ is directed in $Y$. Also, since $\bigsqcup D \in f^{\leftarrow}(U)$, then $f(\bigsqcup D) \in U$. Further, since $f$ is Scott continuous in the order theoretic sense, then $f(\bigsqcup D)$ is a least upper bound of $f^{\rightarrow}(D)$. Hence, we have a directed set $f^{\rightarrow}(D)$ in $Y$ and we have the least upper bound, $f(\bigsqcup D)$, in the open set $U$. It follows that $f^{\rightarrow}(D) \cap U \neq \emptyset$, and thus, there exists $d \in D$ with $f(d) \in U$. Therefore, $D \cap f^{\leftarrow}(U) \neq \emptyset$, and $f^{\leftarrow}(U)$ is Scott open in $X$. Hence, $f$ is continuous with respect to the Scott topologies on $X$ and $Y$.

Suppose $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous in the topological sense. Let $D$ be a directed subset of $(X, \leq)$ such that $\bigsqcup D$ exists. Since $f$ is order-preserving, then $f^{\rightarrow}(D)$ is directed, and $f(\bigsqcup D)$ is an upper bound for $f^{\rightarrow}(D)$. Suppose $f(\bigsqcup D)$ is not a least upper bound. Then there exists $y \in Y$ such that $y$ is an upper bound for $f^{\rightarrow}(D)$, and $y < f(\bigsqcup D)$. Let $V = (Y - \downarrow y)$. By Lemma 2.17, $V$ is open in $Y$, and thus, $f^{\leftarrow}(V)$ is open in $X$. Since $f^{\rightarrow}(D) \cap V = \emptyset$ because $f^{\rightarrow}(D) \subset \downarrow y$, then $D \cap f^{\leftarrow}(V) = \emptyset$. However, since $f^{\leftarrow}(V)$ is open in $X$ and $\bigsqcup D \in f^{\leftarrow}(V)$, then $D \cap f^{\leftarrow}(V) \neq \emptyset$. This contradiction shows that $f(\bigsqcup D)$ is the least upper bound for $f^{\rightarrow}(D)$, and therefore, $f : (X, \leq) \to (Y, \sqsubseteq)$ is Scott continuous in the order theoretic sense. □

**Notation 2.19.** Let $X$ and $Y$ be sets. We denote the set of all functions from $X$ to $Y$ by $[X \to Y]$. If $D \subset [X \to Y]$, we let $D_{x\mapsto}$ denote the set $\{f(x) \mid f \in D\}$. We use $[(X, \leq) \to (Y, \sqsubseteq)]_S$ or just $[X \to Y]_S$ for the set of Scott continuous functions from $(X, \leq)$ to $(Y, \sqsubseteq)$. •

**Lemma 2.20.** *Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be presets. We define $\leq$ on $[X \to Y]$ pointwise, i.e., for $f, g \in [X \to Y]$, $f \leq g$ if and only if $\forall x \in X$, $f(x) \sqsubseteq g(x)$. If $D$ is a directed subset of $[X \to Y]$, then for each $x \in X$, $D_{x\mapsto}$ is a directed subset of $Y$.* •

*Proof.* Let $x \in X$. Since $D$ is non-empty, then $D_{x\mapsto}$ is also non-empty.

Suppose that $y_1$ and $y_2$ are arbitrary elements in $D_{x\mapsto}$. Then there exist $f_1, f_2 \in D$ such that $f_1(x) = y_1$ and $f_2(x) = y_2$. Since $D$ is directed, there exists $h \in D$ such that $f_1 \leq h$ and $f_2 \leq h$. Therefore, by definition of $\leq$ on $[X \to Y]$, $y_1 = f_1(x) \sqsubseteq h(x)$ and $y_2 = f_2(x) \sqsubseteq h(x)$, and thus, $D_{x\mapsto}$ is a directed subset of $Y$. □

In the proof of the next theorem, the axiom of choice is used in the definition of the function $h$.

**Theorem 2.21.** *Let* $(X, \leq)$ *be a preset, and let* $(Y, \sqsubseteq)$ *be a cpro. Then* $([X \to Y]_S, \leq)$ *is also a cpro.* $\bullet$

*Proof.* Since $(Y, \sqsubseteq)$ is a preset, then $([X \to Y], \leq)$ is also a preset.

Let $D_{[]_S}$ be a directed subset of $[X \to Y]_S$. Define $h : X \to Y$ such that for each $x \in X$, $h(x) = \bigsqcup\{f(x) \mid f \in D_{[]_S}\}$. We claim that $h$ is well defined; that $h \in [X \to Y]_S$; and that $\bigsqcup D_{[]_S} = h$.

Since $D_{[]_S}$ is directed in $[X \to Y]_S$, then by the proof of Lemma 2.20, $\{f(x) \mid f \in D_{[]_S}\}$ is directed for each $x \in X$. Thus, $h$ is well defined since $(Y, \sqsubseteq)$ is directed complete.

Let $a, b \in X$ such that $a \leq b$. Since $a \leq b$, then for each $f \in D_{[]_S}$, $f(a) \sqsubseteq f(b)$. It follows that $\bigsqcup\{f(a) \mid f \in D_{[]_S}\} \sqsubseteq \bigsqcup\{f(b) \mid f \in D_{[]_S}\}$, i.e., $h(a) \sqsubseteq h(b)$.

Let $D$ be a directed subset of $X$ such that $\bigsqcup D$ exists. Since $h$ is order-preserving, then $h^{\to}(D)$ is directed, and $\bigsqcup h^{\to}(D) \sqsubseteq h(\bigsqcup D)$. (We know that $\bigsqcup h^{\to}(D)$ exists because $Y$ is a cpro.)

Suppose that $\bigsqcup h^{\to}(D) < h(\bigsqcup D)$. Then the Scott open set $V = Y - \downarrow \bigsqcup h^{\to}(D)$ contains no elements in $h^{\to}(D)$, i.e., $h^{\to}(D) \cap V = \emptyset$. However, since $h(\bigsqcup D) \in V$, then $\bigsqcup D \in h^{\leftarrow}(V)$, and thus, $D \cap h^{\leftarrow}(V) \neq \emptyset$. Therefore, $h^{\to}(D) \cap V \neq \emptyset$. This contradiction means that $\bigsqcup h^{\to}(D) = h(\bigsqcup D)$, and therefore, $h : (X, \leq) \to (Y, \sqsubseteq)$ is Scott continuous. Since the ordering on $[X \to Y]_S$ is pointwise, then $\bigsqcup D_{[]_S} = h$.

The bottom element of $([X \to Y]_S, \leq)$ is the function $f : X \to Y$ where $f(x) = \perp_Y$ for each $x \in X$. It follows that $([(X, \leq) \to (Y, \sqsubseteq)]_S, \leq)$ is a cpro. $\square$

## 3. Introducing Function Spaces with $L$-Preorders

In this section, we work with $L$-valued relations.

**Definition 3.1.** Let $X$ be a set, and let $(L, \leq)$ be a frame with largest element $\top_L$. An *$L$-valued relation* $R$ on $X$ is a function $R : X \times X \to L$. $\bullet$

**Definition 3.2.** Let $R : X \times X \to L$ be an $L$-valued relation on $X$. $R$ is an *$L$-preorder* if

- $R$ is *reflexive*, i.e., $\forall x \in X,\ R(x, x) = \top_L$, and
- $R$ is *transitive*, i.e., $\forall x, y, z \in X,\ R(x, y) \wedge R(y, z) \leq R(x, z)$.

$R$ is an *$L$-partial order* if, additionally,

- $R$ is *antisymmetric*, i.e., $\forall x, y \in X, R(x, y) \wedge R(y, x) = \top_L \Rightarrow x = y$.

If $R$ is an $L$-preorder on $X$, then $(X, R)$ or just $X$ is an *$L$-preordered set* or an *$L$-preset*, and if $R$ is an $L$-partial order, then $(X, R)$ or just $X$ is an *$L$-partially ordered set* or an *$L$-poset*. •

Of course, the antisymmetry condition that $R(x, y) \wedge R(y, x) = \top_L$ is equivalent to $R(x, y) = \top_L$ and $R(y, x) = \top_L$. For more on our antisymmetry property, please see [2].

The axioms of Definition 3.2 are equivalent to those of [11]. The axioms of Definition 3.2 are also given in [2], and there they are motivated from enriched categories.

For an $L$-preset $(X, R)$, when $R(x, y) = \alpha \in L$, we consider $x$ to be $R$-related to $y$ to degree $\alpha$. However, when $R(x, y) = \alpha$, we normally say either $x$ is less than or equal to $y$ to degree $\alpha$, or $y$ is greater than or equal to $x$ to degree $\alpha$.

When $R : X \times X \to L$ is an $L$-valued relation, we may, at times, use the expressions "$L$-reflexive", "$L$-antisymmetric", and "$L$-transitive", respectively, in place of "reflexive", "antisymmetric", and "transitive".

In [6], Hongliang Lai and Dexue Zhang define $L$-preorders and $L$-partial orders. In their definitions, $L$ is restricted to the closed unit interval $[0, 1]$, and in their transitivity definition, they use a triangular or t-norm instead of the meet operation. Additionally, they define $R$ to be a fuzzy partial order on $X$ if it is a fuzzy preorder on $X$ and if

$$\forall x, y \in X, [x = y \text{ iff } \forall z \in X, R(z, x) = R(z, y)].$$

**Proposition 3.3.** *Let $X$ be a set; let $L$ be a frame; and $R$ an $L$-valued relation on $X$, i.e., $R : X \times X \to L$. The Lai-Zhang condition*

$$\forall x, y \in X, [x = y \textit{ iff } \forall z \in X, R(z, x) = R(z, y)]$$

*and our antisymmetry condition*

$$\forall x, y \in X, R(x, y) \wedge R(y, x) = \top_L \Rightarrow x = y$$

*are not equivalent. However, if $(X, R)$ is an $L$-preset, then the two conditions are equivalent.* •

*Proof.* This proof comprises the following two examples and two claims.

**Example 1:** Let $X = \{a, b\}$ where $a$ and $b$ are distinct elements, and let $R : X \times X \to [0, 1]$ where $[0, 1]$ has the normal ordering. Define $R$ such that for all $x, y \in X$, $R(x, y) = r$, where $r$ is a fixed element in $[0, 1)$. $R$ is an $L$-transitive relation. Our antisymmetry condition holds, but the Lai-Zhang condition does not hold.

**Claim 1:** If $X$ is a set, if $L$ is a frame, and if $R$ is an $L$-transitive relation on $X$, then the Lai-Zhang condition implies our antisymmetry condition.

Suppose $x, y \in X$ and further suppose that $R(x,y) \wedge R(y,x) = \top_L$, i.e., $R(x,y) = \top_L$ and $R(y,x) = \top_L$. For $z \in X$,

$$R(z,x) = R(z,x) \wedge \top_L = R(z,x) \wedge R(x,y) \leq R(z,y).$$

Similarly,

$$R(z,y) = R(z,y) \wedge \top_L = R(z,y) \wedge R(y,x) \leq R(z,x).$$

Therefore, $R(z,x) = R(z,y)$, and by the Lai-Zhang condition, $x = y$. Therefore, our antisymmetric condition holds.

**Example 2:** Let $X = \{a, b, c\}$ where $a$, $b$, and $c$ are distinct elements, and let $R : X \times X \to \{0,1\}$ where $\{0,1\}$ has the usual ordering. Define $R(a,b) = 1$; $R(a,c) = 0$; $R(b,a) = 1$; $R(b,c) = 0$; $R(c,a) = 0$; and $R(c,b) = 1$. For each $x \in X$, $R(x,x) = 1$. $R$ is an $L$-reflexive relation. The Lai-Zhang-condition holds, but our condition does not hold because $a \neq b$.

**Claim 2:** If $X$ is a set, if $L$ is a frame, and if $R$ is an $L$-reflexive relation on $X$, then our antisymmetry condition implies the Lai-Zhang condition.

Suppose $x, y \in X$ and further suppose for each $z \in X$, that $R(z,x) = R(z,y)$. Since $R$ is $L$-reflexive, when $z = y$, then $R(y,x) = R(y,y) = \top_L$, and similarly, when $z = x$, then $R(x,y) = R(y,y) = \top_L$. It follows that $R(x,y) \wedge R(y,x) = \top_L$, and therefore, $x = y$.

By Claims 1 and 2, if $R$ is an $L$-preset, then the Lai-Zhang condition and our antisymmetry condition are equivalent. □

**Definition 3.4.** Let $(X, R)$ be an $L$-preset; let $Y \subset X$; and let $\alpha \in L$. An element $x \in X$ is an *$\alpha$-upper bound* for $Y$ if $\forall y \in Y, R(y,x) \geq \alpha$.

The element $x$ is an *$\alpha$-least upper bound* for $Y$ if it is an $\alpha$-upper bound for $Y$ and if for every $\alpha$-upper bound $z$ of $Y$, $R(x,z) \geq \alpha$.

If $(X, R)$ is an $L$-preset and if $Y$ has an $\alpha$-least upper bound, we denote this element by $\bigsqcup_\alpha Y$. We may also call $\bigsqcup_\alpha Y$ an *$\alpha$-supremum* or an *$\alpha$-limit* of $Y$ in $X$. The $\alpha$-suprema of a subset $Y$ need not be unique. •

**Definition 3.5.** Let $(D, R)$ be an $L$-preset, and let $\alpha \in L$. $(D, R)$ or just $D$ is an *$\alpha$-directed set* if every finite subset of $D$ has an $\alpha$-upper bound in $D$. Since the finite subset may be empty, then $D$ must be non-empty.

If $(X, R)$ is an $L$-preset, then $D \subset X$ is an *$\alpha$-directed subset* of $X$ if $(D, R_D)$ is an $\alpha$-directed set where $R_D$ is the restriction of $R$ to $D \times D$. •

**Definition 3.6.** Let $(X, R)$ be an $L$-preset, and let $\alpha \in L$. $(X, R)$ or $X$ is an *$\alpha$-directed complete preset* or an *$\alpha$-dcpro* if every $\alpha$-directed subset $D$ of $X$ has an $\alpha$-least upper bound or $\alpha$-supremum $\bigsqcup_\alpha D$ in $X$. •

**Definition 3.7.** Let $(X,R)$ be an $L$-preset, and let $\alpha \in L$. $X$ has an *$\alpha$-bottom element* if there is an element $\perp_{(X,R)_\alpha} \in X$ such that for each $x \in X$, $R(\perp_{(X,R)_\alpha}, x) \geq \alpha$. An $L$-preset $(X,R)$ is said to have an *$L$-bottom element* or simply a *bottom element* if it has a $\top_L$-bottom element, and then $\perp_{(X,R)_{\top_L}}$ denotes an $L$-bottom element. •

**Definition 3.8.** An $L$-preset $(X,R)$ is an *$\alpha$-complete preset* or an *$\alpha$-cpro* if it is $\alpha$-directed complete and if it has an $\alpha$-bottom element. •

**Example 3.9.** Let $(X,R)$ be an $L$-preset, and let $\perp_L$ be the least or bottom element of $L$. For every $x,y \in X$, $R(y,x) \geq \perp_L$. Thus, if $Y \subset X$, each element of $X$ is a $\perp_L$-upper bound for $Y$. Therefore, if $D$ is a non-empty subset of $X$, then $D$ is a $\perp_L$-directed subset of $X$. Further, if $w$ and $z$ are $\perp_L$-upper bounds for $D$, then since $R(w,z) \geq \perp_L$, we have that $w$ is a $\perp_L$-least upper bound for $D$. Thus, each element of $X$ is a $\perp_L$-least upper bound for $D$, and $(X,R)$ is a $\perp_L$-directed complete preset. Additionally, since for every $x,y \in X$, $R(y,x) \geq \perp_L$, then each element in $X$ is a $\perp_L$-bottom element for $(X,R)$; thus, $(X,R)$ is a $\perp_L$-complete preset.

Let $\top_L$ be the largest element in $L$. Define $(X,\leq)$ such that for $x,y \in X$, $y \leq x$ if and only if $R(y,x) = \top_L$. Then $(X,R)$ is a $\top_L$-directed complete preset if and only if $(X,\leq)$ is a directed complete preset. Further, when $(X,R)$ has a least or bottom element, then $(X,R)$ is a $\top_L$-cpro if and only if $(X,\leq)$ is a cpro. •

**Definition 3.10.** Let $(X,R)$ and $(Y,S)$ be $L$-presets, and let $\alpha \in L$. A function $f : X \to Y$ is *$\alpha$-order-preserving* if whenever $a,b \in X$ with $R(a,b) \geq \alpha$, then $S(f(a),f(b)) \geq \alpha$. The function $f$ is $L$-order-preserving if for all $a,b \in X$, $R(a,b) \leq S(f(a),f(b))$. •

**Proposition 3.11.** *Let $(X,R)$ and $(Y,S)$ be $L$-presets, and let $f : X \to Y$. The function $f$ is $L$-order-preserving if and only if it is $\alpha$-order-preserving for each $\alpha \in L$.* •

*Proof.* If $f$ is $L$-order-preserving, then clearly it is $\alpha$-order-preserving for each $\alpha \in L$.

Suppose $f$ is $\alpha$-order-preserving for each $\alpha \in L$. Assume $f$ is not $L$-order-preserving. Thus, there exists $a,b \in X$ such that $R(a,b) \not\leq S(f(a),f(b))$. It follows that $f$ is not $R(a,b)$-order-preserving. This contradiction gives us that $f$ is $L$-order-preserving. □

**Example 3.12.** There exist $L$-posets $(X,R)$, $(Y,S)$, and $(Z,T)$; functions $f : X \to Y$, and $g : X \to Z$; and $\alpha_2, \alpha_3 \in L$ with $\alpha_2 < \alpha_3$ such that $f$ is $\alpha_3$-order-preserving but not $\alpha_2$-order-preserving and $g$ is $\alpha_2$-order-preserving but not $\alpha_3$-order-preserving.

Let $L$ be a frame with $\alpha_1, \alpha_2, \alpha_3 \in L$ such that $\alpha_1 < \alpha_2 < \alpha_3$. Define $(X, R)$, $(Y, S)$, and $(Z, T)$ as follows: $X = Y = Z = \{a, b, c\}$. For each $x \in X$, for each $y \in Y$, and for each $z \in Z$, $R(x, x) = S(y, y) = T(z, z) = \top_L$; $R(b, c) = S(b, c) = \alpha_3$; and otherwise for $x_1, x_2 \in X$, $R(x_1, x_2) = \alpha_2$; for $y_1, y_2 \in Y$, $S(y_1, y_2) = \alpha_1$; and for $z_1, z_2 \in Z$, $T(z_1, z_2) = \alpha_2$.

Define $f$ such that for each $x \in X$, $f(x) = x$; and define $g$ such that for each $x \in X$, $g(x) = x$. The function $f$ is $\alpha_3$-order-preserving, but it is not $\alpha_2$-order-preserving. The function $g$ is $\alpha_2$-order-preserving, but it is not $\alpha_3$-order-preserving. It is also the case that neither $f$ nor $g$ is $L$-order-preserving. •

**Definition 3.13.** Let $(X, R)$ and $(Y, S)$ be $L$-presets, and let $\alpha \in L$. A function $f : X \to Y$ is *$\alpha$-Scott continuous* if it preserves $\alpha$-suprema of $\alpha$-directed sets. That is, if $D$ is an $\alpha$-directed subset of $X$ and if $\bigsqcup_\alpha D$ exists in $X$, then $f(\bigsqcup_\alpha D)$ is an $\alpha$-supremum of $f^{\to}(D)$. •

**Definition 3.14.** Let $(X, R)$ and $(Y, S)$ be $L$-presets, and let $f : X \to Y$. The function $f$ is $L$-Scott continuous if and only if it is $\alpha$-Scott continuous for each $\alpha \in L$. •

As we will see below, calling the functions in Definition 3.13 $\alpha$-Scott continuous is, in fact, justifiable from a topological perspective.

**Proposition 3.15.** *Let $(X, R)$ and $(Y, S)$ be $L$-presets; let $\alpha \in L$; and let $f : X \to Y$ be a function. If the function $f$ is $\alpha$-Scott continuous, then it is $\alpha$-order-preserving.* •

*Proof.* Let $a, b \in X$ such that $R(a, b) \geq \alpha$. Then $D = \{a, b\}$ is an $\alpha$-directed subset of $X$ with $b = \bigsqcup_\alpha D$. Since $f$ is $\alpha$-Scott continuous, then $f(b) = \bigsqcup_\alpha f^{\to}(D)$. It follows that $S(f(a), f(b)) \geq \alpha$, and thus, $f$ is $\alpha$-order-preserving. □

**Definition 3.16.** Let $(X, R)$ be an $L$-preset; let $V \subset X$; and let $\alpha \in L$. The *$\alpha$-up-closure* of $V$ in $X$ is $\{x \in X \,|\, \exists v \in V \,.\, R(v, x) \geq \alpha\}$. The $\alpha$-up-closure of $V$ is denoted by $\uparrow_\alpha V$. $V$ is said to be *$\alpha$-up-closed* if $V = \uparrow_\alpha V$. •

**Definition 3.17.** Let $(X, R)$ be an $L$-preset, and let $\alpha \in L$. A subset $U \subset X$ is *$\alpha$-Scott open* if $U$ is $\alpha$-up-closed and if it non-trivially intersects every $\alpha$-directed set whose limit it contains. Thus, $U$ is $\alpha$-Scott open if $U = \uparrow_\alpha U$ and whenever $D$ is an $\alpha$-directed subset of $X$ with $\bigsqcup_\alpha D$ existing and in $U$, i.e., $\bigsqcup_\alpha D \in U$, then $U \cap D \neq \emptyset$. •

**Proposition 3.18.** *Let $(X, R)$ be an $L$-preset; let $\alpha \in L$; and let $\tau_\alpha = \{U \subset X \,|\, U$ is $\alpha$-Scott open$\}$. Then $\tau_\alpha$ is a topology on $X$.* •

*Proof.* The empty set and $X$ are in $\tau_\alpha$.

Suppose $U, V \in \tau_\alpha$. Since $U$ and $V$ are both $\alpha$-up-closed, then $U \cap V$ is also $\alpha$-up-closed. Suppose $D$ is an $\alpha$-directed subset of $X$ with $\bigsqcup_\alpha D \in U \cap V$. Since both $U$ and $V$ are $\alpha$-Scott open, there exist $d_U \in D \cap U$ and $d_V \in D \cap V$. Since $D$ is directed, there is also $d \in D$ such that $d$ is an $\alpha$-upper bound for $d_U$ and $d_V$. Since both $U$ and $V$ are $\alpha$-up-closed, then $d \in U \cap V$. It follows that $d \in D \cap (U \cap V)$, and $U \cap V$ is $\alpha$-Scott open.

Suppose $\mathcal{U} = \{U_i \mid i \in I\} \subset \tau_\alpha$. Since each $U_i$ is $\alpha$-up-closed, then $\bigcup \mathcal{U}$ is also $\alpha$-up-closed. Suppose that $D$ is an $\alpha$-directed subset of $X$ and $\bigsqcup_\alpha D \in \bigcup \mathcal{U}$. There is $j \in I$ such that $\bigsqcup_\alpha D \in U_j$, and since $U_j$ is $\alpha$-Scott open, then $D \cap U_j \neq \emptyset$. It follows that $D \cap (\bigcup \mathcal{U}) \neq \emptyset$, and therefore, $\bigcup \mathcal{U}$ is $\alpha$-Scott open. Hence, $\tau_\alpha$ is a topology. $\square$

**Definition 3.19.** The topology $\tau_\alpha$ defined in Proposition 3.18 is the $\alpha$-*Scott topology* on the $L$-preset $(X, R)$ or just $X$. $\bullet$

**Lemma 3.20.** *Let $(X, R)$ be an $L$-preset; let $\alpha \in L$; and let $a \in X$. Let $\downarrow_\alpha a = \{x \in X \mid R(x, a) \geq \alpha\}$, and let $U = X - \downarrow_\alpha a$. Then $U$ is an $\alpha$-Scott open subset of $X$.* $\bullet$

*Proof:* Suppose that $c, d \in X$ such that $c \in U$ and $R(c, d) \geq \alpha$. If $d \notin U$, then $d \in \downarrow_\alpha a$ which means that $R(d, a) \geq \alpha$. Since $R(c, d) \geq \alpha$ and $R(d, a) \geq \alpha$, then $R(c, a) \geq R(c, d) \wedge R(d, a) \geq \alpha$. Therefore, $c \in \downarrow_\alpha a$ and $c \notin U$. This contradiction shows that $d \in U$, and thus, $U$ is $\alpha$-up-closed.

Suppose that $D$ is an $\alpha$-directed subset of $X$ and that $\bigsqcup_\alpha D \in U$. If $D \cap U = \emptyset$, then $D \subset \downarrow_\alpha a$. If $D \subset \downarrow_\alpha a$, then clearly $a$ is an $\alpha$-upper bound for $D$. Since $\bigsqcup_\alpha D$ is an $\alpha$-least upper bound for $D$, then $R(\bigsqcup_\alpha D, a) \geq \alpha$. Therefore, $\bigsqcup_\alpha D \in \downarrow_\alpha a$, and $\bigsqcup_\alpha D \notin U$. This contradiction shows that $D \cap U \neq \emptyset$, and hence, $U$ is $\alpha$-Scott open. $\square$

**Lemma 3.21.** *Let $(X, R)$ and $(Y, S)$ be $L$-presets; let $\alpha \in L$; and let $f : (X, \tau_R) \to (Y, \tau_S)$ be continuous when $\tau_R$ and $\tau_S$ are the $\alpha$-Scott topologies on $(X, R)$ and $(Y, S)$, respectively. Then $f : (X, R) \to (Y, S)$ is $\alpha$-order-preserving.* $\bullet$

*Proof.* Let $a, b \in X$ such that $R(a, b) \geq \alpha$, and let $V \subset Y$ such that $V = Y - \downarrow_\alpha f(b)$. We claim that $S(f(a), f(b)) \geq \alpha$, i.e., we claim that $f(a) \in \downarrow_\alpha f(b)$. If this claim is false, then $f(a) \in V$, and if $f(a) \in V$, then $a \in f^{\leftarrow}(V)$. However, since $f$ is continuous with respect to $\tau_R$ and $\tau_S$ and since $V \in \tau_S$, then $f^{\leftarrow}(V)$ is $\alpha$-Scott open in $X$, which means it is $\alpha$-up-closed. Therefore, since $a \in f^{\leftarrow}(V)$ and $R(a, b) \geq \alpha$, then $b \in f^{\leftarrow}(V)$. However, this would imply that $f(b) \in V$, which is not true. Therefore, $f(a) \in \downarrow_\alpha f(b)$, and $S(f(a), f(b)) \geq \alpha$, i.e., $f$ is $\alpha$-order-preserving. $\square$

**Theorem 3.22.** *Let $(X, R)$ and $(Y, S)$ be $L$-presets; let $\alpha \in L$; and let $f : X \to Y$ be a function. $f : (X, R) \to (Y, S)$ is $\alpha$-Scott continuous if and only if $f : (X, \tau_R) \to (Y, \tau_S)$ is continuous when $\tau_R$ and $\tau_S$ are the $\alpha$-Scott topologies on $(X, R)$ and $(X, S)$, respectively.* •

*Proof.* Suppose $f : (X, R) \to (Y, S)$ is $\alpha$-Scott continuous. Let $V \in \tau_S$. We claim that $f^{\leftarrow}(V) \in \tau_R$. Thus, we need to show that $f^{\leftarrow}(V)$ is $\alpha$-up-closed and that $f^{\leftarrow}(V)$ behaves correctly with respect to least upper bounds of $\alpha$-directed subsets in $(X, R)$.

Suppose that $a, b \in X$, that $R(a, b) \geq \alpha$, and that $a \in f^{\leftarrow}(V)$. By Proposition 3.15, $S(f(a), f(b)) \geq \alpha$. Since $f(a) \in V$ and since $V$ is $\alpha$-up-closed, then $f(b) \in V$. Hence, $b \in f^{\leftarrow}(V)$, and $f^{\leftarrow}(V)$ is $\alpha$-up-closed.

Suppose that $D$ is an $\alpha$-directed subset of $X$, that $\bigsqcup_\alpha D$ exists, and that $\bigsqcup_\alpha D \in f^{\leftarrow}(V)$. Then $f(\bigsqcup_\alpha D) \in V$. Further, $f$ is $\alpha$-Scott continuous, then by Definition 3.13 $f^{\rightarrow}(D)$ is an $\alpha$-directed subset of $Y$, and $f(\bigsqcup_\alpha D)$ is an $\alpha$-least upper bound of $f^{\rightarrow}(D)$. Thus, $f^{\rightarrow}(D) \cap V \neq \emptyset$. Let $d \in D$ such that $f(d) \in f^{\rightarrow}(D) \cap V$. Then $d \in D \cap f^{\leftarrow}(V)$. It follows that $f^{\leftarrow}(V)$ is $\alpha$-Scott open in $X$, and $f : (X, \tau_R) \to (Y, \tau_S)$ is continuous with respect to the $\alpha$-topologies.

Now suppose that $f : (X, \tau_R) \to (Y, \tau_S)$ is continuous with respect to the $\alpha$-topologies. Further, suppose that $D$ is an $\alpha$-directed subset of $X$ and that $\bigsqcup_\alpha D$ exists. We need to show that $f(\bigsqcup_\alpha D)$ is an $\alpha$-upper bound for $f^{\rightarrow}(D)$ and if $y \in Y$ is an $\alpha$-upper bound for $f^{\rightarrow}(D)$, then $S(f(\bigsqcup_\alpha D), y) \geq \alpha$.

By Lemma 3.21, $f$ is $\alpha$-order-preserving, and thus, $f^{\rightarrow}(D)$ is $\alpha$-directed, and $f(\bigsqcup_\alpha D)$ is an $\alpha$-upper bound for $f^{\rightarrow}(D)$.

Suppose that $y \in Y$ is an $\alpha$-upper bound for $f^{\rightarrow}(D)$. Then $f^{\rightarrow}(D) \subset {\downarrow_\alpha} y$.

Let $U = Y - {\downarrow_\alpha} y$. Then $U$ is $\alpha$-Scott open in $Y$, and $f^{\leftarrow}(U)$ is $\alpha$-Scott open in $X$. If we assume that $f(\bigsqcup_\alpha D) \notin {\downarrow_\alpha} y$, then $f(\bigsqcup_\alpha D) \in U$, and $\bigsqcup_\alpha D \in f^{\leftarrow}(U)$. It then follows that $D \cap f^{\leftarrow}(U) \neq \emptyset$, and therefore, there exists $d \in D$ with $f(d) \in U$. Thus, $S(f(d), y) \ngeq \alpha$, and $f^{\rightarrow}(D) \not\subset {\downarrow_\alpha} y$. This contradiction shows that $f(\bigsqcup_\alpha D) \in {\downarrow_\alpha} y$ and $S(f(\bigsqcup_\alpha D), y) \geq \alpha$. □

**Notation 3.23.** Let $X$ be a set and $(Y, S)$ an $L$-preset. Define $R_{X \to (Y,S)} : [X \to Y] \times [X \to Y] \to L$ by

$$R_{X \to (Y,S)}(f, g) = \bigwedge\nolimits_{x \in X} S(f(x), g(x)).$$

Let $(X, R)$ and $(Y, S)$ be $L$-presets, and let $\alpha \in L$. We let $[X \to Y]_\alpha$ or $[(X, R) \to (Y, S)]_\alpha$ denote the set of all $\alpha$-Scott continuous functions from $(X, R)$ to $(Y, S)$. •

**Lemma 3.24.** *Let $X$ be a set and $(Y,S)$ an $L$-preset. Then $([X \to Y], R_{X\to(Y,S)})$ is an $L$-preset. If $(Y,S)$ is an $L$-poset, then so is $([X \to Y], R_{X\to(Y,S)})$.* •

*Proof.* Let $f \in [X \to Y]$. Since $\forall x \in X$, $S(f(x), f(x)) = \top_L$, then we have $R_{X\to(Y,S)}(f,f) = \top_L$.

Let $f, g, h \in [X \to Y]$.

$$\begin{gathered} R_{X\to(Y,S)}(f,g) \wedge R_{X\to(Y,S)}(g,h) = \\ (\bigwedge\nolimits_{x\in X} S(f(x), g(x))) \wedge (\bigwedge\nolimits_{x\in X} S(g(x), h(x))) = \\ \bigwedge\nolimits_{x\in X} (S(f(x), g(x)) \wedge S(g(x), h(x))) \leq \\ \bigwedge\nolimits_{x\in X} S(f(x), h(x)) = \\ R_{X\to(Y,S)}(f,h). \end{gathered}$$

Let $f, g \in [X \to Y]$. If

$$R_{X\to(Y,S)}(f,g) \wedge R_{X\to(Y,S)}(g,f) = \top_L,$$

then $\forall x \in X$,

$$S(f(x), g(x)) \wedge S(g(x), f(x)) = \top_L,$$

and therefore, for $\forall x \in X$, $f(x) = g(x)$. Hence, $f = g$. ☐

**Lemma 3.25.** *Let $(X,R)$ be an $L$-preset, $x \in X$, $\alpha \in L$, $(Y,S)$ an $\alpha$-cpro, and $D$ an $\alpha$-directed subset of $([(X,R) \to (Y,S)]_\alpha, R_{X\to(Y,S)})$. Then $D_{x\mapsto}$ is an $\alpha$-directed subset of $Y$.* •

*Proof.* Let $\{f_1, \ldots, f_n\}$ be an arbitrary finite subset of $D$. It follows that $\{f_1(x), \ldots, f_n(x)\}$ is a finite subset of $Y$. Since $\{f_1, \ldots, f_n\}$ is a finite subset of $D$, there exists $f \in D$ such that for each $i$, $R_{X\to(Y,S)}(f_i, f) \geq \alpha$. Therefore, for each $i$, $S(f_i(x), f(x)) \geq \alpha$, and thus, $D_{x\mapsto}$ is an $\alpha$-directed subset of $Y$. ☐

In the proof of the next theorem, the axiom of choice is used in the definition of the function $h$.

**Theorem 3.26.** *Let $(X,R)$ be an $L$-preset; let $\alpha \in L$; and let $(Y,S)$ be an $\alpha$-cpro. Then $([(X,R) \to (Y,S)]_\alpha, R_{X\to(Y,S)})$ is also an $\alpha$-cpro.* •

*Proof:* Let $\perp_{(Y,S)_\alpha}$ be an $\alpha$-bottom element of $(Y,S)$, and define $\perp_{[X\to Y]_\alpha} : (X,R) \to (Y,S)$ by $\perp_{[X\to Y]_\alpha}(x) = \perp_{(Y,S)_\alpha}$, for all $x \in X$. Then $\perp_{[X\to Y]_\alpha}$ is an $\alpha$-bottom element of $[X \to Y]_\alpha$ because $\forall y \in Y, S(\perp_{(Y,S)_\alpha}, y) \geq \alpha$.

Let $D$ be an $\alpha$-directed subset of $([(X,R) \to (Y,S)], R_{X\to(Y,S)})$. For each $x \in X$, $D_{x\mapsto}$ is an $\alpha$-directed subset of $Y$, and thus, $\bigsqcup_\alpha D_{x\mapsto}$ exists because $(S,Y)$ is an $\alpha$-cpro. Define $h : (X,R) \to (Y,S)$ so that $\forall x \in X$,

$$h(x) = \bigsqcup\nolimits_\alpha D_{x\mapsto}. \tag{3.1}$$

We claim that $h$ is $\alpha$-Scott continuous and that $h = \bigsqcup_\alpha D$.

Let $a, b \in X$ with $R(a, b) \geq \alpha$. For each $f \in D$, $S(f(a), f(b)) \geq \alpha$ because $f$ is $\alpha$-Scott continuous. Since for each $f \in D$, $S(f(b), \bigsqcup_\alpha D_{b\mapsto}) \geq \alpha$, then by the $L$-transitivity of $S$, $S(f(a), \bigsqcup_\alpha D_{b\mapsto}) \geq \alpha$ for each $f \in D$. Thus, $\bigsqcup_\alpha D_{b\mapsto}$ is an $\alpha$-upper bound for $D_{a\mapsto}$, and therefore, it follows that $S(\bigsqcup_\alpha D_{a\mapsto}, \bigsqcup_\alpha D_{b\mapsto}) \geq \alpha$. Hence, $S(h(a), h(b)) \geq \alpha$, and $h$ is $\alpha$-order-preserving.

Let $E$ be an $\alpha$-directed subset of $(X, R)$ such that $\bigsqcup_\alpha E$ exists. Since $h$ is $\alpha$-order-preserving, then $h^\rightarrow(E)$ is an $\alpha$-directed subset of $(Y, S)$ and $h(\bigsqcup_\alpha E)$ is an $\alpha$-upper bound for $h^\rightarrow(E)$.

Let $y \in Y$ be an $\alpha$-upper bound for $h^\rightarrow(E)$. Let $f \in D$, and let $x \in X$. From Equation (3.1), we get $S(f(x), h(x)) \geq \alpha$. Therefore, for each $x \in E$, $S(f(x), h(x)) \geq \alpha$, and thus, $y$ is also an $\alpha$-upper bound for $f^\rightarrow(E)$. Thus, since each $f \in D$ is $\alpha$-Scott continuous, we have $S(f(\bigsqcup_\alpha E), y) \geq \alpha$. Since $D$ is an $\alpha$-directed subset of $([(X, R) \to (Y, S)], R_{X\to(Y,S)})$ and since $\bigsqcup_\alpha E \in X$, then $D_{\bigsqcup_\alpha E\mapsto}$ is an $\alpha$-directed subset of $Y$, and since $(Y, S)$ is an $\alpha$-cpro, then $\bigsqcup_\alpha(D_{\bigsqcup_\alpha E\mapsto})$ exists. Since for each $f \in D$, $S(f(\bigsqcup_\alpha E), y) \geq \alpha$, then $S(\bigsqcup_\alpha(D_{\bigsqcup_\alpha E\mapsto}), y) \geq \alpha$, i.e., $S(h(\bigsqcup_\alpha E), y) \geq \alpha$. It follows that, $h(\bigsqcup_\alpha E)$ is an $\alpha$-least upper bound of $h^\rightarrow(E)$, and $h$ is $\alpha$-Scott continuous.

We have yet to show that $h$ is $\bigsqcup_\alpha D$.

Let $f \in D$. For each $x \in X$, $S(f(x), \bigsqcup_\alpha D_{x\mapsto}) \geq \alpha$, and thus, $R_{X\to(Y,S)}(f, h) \geq \alpha$. Suppose $l \in [(X, R) \to (Y, S)]_\alpha$ with $R_{X\to(Y,S)}(k, l) \geq \alpha$ for each $k \in D$. It follows for each $x \in X$ and each $k \in D$, that $S(k(x), l(x)) \geq \alpha$, and therefore, $S(\bigsqcup_\alpha D_{x\mapsto}, l(x)) \geq \alpha$, for all $x \in X$, i.e., $S(h(x), l(x)) \geq \alpha$ for all $x \in X$. Hence, $R_{X\to(Y,S)}(h, l) \geq \alpha$, and $h$ is an $\alpha$-least upper bound of $D$. □

## 4. Conclusion

Working in an $L$-preordered setting, we have defined Scott continuous functions and domain theory-like function spaces with properties which may potentially allow for a fully developed domain theory in a many-valued setting. We have further shown that our generalized setting for defining Scott continuous functions is in some sense maximal. Also, we have shown that when working with $L$-presets, then our antisymmetry condition and the one of Lai and Zhang [6] are equivalent.

We want to thank the referee for many helpful comments and suggestions.

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