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METRIC AXIOMS: A STRUCTURAL STUDY

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Abstract. For a fixed set X, an arbitrary *weight structure* $d \in [0,\infty]^{X\times X}$ can be interpreted as a distance assignment between pairs of points on X. Restrictions (i.e., *metric axioms*) on the behaviour of any such d naturally arise, such as separation, triangle inequality and symmetry. We present an order-theoretic investigation of various collections of weight structures, as naturally occurring subsets of $[0,\infty]^{X\times X}$ satisfying certain metric axioms. Furthermore, we exploit the categorical notion of adjunctions when investigating connections between the above collections of weight structures. As a corollary, we present several lattice-embeddability theorems on a well-known collection of weight structures on X.

1. Introduction

For a fixed set X, a **standard metric** on it is any $d \in [0,\infty]^{X\times X}$ for which:

(i) $\forall x \in X$, $d(x,x) = 0$
(ii) $\forall x, y \in X$, $d(x,y) = 0 = d(y,x) \Rightarrow x = y$ (*separation*)
(iii) $\forall x, y \in X$, $d(x,y) = d(y,x)$ (*symmetry*)
(iv) $\forall x, y, z \in X$, $d(x,y) + d(y,z) \geq d(x,z)$ (*triangle inequality*).

The collection of all standard metrics on X can then be identified with a particular subset of $[0,\infty]^{X\times X}$. For convenience, we shall refer to axioms (i), (ii), (iii) and (iv) as $0, s, \Sigma$ and Δ respectively. By letting P

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