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ON HOMOGENEITY AND THE H-CLOSED PROPERTY

N.A. CARLSON, J.R. PORTER, AND G.J. RIDDERBOS

Abstract. We establish several results concerning topological homogeneity and the weakening of compactness known as the H-closed property. First, it is shown that every Hausdorff space can be embedded in a homogeneous space that is the countable union of H-closed spaces. Second, it is shown that if X is an H-closed Urysohn homogeneous space then for every H-set set $A \subseteq X$, $x \in A$, and $y \notin A$, there exists a homeomorphism $h : X \to X$ such that $h(y) \in A$ and $h(x) \notin A$. This is an extension of Motorov's result that every compact homogeneous space is 1.5-homogeneous. Third, we show that the cardinality bound $2^{t(X)}$, shown to hold for a compact homogeneous space X by De La Vega, does not hold in general for H-closed homogeneous spaces. Last, we show the Katětov H-closed extension κX is never homogeneous if X is non-H-closed, and the remainder $\sigma X \backslash X$ in the H-closed Fomin extension σX is never power homogeneous if X is locally H-closed.

1. Introduction

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. X is *power homogeneous* if there exists a cardinal κ such that X^κ is homogeneous. Many intriguing results have been obtained in the theory of compact homogeneous spaces; for example, De la Vega [11] showed that the cardinality of such a space X is at most $2^{t(X)}$, where $t(X)$ is the tightness of X. Motorov showed that a compact homogeneous space has a stronger form of homogeneity known as $1^1/_2$-homogeneity (see [2]). Many deep questions concerning these spaces are still open (see, for example Jan van Mill's survey in [17]).

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We focus here on questions of homogeneity of a space X in the case where X has the weaker compactness-like property known as *H-closed*. Recall that X is H-closed if every open cover of X has a finite subcover whose union is dense in X. A natural question is to ask whether results such as that of Motorov and De la Vega still hold in the case where X is H-closed and homogeneous. We show in §3 that if an H-closed homogeneous space is additionally Urysohn, then it exhibits a stronger form of homogeneity we denote as $1^1/_4$-homogeneity (Theorem 3.4). This result in fact generalizes Motorov's result. The cardinality bound $2^{t(X)}$ given in De la Vega's Theorem is shown not to hold in general for H-closed homogeneous spaces. We demonstrate this in §4 by giving an example of an H-closed, homogeneous, countably compact, countably tight, separable space X such that $|X| > 2^{t(X)}$ (Theorem 4.4). In §1 an embedding result is given. We show that every Hausdorff space X can be embedded in a homogeneous space that is the countable union of H-closed spaces. (Theorem 2.3). In §5 we focus on homogeneity questions concerning H-closed extensions and their remainders. It is shown in Corollary 5.6 that the Katětov H-closed extension κX and the Fomin H-closed extension σX are never homogeneous for a non-H-closed space X.

Recall that the *semiregularization* of a space X is the space X_s with underlying set X with the regular-open sets of X as a basis. If $h : X \to X$ is a homeomorphism, it is not hard to show that h viewed as a function from X_s to X_s is still a homeomorphism. (See the proof of Proposition 2.1 in [5]). It follows that if X is homogeneous so is X_s. However, it is not necessarily the case that if X_s is homogeneous then X is homogeneous, as is demonstrated in Example 1.1.

A simple process to generate examples of H-closed homogeneous spaces that are not compact is to start with a compact homogeneous space Y and uniformly modify the topology on Y to form a new space X such that $Y = X_s$. As the semiregularization of X is compact, X remains H-closed (see 4.8h(8) in [18], for example). The following example demonstrates this process.

Example 1.1. Let Y be the unit circle with its usual compact, homogeneous topology. Let X be the space formed on the underlying set Y with the following as a basis:

$$\mathcal{B} = \{U \backslash C : U \in \tau(Y), C \in [Y]^\omega\}.$$

One can check that X remains homogeneous and that $Y = X_s$. Although X is clearly not compact, it is still H-closed as its semiregularization is compact. In addition, X has the Urysohn property as X_s is Urysohn.

Now let Z be the space with the same set as Y but with the topology generated by the following basis:

$$\mathcal{B}' = \{U \backslash C : U \in \tau(Y), C \in [Y]^{\omega}, \text{ and } C \text{ converges to } (1,0)\}.$$

One can see that $\tau(X) \supset \tau(Z) \supset \tau(Y)$, $Z_s = Y$, and that Z is H-closed. Let $h : Y \to Y$ be the rigid rotation that takes $(-1, 0)$ to $(1, 0)$. Then $h : Z \to Z$ is not a homeomorphism as it is not continuous, although $h : X \to X$ is a homeomorphism. This shows that Z is not homogeneous even though $Z_s = Y$ is homogeneous. In general for a space W, a homeomorphism $f : W_s \to W_s$ may not remain a homeomorphism viewed as function $f : W \to W$.

The above process of constructing non-compact H-closed homogeneous spaces (such as the space X in Example 1.1) will always generate examples that are Urysohn. In fact, all examples of H-closed homogeneous spaces that the authors are aware of are also Urysohn, including the example given in Theorem 4.4 in §2 below. In view of this, we ask Question 1.2. We would be surprised, however, if the answer to this question were negative.

Question 1.2. *Is there an example of an H-closed homogeneous space that is not Urysohn?*

However, many examples of non-Urysohn homogeneous spaces that are the countable union of H-closed spaces can be generated by the embedding result given in Theorem 2.3.

All spaces under consideration in this paper are Hausdorff. See [13] for any notions not defined here.

2. An Embedding into a Homogeneous Space

In this section we construct an embedding of any Hausdorff space into a homogeneous space which is the union of countably many H-closed subspaces. Uspenskiĭ's showed in [20] that for any space X there exists a cardinal κ and a nonempty subspace $Y \subseteq X^{\kappa}$ such that $X \times Y$ is homogeneous. The space Y is found by selecting a set A such that $|A| = |X|$ and letting $Y = \{f \in X^A : \text{for each } x \in X, |f^{\leftarrow}(x)| = |A|\}$. Both Y and $X \times Y$ are homogeneous and homeomorphic. (See [20] for details.) For our construction we write $H(X) = X \times Y$ and consider X as a subspace of $H(X)$. We begin with the following lemma, the proof of which is straightforward.

Lemma 2.1. *Let X be a space and $h : X \to X$ be a homeomorphism and let id_Y be the identity function on Y. Then the function $h \times id_Y : H(X) \to H(X)$ is also a homeomorphism that extends h.*

Recall that the Katětov H-closed extension of X is defined by $\kappa X = X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$ with basis given by

$$\{U : U \text{ is open in } X\} \cup \{U \cup \{\mathcal{U}\} : U \in \mathcal{U}, \mathcal{U} \in \kappa X \backslash X\}.$$

κX is the projective maximum of all H-closed extensions of X.

Lemma 2.2. *Let X be a Hausdorff space and $h : X \to X$ a homeomorphism. Then there is a homeomorphism $\kappa h : \kappa X \to \kappa X$ that extends h.*

Proof. Let $p \in \kappa X \backslash X$. As p is a free open ultrafilter on X and h is a homeomorphism, $\{h[U] : U \in p\}$ is also a free open ultrafilter on X; denote this free open ultrafilter by $\kappa h(p)$. For $x \in X$, define $\kappa h(x) = h(x)$. It is straightforward to verify that $\kappa h : \kappa X \to \kappa X$ is a homeomorphism that extends h. □

We arrive now at our embedding theorem.

Theorem 2.3. *Let X be a Hausdorff space. Then X can be embedded in a homogeneous space H such that H is the countable union of H-closed spaces.*

Proof. Let $H_1 = H(\kappa X)$. If H_n is defined, let $H_{n+1} = H(\kappa H_n)$ and $H = \bigcup_n H_n$. A subset $U \subseteq H$ is defined to be open in H iff $U \cap H_n \in \tau(H_n)$ for all n. The space H is clearly the countable union of H-closed spaces.To show that H is homogeneous, let $p, q \in H$. There is some n such that $p, q \in H_n$. There is a homeomorphism $h : H_n \to H_n$ such that $h(p) = q$. By applying Lemmas 2.1 and 2.2, h can be extended to a homeomorphism $h_1 : H_{n+1} \to H_{n+1}$. By induction, h can be extended to a homeomorphism $h_k : H_{n+k} \to H_{n+k}$ for $k \in \mathbb{N}$. The function $g = \cup_k h_k : H \to H$ extends h and is a homeomorphism. Thus, H is homogeneous. □

If we start with a space X that is not Urysohn then the homogeneous space H constructed in Theorem 2.3 is also not Urysohn. This leads to many examples of non-Urysohn homogeneous spaces that are the countably union of H-closed spaces. Nevertheless, Question 1.2 remains open.

3. $1^1\!/\!_2$-homogeneity and $1^1\!/\!_4$-homogeneity

Definition 3.1. A space X is *2-homogeneous* if for every $(x_1, y_1) \in X^2$ and $(x_2, y_2) \in X^2$ there exists a homeomorphism $h : X \to X$ such that $h(x_1) = x_2$ and $h(y_1) = y_2$. X is $1^1\!/\!_2$-*homogeneous* if for every closed subset A of X, $x \in A$, and $y \notin A$, there is a homeomorphism $h : X \to X$ such that $h(x) \notin A$ and $h(y) \in A$.

We observe that a 2-homogeneous space is $1^1/_2$-homogeneous and that a $1^1/_2$-homogeneous space is homogeneous. Motorov introduced the notion of $1^1/_2$-homogeneity and established that if a homogeneous space is compact it in fact has the stronger $1^1/_2$-homogeneous property. See [2] and [19].

Theorem 3.2 (Motorov)**.** *Every compact homogeneous space is* $1^1/_2$-*homogeneous.*

We aim to establish an extension of this theorem for H-closed homogeneous spaces and to do so we work with the notion of an H-set. A subset A of a space X is an *H-set* if every cover of A by sets open in X has a finite subfamily whose closures cover A. It can be shown that an H-set is always closed and clearly in a compact space every closed set is an H-set. We define another form of homogeneity known as $1^1/_4$-homogeneity:

Definition 3.3. A space X is $1^1/_4$-*homogeneous* if for every H-set A of X, $x \in A$, and $y \notin A$, there is a homeomorphism $h : X \to X$ such that $h(x) \notin A$ and $h(y) \in A$.

As singleton sets are H-sets, a $1^1/_4$-homogeneous space is homogeneous. As H-sets are closed, a $1^1/_2$-homogeneous space is $1^1/_4$-homogeneous. The proof of the following theorem is similar to the proof of Theorem 3.2 given in [19, Theorem 4.1.2].

Theorem 3.4. *Every H-closed, Urysohn, homogeneous space is* $1^1/_4$-*homogeneous.*

Proof. We will work in the semiregularization X_s of X. Since X is Urysohn and H-closed, so is X_s by [18, 4K(7)] and [18, 4.8(h)(8)]. Therefore X_s is compact by [18, 4.8(k)].

Let A be an H-set of X. As X is H-closed and Urysohn, A is compact in X_s by [18, 4N(11)]. Recall that a homeomorphism $h : X \to X$ is still a homeomorphism viewed as a function $h : X_s \to X_s$. Let $\mathcal{A} = \{h[A] : h \in \mathcal{H}(X)\}$, where $\mathcal{H}(X)$ is the collection of homeomorphisms on X, and let $\mathcal{B}$ be the collection of all non-empty intersections of subfamilies of $\mathcal{A}$ ordered by inclusion. If $\mathcal{C}$ is a decreasing chain in $\mathcal{B}$, then by compactness of X_s, the intersection $\bigcap \mathcal{C}$ is non-empty. Thus every decreasing chain in $\mathcal{B}$ has a lower bound in $\mathcal{B}$. It follows from Zorn's Lemma that the collection $\mathcal{E}$ which consists of minimal elements of $\mathcal{B}$ is non-empty. Note that if h is a homeomorphism of X, and $E \in \mathcal{E}$, then also $h[E] \in \mathcal{E}$.

Since $\mathcal{H}(X)$ acts transitively on X_s, it follows that $\mathcal{E}$ forms a cover of X. Also, since $\mathcal{B}$ is closed under taking non-empty intersections, the collection $\mathcal{E}$ forms a partition of X.

Now let $x \in A$ and $y \notin A$ be arbitrary. For $z \in X$, we let E_z be the unique element of $\mathcal{E}$ that contains z. Note that $E_x \subseteq A$ and hence $y \notin E_x$ and therefore $x \notin E_y$. Since $x \notin E_y$ there is some $h \in \mathcal{H}(X)$ such that $x \notin h[A]$ and $y \in h[A]$. But then we have $h^{-1}(x) \notin A$ and $h^{-1}(y) \in A$. So h^{-1} is the required homeomorphism. $\square$

Since in a compact space every closed set is an H-set, Theorem 3.4 generalizes Motorov's Theorem 3.2.

The Urysohn condition in Theorem 3.4 is required to ensure that X_s is compact. We ask if this condition can be dropped.

Question 3.5. *Is every H-closed homogeneous space* $1^1/_4$*-homogeneous?*

Definition 3.6. We call a space X *weakly* $1^1/_4$*-homogeneous* if for every H-set A of X, and $x \in A$ and $y \notin A$, there is a continuous function $h : X \to X$ such that $h(y) \in A$ and $h(x) \notin A$.

We have the following corollary to Theorem 3.4.

Corollary 3.7. *If* $X \times Y$ *is H-closed, Urysohn, and homogeneous, then* X *is weakly* $1^1/_4$*-homogeneous.*

Proof. Note that X is H-closed by [18, 4.8(l)] and is also Urysohn. Let $f : X \times Y \to X$ be the projection map, let A be an H-set in X, $x \in A$ and $y \notin A$. By [18, 4N(11)] A is compact in X_s. As f is continuous viewed as a function from the compact space $(X \times Y)_s$ to X_s, we see that $f^{-1}[A]$ is compact in $(X \times Y)_s$. Therefore $f^{-1}[A]$ is an H-set in $X \times Y$, again by [18, 4N(11)]. Now fix $p \in Y$. Then $(x, p) \in f^{-1}[A]$ and $(y, p) \notin f^{-1}[A]$. Since $X \times Y$ is $1^1/_4$-homogeneous by Theorem 3.4 and $f^{-1}[A]$ is an H-set, there is a homeomorhpism h of $X \times Y$ such that $h(x, p) \notin f^{-1}[A]$ and $h(y, p) \in f^{-1}[A]$.

Now we define $g : X \to X$ by $g(z) = f(h(z, p))$. Then g is continuous and has the desired properties. $\square$

An immediate corollary of Corollary 3.7 is the following "reflection" property of power homogeneity:

Corollary 3.8. *If* X *is H-closed, Urysohn, and power homogeneous, then* X *is weakly* $1^1/_4$*-homogeneous.*

4. A Counterexample to a Cardinality Bound

A celebrated result in the theory of compact homogeneous spaces is that any such space X has cardinality at most $2^{t(X)}$, where $t(X)$ is the tightness of X. This was shown by De la Vega in [11] and answered a long-standing question of Arhangel′skiĭ. It was generalized to compact power homogeneous spaces in [3]. It has also been generalized in various ways to the Hausdorff setting in [8] and [9], and the Urysohn setting in [10]. A natural question is whether the cardinality bound $2^{t(X)}$ can be generalized to all H-closed homogeneous spaces. This question was asked by Jeffrey Norden in a personal communication to Jack Porter in 1992. In this section we use the countable tightness modification of the Cantor cube $2^{\mathfrak{c}}$ to demonstrate that the cardinality bound $2^{t(X)}$ does not necessarily hold even for an H-closed, countably compact, Urysohn, separable homogeneous space X.

Definition 4.1. Let (X, τ) be a space and κ a cardinal. Define a topology σ on X by declaring the closure of any set $A \subseteq X$ to be as follows:

$$cl_\sigma A = \bigcup_{B \in [A]^{\leq \kappa}} cl_\tau B.$$

Denote the space (X, σ) by X^τ_κ, or by X_κ when τ is understood. We call X^τ_κ the *κ-tightness modification of* (X, τ).

It is clear that the tightness of X_κ is at most κ, and for $\kappa < t(X)$, the space X_κ has a finer topology than X. Thus for a compact space X that is not countably tight, the countable-tightness modification X_ω will no longer be compact as it is not minimal Hausdorff. That is, the topology on X_ω is not minimal in the partial order of all Hausdorff topologies on X and so cannot be compact. However, by the following theorem if X is compact and additionally separable, the space X_ω will still be H-closed and countably compact. This is Lemma 3.6 in [6].

Theorem 4.2 (Carlson)**.** *If X is an H-closed, countably compact, separable space then the countable-tightness modification X_ω is an H-closed, countably compact, countably tight, separable space.*

Proposition 4.3. *If a space X is homogeneous then the κ-tightness modification X_κ is homogeneous for any cardinal κ.*

Proof. We simply need to show that a homeomorphism $h : X \to X$ is still a homeomorphism $h : X_\kappa \to X_\kappa$. Towards showing that $h : X_\kappa \to X_\kappa$ is an open map, pick U open in X_κ and consider a set $A \in [X]^{\leq \kappa}$. As $h^{\leftarrow}[A] \in [X]^{\leq \kappa}$, by Proposition 3.3 in [6] there exists an open set V in X such that $U \cap h^{\leftarrow}[A] = V \cap h^{\leftarrow}[A]$. Hence $h[U] \cap A = h[V] \cap A$. As $h[V]$ is open in X it follows that $h[U]$ is open in X_κ. Establishing that $h^{\leftarrow} : X_\kappa \to X_\kappa$ is open is similar. □

Theorem 4.4. *There exists an H-closed, countably compact, Urysohn, separable, countably tight, homogeneous space X such that $|X| = 2^{\mathfrak{c}} > 2^{t(X)}$.*

Proof. Let Y be the Cantor Cube $2^{\mathfrak{c}}$ with its usual topology and let $X = Y_\omega$. Now Y is compact and separable and hence X is H-closed, countably compact, countably tight, and separable by Theorem 4.2. Since Y is compact, by Lemma 3.5 in [6] Y is the semiregularization of X. Since Y is Urysohn it follows that X is also Urysohn by 4K(7) in [18]. □

Given Theorem 4.4, the question remains as to what may be a suitable cardinality bound for H-closed homogeneous spaces. A fundamental generalization of De la Vega's Theorem to the Hausdorff setting is the cardinality bound $2^{L(X)t(X)pct(X)}$ for power homogeneous Hausdorff spaces X, given in Corollary 3.11 in [8]. Here $pct(X)$ is the *point-compactness type* of X, defined to be the least cardinal κ for which X can be covered by compact subsets K such that $\chi(K, X) \leq \kappa$. (Note every compact space X has $pct(X) = 1$). This bound was improved to the bound $2^{aL_c(X)t(X)pct(X)}$ for power homogeneous Hausdorff spaces X, given in Theorem 3.10 in [9], where $aL_c(X)$ is the *almost Lindelöf degree with respect to closed sets.* (See [9] for the definitions of $aL_c(X)$ and $aL(X)$). It is clear that $aL(X) \leq aL_c(X) \leq L(X)$. However, despite the fact that H-closed spaces have finite $aL(X)$, there are H-closed spaces X with arbitrarily large $aL_c(X)$ and so this cardinality bound does not give any improvement in the H-closed setting. Nevertheless, the following question suggests a possible cardinality bound for H-closed homogeneous spaces.

Question 4.5. *Is the cardinality of an H-closed homogeneous space at most $2^{t(X)pct(X)}$?*

De la Vega's Theorem is closely related to Arhangel'skii's Theorem that a compact space has cardinality at most $2^{\chi(X)}$. Both results follow from a generalized result given by Arhangel′skiĭ in [4]. Dow and Porter [12] showed that the bound $2^{\chi(X)}$ holds for all H-closed spaces using remainders of H-closed extensions of discrete spaces. Hodel [15] gave another proof of this bound for H-closed spaces using κ-nets. We conjecture that if one wishes to generalize De la Vega's Theorem to the H-closed setting by answering Question 4.5 in the affirmative, then a proof may hinge upon using the techniques Dow/Porter or Hodel used in generalizing Arhangel'skii's Theorem to the H-closed setting.

5. Non-homogeneity of H-closed Extensions and Remainders

For Tychonoff spaces X, the homogeneity of βX and the remainder $\beta X \backslash X$ has been extensively studied. For example, if X is an F-space, that is every cozero set of X is C^*-embedded, then no power of βX is homogenous. This follows from the fact that no compact F-space is power homogeneous (Kunen [16]) and that X is an F-space iff βX is an F-space. If X is a locally compact F-space we recall the following well-known result.

Theorem 5.1. *If a Tychonoff space X is a non-compact locally compact F-space then the remainder $\beta X \backslash X$ is not power homogeneous.*

Proof. As X is locally compact the remainder $\beta X \backslash X$ is compact. In addition $\beta X \backslash X$ is an F-space. (See, for example, Exercise 14O(3) in [14]). Finally, no compact F-space can be power homogeneous. □

In this section we seek to establish similar results for the Katětov H-closed extension κX and the Fomin H-closed extension σX for non-H-closed Hausdorff spaces X. σX is the strict extension $(\kappa X)^{\#}$ of X. See [18] for the construction of σX. We will need the following definition.

Definition 5.2. A *maximal point* of a space X is a point $p \in X$ such that if $p \in clU$ for an open set $U \subseteq X$, then $\{p\} \cup U$ is open in X.

Proposition 5.3. *Let X be a space with a maximal point p such that the semiregularization X_s is homogeneous. Then X_s is extremally disconnected.*

Proof. Let U and V be disjoint open subsets of X_s. We wish to show $cl_{X_s}U \cap cl_{X_s}V = \varnothing$. Suppose there exists $x \in cl_{X_s}U \cap cl_{X_s}V$. As X_s is homogeneous, there exists a homeomorphism $h : X_s \to X_s$ such that $h(x) = p$. Then, as $h[U]$ and $h[V]$ are open in X_s and also X, we have

$$\begin{aligned} p = h(x) &\in h[cl_{X_s}U \cap cl_{X_s}V] \\ &= cl_{X_s}[h[U]] \cap cl_{X_s}[h[V]] \\ &= cl_X[h[U]] \cap cl_X[h[V]]. \end{aligned}$$

As p is a maximal point of X and $h[U]$ is open in X, it follows that $\{p\} \cup h[U]$ is open in X. As $p \in cl_X[h[V]]$, we have that $(\{p\} \cup h[U]) \cap h[V] \neq \varnothing$. But $h[U] \cap h[V] = \varnothing$ so $p \in h[V]$. Likewise, $p \in h[U]$. This is a contradiction so $cl_{X_s}U \cap cl_{X_s}V = \varnothing$. □

Corollary 5.4. *If X is an H-closed space with a maximal point, then neither X_s nor X is homogeneous.*

Proof. The semiregularization of an H-closed space is H-closed, and by Proposition 5.3, X_s is also extremally disconnected. It was shown in Theorem 3.3 in [5] that no H-closed extremally disconnected space is homogeneous, thus X_s is not homogeneous. Finally, by Proposition 2.1 in [5] X cannot be homogeneous. □

We note, however, that if the H-closed space in Corollary 5.4 is in fact compact, it cannot have a maximal point at all. Alas and Wilson [1] showed that no compact space can contain a maximal point, and this was extended in [7] to any locally countably compact space. Yet H-closed spaces may have maximal points. The following characterization of maximal points given in [1] shows that every point of $\kappa X \backslash X$ is a maximal point of the H-closed space κX.

Theorem 5.5 (Alas and Wilson)**.** *A point $p \in X$ is a maximal point if and only if the trace of its open neighborhood filter on $X \backslash \{p\}$ is an open ultrafilter.*

Corollary 5.6. *κX, $(\kappa X)_s$, and σX are never homogeneous for any non-H-closed space X.*

Proof. Each point p of $\kappa X \backslash X$ is an open ultrafilter on X and furthermore the trace of its open neighborhood filter on $\kappa X \backslash \{p\}$ is an open ultrafilter. By Theorem 5.5, p is a maximal point of κX. By Corollary 5.4, neither κX nor $(\kappa X)_s$ are homogeneous. Finally, as $(\sigma X)_s = (\kappa X)_s$, it follows that σX is not homogeneous by Proposition 2.1 in [5]. □

The above corollary should be contrasted with the fact that every homeomorphism on a space X can be extended to a homeomorphism on κX. (Lemma 2.2). We ask the following question.

Question 5.7. *Can κX be power homogeneous for a non-H-closed space X?*

We turn now to the question of homogeneity of remainders in H-closed extensions. First we note that $\kappa X \backslash X$ is always a discrete space and so is trivially homogeneous. We investigate here the homogeneity of $\sigma X \backslash X$. We will use the Iliadis absolute EX and the Gleason space ΘX (see [18]) in the following proposition. Recall that a Hausdorff space X is *locally H-closed* if every point has a neighborhood that is H-closed.

Proposition 5.8. *If X is locally H-closed then EX is locally compact.*

Proof. If X is locally H-closed then $\sigma X \backslash X$ is compact by 7.3(c) in [18]. As the absolute $\Theta X \approx \beta(EX)$ by 6.6(e)(1) in [18], $\sigma X \backslash X \approx \beta(EX) \backslash EX$ by 7.2(c)(1) in [18], it follows that $\sigma X \backslash X \approx \Theta X \backslash EX$. Thus $\Theta X \backslash EX$ is compact and hence closed in ΘX. As EX is dense in ΘX, this makes EX open in its extension ΘX. Thus EX is locally compact. □

We now give an analogue to Theorem 5.1 above.

Theorem 5.9. *If X is locally H-closed then $\sigma X \backslash X$ is not power homogeneous.*

Proof. As X is locally H-closed, by Proposition 5.8 EX is locally compact. Additionally, EX is an F-space as it is extremally disconnected. Thus, $\beta(EX) \backslash EX \approx \sigma X \backslash X$ is not power homogeneous by Theorem 5.1. □

We note that in contrast to Theorem 5.1, in Theorem 5.9 X is not required to be an F-space or any variation of an F-space suitably defined for Hausdorff spaces.

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