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A STABILITY CONJECTURE FOR THE COLORED JONES POLYNOMIAL

STAVROS GAROUFALIDIS AND THAO VUONG

Abstract. We formulate a stability conjecture for the coefficients of the colored Jones polynomial of a knot when the color lies in a fixed ray of a simple Lie algebra. Our conjecture is motivated by a structure theorem for the degree and the coefficients of a q-holonomic sequences given in [6] and by a stability theorem of the colored Jones polynomial of an alternating knot given in [8]. We prove our conjecture for all torus knots and all simple Lie algebras of rank 2. Finally, we illustrate our results with a few explicit q-series.

1. Introduction

1.1. The degree and coefficients of a q-holonomic sequence. Our goal is to formulate a stability conjecture for the coefficients of q-holonomic sequences that appear naturally in Quantum Knot Theory [7]. Our conjecture is motivated by

(a) a structure theorem for the degree and coefficients of a q-holonomic sequence of polynomials given in [6],
(b) a stability theorem of the colored Jones polynomial of an alternating knot [8].

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To discuss our first motivation, recall [22] that a sequence $(f_n(q))$ is q-*holonomic* if it satisfies a linear recursion

$$\sum_{j=0}^{d} c_j(q^n, q) f_{n+j}(q) = 0$$

for all n where $c_j(u,v) \in \mathbb{Z}[u,v]$ and $c_d \neq 0$. Here, $f_n(q)$ is either in $\mathbb{Z}[q^{\pm 1}]$, the ring of Laurent polynomials with integer coefficients, or more generally in $\mathbb{Q}(q)$, the field of rational functions with rational coefficients or even $\mathbb{Z}((q))$, the ring of Laurent power series in q $\sum_{j\in\mathbb{Z}} a_j q^j$ (with a_j integers, vanishing when j is small enough). $\mathbb{Z}((q))$ has a subring $\mathbb{Z}[[q]]$ of formal power series in q, where $a_j = 0$ for $j < 0$. The *degree* $\delta^*(f(q))$ of $f(q) \in \mathbb{Z}((q))$ is the smallest integer m such that $q^m f(q) \in \mathbb{Z}[[q]]$.

Thus, we can expand every non-zero sequence $(f_n(q))$ in the form

$$f_n(q) = a_0(n) q^{\delta^*(n)} + a_1(n) q^{\delta^*(n)+1} + a_2(n) q^{\delta^*(n)+2} + \dots \tag{1.1}$$

where $\delta^*(n)$ is the degree of $f_n(q)$ and $a_k(n)$ is the k-th coefficient of $q^{-\delta^*(n)} f_n(q)$, reading from the left. We will often call $a_k(n)$ the k-th *stable coefficient* of the sequence $(f_n(q))$.

In [6] it was proven that if $(f_n(q))$ is q-holonomic, then

- $\delta^*(n)$ is a *quadratic quasi-polynomial* for all but finitely many values of n,
- for every $k \in \mathbb{N}$, $a_k(n)$ is *recurrent* for all but finitely many values of n.

Recall that a quasi-polynomial (of degree at most d) is a function of the form

$$p : \mathbb{N} \longrightarrow \mathbb{Z}, \qquad n \mapsto p(n) = \sum_{j=0}^{d} c_j(n) n^j$$

where $c_j : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic functions. Let $\mathcal{P}$ denote the ring of integer-valued quasi-polynomials. A recurrent sequence is one that satisfies a linear recursion with constant coefficients. Recurrent sequences are well-known in number theory under the name of *Generalized Exponential Sums* [21, 5]. The latter are expressions of the form

$$a(n) = \sum_{i=1}^{m} A_i(n) \alpha_i^n$$

with *roots* α_i, $1 \leq i \leq m$ distinct algebraic numbers and polynomials A_i. Integer-valued generalized exponential sums form a ring $\mathcal{E}$, which contains a subring $\mathcal{P}$ that consists of integer-valued exponential sums whose roots are complex roots of unity.

1.2. **Stability of the colored Jones polynomial of an alternating link.** The second motivation of our Conjecture 1.5 below comes from the stability theorem of [8] that concerns the colored Jones polynomial of an alternating link. Let $\mathbb{Z}((q))$ denote the ring of formal Laurent power series in q with integer coefficients. Every element of $\mathbb{Z}((q))$ has the form $f(q) = \sum_{n=n_0}^{\infty} a_n q^n$ for some $n_0 \in \mathbb{Z}$ and $a_n \in \mathbb{Z}$. If $f(q) \neq 0$, the smallest n such that $a_n \neq 0$ is denoted by $\delta^*(f)$. Given $f_n(q), f(q) \in \mathbb{Z}((q))$, we say that

$$\lim_{n\to\infty} f_n(q) = f(q)$$

if there exists C such that $\delta^*(f_n(q)) > C$ for all n, and for every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that

$$f_n(q) - f(q) \in q^m \mathbb{Z}[[q]]$$

for all $n \geq n_m$. The next definition of stability appears in [7] and the notion of its tail is inspired by Dasbach-Lin [4].

Definition 1.1. We say that a sequence $f_n(q) \in \mathbb{Z}[[q]]$ is *stable* if there exists a series $F(x,q) = \sum_{k=0}^{\infty} \Phi_k(q) x^k \in \mathbb{Z}((q))[[x]]$ such that for every $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) = 0\,. \tag{1.2}$$

We will call $F(x,q)$ the (x,q)-*tail* (in short, the tail) of the sequence $(f_n(q))$.

Examples of stable sequences are the shifted colored Jones polynomials of an alternating link. Let $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1/2}]$ denote the colored Jones polynomial of a link K colored by the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ (see [19, 20]). Let $\delta^*_K(n)$ and $a_{K,0}(n)$ denote the degree and the 0-th stable coefficient of $J_{K,n}(q)$. It is well-known that $a_{K,0}(n) = (-1)^{c_- n}$ where c_- is the number of negative crossings of K [15].

Theorem 1.2. [8] *If K is an alternating link, then the sequence $a_{K,0}(-n) q^{-\delta^*_K(n)} J_{K,n}(q) \in \mathbb{Z}[q]$ is stable.*

1.3. *c***-stability.** We are now ready to introduce the notion of c-stability.

Definition 1.3. We say that a sequence $f_n(q) \in \mathbb{Z}((q))$ with q-degree $\delta^*(n)$ is *c-stable* (i.e., *cyclotomically stable*) if there exists a series

$$F(n,x,q) = \sum_{k=0}^{\infty} \Phi_k(n,q) x^k \in \mathcal{P}((q))[[x]]$$

such that for every $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} q^{-k(n+1)} \left(q^{-\delta^*(n)} f_n(q) - \sum_{j=0}^{k} \Phi_j(n,q) q^{j(n+1)} \right) = 0 \,. \tag{1.3}$$

We will call $F(n, x, q)$ the (n, x, q)-tail (in short, tail) of the sequence $(f_n(q))$.

Remark 1.4. The stable coefficients of a c-stable sequence $(f_n(q))$ are quasi-polynomials. I.e., with the notation of Equation (1.1), we have that $a_k \in \mathcal{P}$ for all k. In fact, if $(f_n(q))$ is c-stable and $l \in \mathbb{N}$, the stable coefficients of the sequence

$$q^{-l(n+1)} \left(f_n(q) - \sum_{j=0}^{l-1} \Phi_j(q) q^{j(n+1)} \right)$$

are quasi-polynomials.

1.4. **Our results.** For a knot K in S^3, colored by an irreducible representation V_λ of a simple Lie algebra $\mathfrak{g}$ with highest weight λ, one can define the colored Jones polynomial $J^{\mathfrak{g}}_{K,V_\lambda}(q) \in \mathbb{Z}[q^{\pm 1}]$ [19, 20]. This requires a rescaled definition of q, which depends only on the Lie algebra and not on the knot, and is discussed carefully in [14]. In [7] it was shown that for every knot K and every simple Lie algebra $\mathfrak{g}$, the function $\lambda \mapsto J^{\mathfrak{g}}_{K,V_\lambda}(q)$ (and consequently the sequence $(J^{\mathfrak{g}}_{K,n\lambda}(q))$) is q-holonomic.

Conjecture 1.5. Fix a knot K, a simple Lie algebra $\mathfrak{g}$ and a dominant weight λ of $\mathfrak{g}$. Then the sequence $(J^{\mathfrak{g}}_{K,n\lambda}(q))$ of colored Jones polynomials is c-stable.

Theorem 1.6. *Conjecture 1.5 holds for all torus knots and all rank 2 simple Lie algebras.*

For a precise formula for the tail, see Theorem 7.2.

An earlier publication [9] gives an alternative proof of Theorem 1.6 for the trefoil and the case of the A_2 simple Lie algebra.

Remark 1.7. Theorem 1.2 implies that if K is an alternating knot with c_- crossings and $k \in \mathbb{N}$, the k-th stable coefficient $a_{K,k}(n)$ of the sequence $(J_{K,n}(q))$ is given by

$$a_{K,k}(n) = (-1)^{c_- n} \mathrm{coeff}(\Phi_{K,0}(q), q^k)$$

and satisfies the first order linear recurrence relation

$$a_{K,k}(n+1) - (-1)^{c_-} a_{K,k}(n) = 0 \,.$$

Here $\mathrm{coeff}(f(q), q^k)$ denotes the coefficient of q^k in $f(q) \in \mathbb{Z}((q))$. The stable coefficients $c_{K,k}$ of an alternating knot K are studied in [11, 10]. In all examples of the colored Jones polynomial of a knot that have been analyzed (this includes alternating knots, torus knots and the 2-fusion knots), the k-stable coefficient is a quasi-polynomial of degree 0, i.e., it is constant on suitable arithmetic progressions. One might think that this holds for all simple Lie algebras. Example 1.10 below shows that this is not the case, hence the notion of c-stability is necessary.

1.5. **A sample of q-series.** In this section we give a concrete sample of tails and q-series that appear in our study.

Example 1.8. Consider the theta series given by [3]

$$\theta_{b,c}(q) = \sum_{s\in\mathbb{Z}} (-1)^s q^{\frac{b}{2}s^2+cs} . \tag{1.4}$$

In Section 10 we will prove the following.

Theorem 1.9. *The tail of the c-stable sequence $(J^{A_2}_{T(2,b),n\lambda_1}(q))$ for $b > 2$ odd is given by*

$$\frac{\theta_{b,\frac{b}{2}-1}(q)(1+q^3x^2) + q^3\theta_{b,\frac{b}{2}+2}(q)x}{(1-q)(1-qx)(1-q^2x)} .$$

In particular, when $b = 3$ (i.e., the case of the trefoil), the tail equals to

$$(q)_\infty \frac{1-qx+q^3x^2}{(1-q)(1-qx)(1-q^2x)} .$$

Here, $(x;q)_\infty = \prod_{k=0}^\infty (1-q^k x)$ and $(q)_\infty = (q;q)_\infty$.

Example 1.10. The tail of the c-stable sequence $(J^{A_2}_{T(4,5),n\rho}(q))$ is given by

$$\frac{1}{(1-xq)^2(1-x^2q^2)} \left(A_0(q) + nA_1(q)\right)$$

where $A_0(q), A_1(q) \in \mathbb{Z}[[q]]$ are given explicitly in Proposition 10.3. The first few terms of those q-series are given by

$$\begin{aligned}
A_0(q) &= 1 - 2q + 2q^3 - q^4 + q^{48} - 2q^{55} - 2q^{57} + 2q^{63} + 2q^{66} + 2q^{69} \\
&\quad + 2q^{75} - q^{76} - 2q^{78} - 2q^{81} - 2q^{82} - q^{84} - 2q^{85} + 2q^{87} + \dots \\
A_1(q) &= 1 - 2q + 2q^3 - q^4 - q^6 + 2q^9 + 2q^{10} - 2q^{12} - 4q^{15} - q^{18} + 2q^{19} \\
&\quad + 2q^{21} + 3q^{22} - 2q^{27} - 2q^{30} - 2q^{33} + 4q^{36} - q^{42} - 2q^{46} + q^{48} \\
&\quad - 2q^{49} + 2q^{51} + 2q^{55} + 4q^{57} - 2q^{58} + 2q^{60} - 4q^{64} - 2q^{66} - 2q^{69} \\
&\quad - 2q^{73} - q^{76} + 4q^{78} + 2q^{81} + 2q^{82} + q^{84} + 2q^{85} - 2q^{87} + \dots
\end{aligned}$$

It follows that for every fixed k, the k-th stable coefficient $a_k(n)$ of $(J^{A_2}_{T(4,5),n\rho}(q))$ satisfies the linear recursion relation

$$a_k(n+2) - 2a_k(n+1) + a_k(n) = 0$$

for all n.

Using the methods of [3], one can show that

$$\begin{aligned}
A_1(q) = &\sum_{m_1,m_2\in\mathbb{Z}} q^{20(m_1^2+3m_1m_2+3m_2^2)+2m_1+3m_2} \\
&\cdot (1-q^{4m_1+1})(1-q^{4m_1+12m_2+1})(1-q^{8m_1+12m_2+2}) \\
= &(q)_\infty \left(\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{15n^2+n}{2}} - \sum_{n\in\frac{3}{5}+\mathbb{Z}} (-1)^n q^{\frac{15n^2+n}{2}} \right) \\
= &(q)_\infty \big((q^7;q^{15})_\infty (q^8;q^{15})_\infty (q^{15};q^{15})_\infty \\
&-q(1-q^2)(q^{13};q^{15})_\infty (q^{15};q^{15})_\infty (q^{17};q^{15})_\infty \big) \ .
\end{aligned}$$

2. Lie algebra notation

We recall some standard Lie algebra notation that we will use throughout the paper [2, 12].

$\mathfrak{g}$ denotes a simple Lie algebra over the complex numbers. W is the Weyl group of $\mathfrak{g}$, Λ and Λ_r are the weight and root lattices of $\mathfrak{g}$. Λ^+ denotes the set of dominant weights, with respect to a choice of Weyl chamber.

$p : \Lambda_r \longrightarrow \mathbb{N}$ denotes the Kostant partition function which is the number of ways to express an element of the root lattice as a linear combination (of nonnegative integer coefficients) of positive roots of $\mathfrak{g}$.

Let ρ denote half of the sum of positive roots.

If $\lambda \in \Lambda^+$ is a dominant weight, V_λ denote the irreducible representation of highest weight λ, and Π_λ denote the set of weights of V_λ.

If r is the rank of $\mathfrak{g}$, we we denote by α_i for $i = 1, \dots, r$ the simple roots of $\mathfrak{g}$, and by λ_i for $i = 1, \dots, r$ the fundamental weights of $\mathfrak{g}$.

3. The colored Jones polynomial of a torus knot

3.1. The Jones-Rosso formula.

3.1. The Jones-Rosso formula. To verify Conjecture 1.5 for all torus knots $T(a,b)$ (where $0 < a < b$ and a and b are coprime integers), we will use the formula of Jones-Rosso [16]. It states that

$$J^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \frac{\theta_\lambda^{-ab}}{d_\lambda} \sum_{\mu\in S_{\lambda,a}} m^\mu_{\lambda,a} d_\mu \theta_\mu^{\frac{b}{a}} \tag{3.1}$$

where

- d_λ is the *quantum dimension* of V_λ and θ_λ is the eigenvalue of the *twist* operator on the representation V_λ given by:

$$d_\lambda = \prod_{\alpha>0} \frac{[(\lambda+\rho,\alpha)]}{[(\rho,\alpha)]}, \qquad \theta_\lambda = q^{\frac{1}{2}(\lambda,\lambda+2\rho)}, \qquad [n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,. \tag{3.2}$$

- $m^\mu_{\lambda,a} \in \mathbb{Z}$ is the multiplicity of V_μ in the a-*plethysm* of V_λ where ψ_a denote the a-Adams operation. I.e., we have:

$$\psi_a(ch_\lambda) = \sum_{\mu\in S_{\lambda,a}} m^\mu_{\lambda,a} ch_\mu \tag{3.3}$$

where ch_λ is the formal character of V_λ.

To describe the plethysm multiplicity $m^\mu_{\lambda,a}$ and the summation set $S_{\lambda,a}$, recall the Kostant multiplicity formula [13] which expresses the multiplicities m^μ_λ of the μ-weight space of V_λ in terms of the Kostant partition function p:

$$m^\mu_\lambda = \sum_{\sigma\in W} (-1)^\sigma p(\sigma(\lambda+\rho) - \mu - \rho)\,. \tag{3.4}$$

As usual, W is the Weyl group of the simple Lie algebra $\mathfrak{g}$ and ρ is half the sum of its positive roots.

Lemma 3.1. (a) We have:

$$m^\mu_{\lambda,a} = \sum_{\sigma\in W} (-1)^\sigma m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{a}} \tag{3.5}$$

where the summation is over the elements $\sigma \in W$ such that $\frac{\mu+\rho-\sigma(\rho)}{a}$ is in the weight lattice (but not necessarily a dominant weight).
(b) It follows that

$$S_{\lambda,a} = \left[\bigcup_{\sigma\in W} (\sigma(\rho) - \rho + a\Pi_\lambda) \right] \cap \Lambda^+ \tag{3.6}$$

where Π_λ is the set of all weights of V_λ and Λ^+ is the set of dominant weights of $\mathfrak{g}$.

Remark 3.2. The Jones-Rosso formula (3.1) combined with Equations (3.4) and (3.5) imply that that we can write

$$J^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \sum_{\sigma,\sigma'\in W} J^{\mathfrak{g}}_{T(a,b),\lambda,\sigma,\sigma'}(q) \tag{3.7}$$

for some rational functions $J^{\mathfrak{g}}_{T(a,b),\lambda,\sigma,\sigma'}(q)$. It is easy to see that the sequences $(J^{\mathfrak{g}}_{T(a,b),n\lambda,\sigma,\sigma'}(q))$ are q-holonomic (with respect to n) and c-stable. If cancellation of the leading and trailing terms did not occur in Equation (3.7), it would imply a short proof of Theorem 1.6 for all torus knots and all simple Lie algebras. Unfortunately, after we perform the sum in Equation (3.7) cancellation occurs and the degree of the summand is much lower than the degree of the sum. This already happens for A_2 and the trefoil, an alternating knot. This cancellation is responsible for the minimizer $\mu_{\lambda,a}$ to be of order $O(\lambda)$ rather than $O(1)$ in case A_2, part (b) of Theorem 3.4.

3.2. **The degree of the colored Jones polynomial.** The Jones-Rosso formula can be written in the form

$$J^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \frac{q^{-\frac{ab}{2}(\lambda,\lambda)-(-1+ab)(\lambda,\rho)}}{\prod_{\alpha\succ 0}(1-q^{(\lambda+\rho,\alpha)})} \cdot \sum_{\mu\in S_{\lambda,a}} q^{\frac{b}{2a}(\mu,\mu)+(-1+\frac{b}{a})(\mu,\rho)} \prod_{\alpha\succ 0}(1-q^{(\mu+\rho,\alpha)}) \,. \tag{3.8}$$

Here the products are taken over the set of positive roots $\{\alpha\}$ of $\mathfrak{g}$. When the dominant weight λ and the torus knot $T(a,b)$ is fixed, the minimum and the maximum degree of the summand are positive-definite quadratic forms $f^*(\mu)$ and $f(\mu)$ given by

(3.9a)
$$f^*(\mu) = \frac{b}{2a}(\mu,\mu) + \left(-1+\frac{b}{a}\right)(\mu,\rho) - \frac{ab}{2}(\lambda,\lambda) - (-1+ab)(\lambda,\rho)$$

(3.9b)
$$f(\mu) = \frac{b}{2a}(\mu,\mu) + \left(1+\frac{b}{a}\right)(\mu,\rho) - \frac{ab}{2}(\lambda,\lambda) - (1+ab)(\lambda,\rho)\,.$$

In Section 8 we will prove the following.

Theorem 3.3. *Fix a simple Lie algebra $\mathfrak{g}$ and a torus knot $T(a,b)$. The quadratic form $f(\mu)$ achieves maximum uniquely at $M_{\lambda,a} = a\lambda \in S_{a,\lambda}$. Moreover, $m_{\lambda,a}^{M_{\lambda,a}} = 1$.*

The next theorem states that $f^*(\mu)$ has a unique minimizer which we denote by $\mu_{\lambda,a}$ and describes $\mu_{\lambda,a}$ explicitly for all simple Lie algebras of rank 2. Below, $\{\lambda_1,\lambda_2\}$ are the dominant weights of a simple Lie algebra of rank 2. Its proof is given in Section 9 using a case-by-case analysis.

Theorem 3.4. *When $\mathfrak{g}$ is a simple Lie algebra of rank 2, then*
(a)The quadratic form $f^*(\mu)$ achieves minimum uniquely at $\mu_{\lambda,a} \in S_{a,\lambda}$ and $m^{\mu}_{\lambda,a} \neq 0$.
(b) For a dominant weight $\lambda = m_1\lambda_1 + m_2\lambda_2$, we have
For A_2:

$$\mu_{\lambda,2} = \begin{cases} (m_1-m_2)\lambda_2 & \text{if } m_1 \geq m_2 \\ (m_2-m_1)\lambda_1 & \text{if } m_1 \leq m_2 \end{cases} \qquad \mu_{\lambda,3} = 0$$

and

$$\mu_{\lambda,a} = \begin{cases} 0 & \text{if } m_1 \equiv m_2 \bmod 3 \\ (a-3)\lambda_1 & \text{if } m_1 \equiv m_2 + 1 \bmod 3 \\ (a-3)\lambda_2 & \text{if } m_1 \equiv m_2 + 2 \bmod 3 \end{cases} \quad \text{for } a \geq 4\,.$$

For B_2:

$$\mu_{\lambda,2} = \begin{cases} \lambda_1 & \text{if } m_1 = 0\,, m_2 \equiv 1 \bmod 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\lambda,3} = \begin{cases} 0 & \text{if } m_1, m_2 \equiv 0 \bmod 2 \\ 2\lambda_2 & \text{if } m_1 \equiv 1 \bmod 2\,, m_2 \equiv 0 \bmod 2 \\ \lambda_1 + \lambda_2 & \text{if } m_2 \equiv 1 \bmod 2 \end{cases}$$

$$\mu_{\lambda,4} = 0 \qquad \mu_{\lambda,a} = \begin{cases} 0 & \text{if } m_2 \equiv 0 \bmod 2 \\ (a-4)\lambda_2 & \text{if } m_2 \equiv 1 \bmod 2 \end{cases} \quad \text{for } a \geq 5\,.$$

For G_2:

$$\mu_{\lambda,a} = 0 \text{ for } a \geq 2\,.$$

Theorem 3.4 part (b) implies the following.

Corollary 3.5. *$\mu_{n\lambda,a}$ is a piecewise quasi-linear (i.e., quasi-polynomial of degree 1) function of n for $n \gg 0$.*

Let $\delta^*_K(\lambda)$ and $\delta_K(\lambda)$ denote the *minimum* and the *maximum* degree of the colored Jones polynomial $J^{\mathfrak{g}}_{K,V_\lambda}(q)$ with respect to q.

Corollary 3.6. *We have:*

$$\delta^*_{T(a,b)}(\lambda) = f^*(\mu_{\lambda,a}) \tag{3.10a}$$

$$\delta_{T(a,b)}(\lambda) = f(a\lambda)\,. \tag{3.10b}$$

4. Some lemmas about stability

In this section we collect some lemmas about stable sequences.

Lemma 4.1. *Fix natural numbers c and d and consider $g_n(q) = \frac{f_n(q)}{1-q^{cn+d}}$. Then $(f_n(q))$ is stable if and only if $(g_n(q))$ is stable. In that case, their corresponding tails $F(x,q)$ and $G(x,q)$ satisfy*

$$G(x,q) = \frac{F(x,q)}{1-q^d x^c}\,. \tag{4.1}$$

Proof. Let

$$F(x,q) = \sum_{k=0}^{\infty} \phi_k(q) x^k, \qquad G(x,q) = \sum_{k=0}^{\infty} \psi_k(q) x^k\,.$$

If F and G satisfy Equation (4.1), collecting powers of x^k on both sides implies that

$$\psi_k(q) = \sum_{i+jc=k} \phi_i(q) q^{jd} \,. \tag{4.2}$$

Assume that $f_n(q)$ is stable, and define $\psi_k(q)$ by Equation (4.2). We will prove by induction on k that $g_n(q)$ is k-stable with corresponding limit $\psi_k(q)$. Let

$$\begin{aligned}
\alpha_{0,n}(q) =& f_n(q) \\
\alpha_{k,n}(q) =& q^{-n}(\alpha_{k-1,n} - \phi_{k-1}(q)) \\
=& q^{-kn} \left(f_n(q) - \sum_{l=0}^{k-1} \phi_l(q) q^{ln} \right), \;\; k \geq 1
\end{aligned}$$

and

$$\begin{aligned}
\beta_{0,n}(q) =& g_n(q) \\
\beta_{k,n}(q) =& q^{-n}(\beta_{k-1,n} - \psi_{k-1}(q)) \\
=& q^{-kn} \left(g_n(q) - \sum_{l=0}^{k-1} \psi_l(q) q^{ln} \right), \;\; k \geq 1 \,.
\end{aligned}$$

For $k = 0$, the limit of $g_n(q)$ is $\lim_{n\to\infty} g_n(q) = \lim_{n\to\infty} \frac{f_n(q)}{1-q^{cn+d}} = \phi_0(q) = \psi_0(q)$. Assuming by induction that $g_n(q)$ is $(k-1)$-stable, we have

$$\begin{aligned}
\beta_{k,n}(q) &= q^{-kn} \left(\frac{f_n(q)}{1-q^{cn+d}} - \sum_{l=0}^{k-1} \sum_{i+jc=l} \phi_i(q) q^{jd} q^{(i+jc)n} \right) \\
&= q^{-kn} \left(f_n(q) \sum_{j=0}^{\infty} q^{j(cn+d)} - \sum_{0 \leq i+jc \leq k-1} \phi_i(q) q^{in} q^{j(cn+d)} \right) \,.
\end{aligned}$$

We continue,

$$\begin{aligned}
\beta_{k,n}(q) &= q^{-kn} \sum_{j=0}^{\lfloor \frac{k-1}{c} \rfloor} q^{j(cn+d)} \left(f_n(q) - \sum_{i=0}^{k-1-jc} \phi_i(q) q^{in} \right) \\
&\quad + q^{-kn} \sum_{j > \lfloor \frac{k-1}{c} \rfloor} q^{j(cn+d)} f_n(q) \\
&= \sum_{j=0}^{\lfloor \frac{k-1}{c} \rfloor} q^{jd} q^{-(k-jc)n} \left(f_n(q) - \sum_{i=0}^{k-1-jc} \phi_i(q) q^{in} \right) \\
&\quad + q^{-kn} \sum_{j > \lfloor \frac{k-1}{c} \rfloor} q^{j(cn+d)} f_n(q) \\
&= \sum_{j=0}^{\lfloor \frac{k-1}{c} \rfloor} q^{jd} \alpha_{k-jc,n}(q) + q^{-kn} \sum_{j > \lfloor \frac{k-1}{c} \rfloor} q^{j(cn+d)} f_n(q) \\
&= \sum_{j=0}^{\lfloor \frac{k-1}{c} \rfloor} q^{jd} \alpha_{k-jc,n}(q) + q^{-kn} \sum_{\lfloor \frac{k-1}{c} \rfloor < j \leq \frac{k}{c}} q^{j(cn+d)} f_n(q) \\
&\quad + q^{-kn} \sum_{j > \frac{k}{c}} q^{j(cn+d)} f_n(q) \\
&= \sum_{i+jc=k} q^{jd} \alpha_{i,n}(q) + \sum_{j > \frac{k}{c}} q^{n(jc-k)+jd} f_n(q) \,.
\end{aligned}$$

Therefore,

$$\lim_{n \to \infty} \beta_{k,n}(q) = \sum_{i+jc=k} q^{jd} \phi_i(q) = \psi_k(q) \,.$$

Conversely, if $(g_n(q))$ is stable, so is $(f_n(q))$. □

Lemma 4.2. *Fix a rational polytope $P \subset [0,\infty)^r$ that intersects the interior of every positive coordinate ray and a positive definite quadratic function $Q : \mathbb{Z}^r \longrightarrow \mathbb{Z}$. Let $c : \mathbb{N} \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$ be such that for each fixed $v \in \mathbb{Z}^r$ and for $n \gg 0$, $c(n,v) = t(n,v)$ where $n \mapsto t(n,v)$ is a quasi-polynomial. For each natural number n define*

$$T_n(q) = \sum_{v \in nP \cap \mathcal{L}} c(n,v) q^{Q(v)} \,,$$

where $\mathcal{L} = \mathbb{Z}^r$. Then $(T_n(q))$ is c-stable and its (n, x, q)-tail is independent of x and given by

$$F(n, x, q) = \sum_{v \in \mathcal{L} \cap \mathbb{R}^r_+} t(n, v) q^{Q(v)}\,.$$

Proof. Let $\phi_0(n, q) = \sum_{v \in \mathcal{L} \cap \mathbb{R}^r_+} t(n, v) q^{Q(v)}$. We need to prove that for all $k \geq 0$, we have

$$\lim_{n \to \infty} q^{-kn}(T_n(q) - \phi_0(n, q)) = 0\,.$$

Let $P_n = nP \cap \mathcal{L}$. We have

$$\begin{aligned}
q^{-kn}(T_n(q) - \phi_0(n, q)) &= q^{-kn} \sum_{v \in P_n} (c(n, v) - t(n, v)) q^{Q(v)} \\
&\quad - \sum_{v \in (\mathcal{L} \cap \mathbb{R}^r_+) \setminus P_n} t(n, v) q^{Q(v) - kn} \\
&= - \sum_{v \in (\mathcal{L} \cap \mathbb{R}^r_+) \setminus P_n} t(n, v) q^{Q(v) - kn}
\end{aligned} \tag{4.3}$$

for n large enough. Let us first assume that Q is a quadratic form and let d be the minimum of Q on $\mathbb{R}^r \setminus P^\circ$, where P° denotes the interior of P. We will prove that $d > 0$. Indeed, since Q is a positive definite form we only need to minimize Q over the union F of the faces of P that are not in the coordinate planes. Since F is compact, Q attains its minimum at some $v_0 \in F$ and $d = Q(v_0) > 0$ since $v_0 \neq 0$. If $v \in \mathbb{R}^r \setminus nP^\circ$ then $v = nv', v' \in \mathbb{R}^r \setminus P^\circ$, so $Q(v) = Q(nv') = n^2 Q(v') \geq dn^2$. Therefore the limit of the right hand side of Equation (4.3) as n approaches infinity is zero.

If Q is not a quadratic form we can write $Q = Q_2 + Q_1$ where Q_2 is the quadratic part of Q. Then if $v \in \mathbb{R}^r \setminus nP^\circ$ we have $Q(v) = Q_2(v) + Q_1(v) \geq dn^2 + Q_1(v) > (d+1)n^2$ for n large enough. ∎

Remark 4.3. Let $p \in P$. The *tangent cone* $\mathrm{Tan}(P, p)$ is defined to be the set of all directions v that one can go and stay in P:

$$\mathrm{Tan}(P, p) = \{v \in \mathbb{R}^r | p + \epsilon v \in P \,\text{for small}\, \epsilon > 0\}\,.$$

Lemma 4.2 still holds if we replace nP with $n(P - p)$ or $nP - p$ and $\mathcal{L}$ with a union of a finite number of translates of $\mathcal{L}$. In this setting, the stable series is

$$F(n, x, q) = \sum_{v \in \mathrm{Tan}(P,p) \cap \mathbb{Z}^r} t(n, v) q^{Q(v)}\,.$$

Remark 4.4. Suppose that $f_n(q)$ satisfies $\delta^*(f_n(q)) \geq cn^2$ for some $c > 0, n \geq 0$ then $g_n(q)$ is c-stable if $g_n(q) + f_n(q)$ is c-stable and they have the same tails.

5. Stability of the multiplicity

5.1. **Lie algebra notation.** Let us recall some standard notation from [2, 12]. Let $\mathfrak{g}$ denote a simple Lie algebra of rank r with weight lattice Λ, root lattice Λ_r and normalized inner product $(\cdot,\cdot)$ on Λ. Let W be its Weyl group and Λ^+ the set of all the dominant weights with respect to a fixed Weyl chamber. Let α_i (resp., λ_i), $1 \leq i \leq r$, be the set of simple roots (resp., fundamental weights) of $\mathfrak{g}$.The root lattice Λ_r has the partial order given by $\beta \prec \alpha$ if and only if $\alpha - \beta = \sum_{i=1}^{r} n_i\alpha_i$ where $n_i \in \mathbb{N}$, $i = 1, \dots, r$.

For a dominant weight $\lambda \in \Lambda^+$, let V_λ denote the corresponding irreducible representation V_λ and let Π_λ denote the set of all of the weights of V_λ.

The *Kostant partition function* $p(\alpha)$ of an element of the root lattice α is the sum of all ways of writing α as a nonnegative integer linear combination of positive roots [13].

5.2. **A formula for the multiplicity of the plethysm.** In this section we prove Lemma 3.1.

Proof. (of Lemma 3.1) (a) We have

$$\psi_a(ch_\lambda) = \psi_a(\sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_\mu) = \sum_{\mu\in\Pi_\lambda} m_\lambda^\mu \psi_a(e_\mu) = \sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu}\,. \tag{5.1}$$

From Equations (3.3) and (5.1) we have

$$\sum_{\mu} m_{\lambda,a}^\mu ch_\mu = \sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu}\,. \tag{5.2}$$

Let us define $\omega(\mu) := \sum_{\sigma\in W} (-1)^\sigma e_{\sigma(\mu)}$ by for $\mu \in \Lambda^+$. The Weyl character formula states that [12]:

$$\omega(\rho) ch_\lambda = \omega(\lambda+\rho)\,.$$

Multiplying both sides of Equation (5.2) with $\omega(\rho)$ and applying Weyl's formula we have

$$\sum_{\mu} m_{\lambda,a}^\mu \omega(\mu+\rho) = (\sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu})\omega(\rho)\,. \tag{5.3}$$

Replacing $\omega(\mu+\rho)$ with $\sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\mu+\rho)}$ and $\omega(\rho)$ with $\sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\rho)}$ in Equation (5.3) we have

$$\sum_{\mu}\sum_{\sigma\in W}(-1)^\sigma m_{\lambda,a}^{\mu} e_{\sigma(\mu+\rho)} = (\sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu})(\sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\rho)}) \tag{5.4}$$

$$= \sum_{\mu\in\Pi_\lambda}\sum_{\sigma\in W}(-1)^\sigma m_\lambda^\mu e_{a\mu+\sigma(\rho)}\,. \tag{5.5}$$

Setting $\sigma(\mu+\rho)=\nu+\rho$ on the left hand side of Equation (5.5) and $a\mu+\sigma(\rho)=\nu+\rho$ on right hand side we have

$$\sum_{\nu}\sum_{\sigma\in W}(-1)^\sigma m_{\lambda,a}^{\sigma^{-1}(\nu+\rho)-\rho} e_{\nu+\rho} = \sum_{\nu}\sum_{\sigma\in W}(-1)^\sigma m_\lambda^{\frac{\nu+\rho-\sigma(\rho)}{a}} e_{\nu+\rho}\,. \tag{5.6}$$

But we want $\sigma^{-1}(\nu+\rho)-\rho$ to be a dominant weight, which can happen only when $\sigma=1$. Therefore Equation (5.6) becomes

$$\sum_{\nu} m_{\lambda,a}^{\nu} e_{\nu+\rho} = \sum_{\nu}\sum_{\sigma\in W}(-1)^\sigma m_\lambda^{\frac{\nu+\rho-\sigma(\rho)}{a}} e_{\nu+\rho}\,. \tag{5.7}$$

Identifying the coefficients of $e_{\nu+\rho}$ on both sides of Equation (5.7) gives us the desired equality.
(b) This follows from the fact that $m_\lambda^\nu \neq 0$ if and only if $\nu\in\Pi_\lambda$. If $\nu=\frac{\mu+\rho-\sigma(\rho)}{a}$ this means that $\mu\in\sigma(\rho)-\rho+a\Pi_\lambda$. □

5.3. **Stability of the plethysm multiplicity.** In this section we will prove that $m_{n\lambda,a}^{\mu+n\nu}$ is a piecewise quasi-polynomial of $n\gg 0$ where $\lambda\in\Lambda^+$, $\mu,\nu\in\Lambda$. A piecewise quasi-polynomial function on a rational vector space is a rational polyhedral fan together with a quasi-polynomial function on each chamber of the fan. Piecewise quasi-polynomials appear naturally as *vector partition functions* [17]. The *Kostant partition function* of a simple Lie algebra $\mathfrak{g}$ is a vector partition function (see [13]), hence a piecewise quasi-polynomial.

Theorem 5.1. *Let $\lambda\in\Lambda^+$, $\mu,\nu\in\Lambda$, then $m_{n\lambda,a}^{\mu+n\nu}$ is a piecewise quasi-polynomial in n for $n\gg 0$.*

Proof. We have

$$m_{n\lambda,a}^{\mu+n\nu} = \sum_{\sigma\in W}(-1)^\sigma m_{n\lambda}^{\frac{\mu+n\nu+\rho-\sigma(\rho)}{a}} \tag{5.8}$$

and by Kostant's multiplicity formula in [13], we have

$$\begin{aligned} m_{n\lambda}^{\frac{\mu+n\nu+\rho-\sigma(\rho)}{a}} &= \sum_{\sigma'\in W} (-1)^{\sigma'} p\left(\sigma'(n\lambda+\rho) - (\frac{\mu+n\nu+\rho-\sigma(\rho)}{a}+\rho)\right) \\ &= \sum_{\sigma'\in W} (-1)^{\sigma'} p\left(n\sigma'(\lambda) - (\frac{\mu+n\nu+\rho-\sigma(\rho)}{a}+\rho-\sigma'(\rho))\right) \\ &= \sum_{\sigma'\in W} (-1)^{\sigma'} p\left(n(\sigma'(\lambda)-\frac{\nu}{a}) - (\frac{\mu+\rho-\sigma(\rho)}{a}+\rho-\sigma'(\rho))\right) \\ &= \sum_{\sigma'\in W} (-1)^{\sigma'} p\left(n\lambda'-\alpha'\right) . \end{aligned}$$

Assume that $n\lambda'-\alpha'$ can be written as sum of positive roots of $\mathfrak{g}$ so that $p(n\lambda'-\alpha')\neq 0$. For $n\gg 0$, $n\lambda'-\alpha'$ stays in some fixed Kostant chamber and it follows from Theorem 1 in [17] that $p(n\lambda'-\alpha')$ is a quasi-polynomial in n. Since $m_{n\lambda,a}^{\mu+n\nu}$ is a finite sum of quasi-polynomials in n, it is also a quasi-polynomial in n. □

6. The summation set

6.1. A lattice point description of the summation set. In this section give a lattice point description of the summation set $S_{\lambda,a}$. Let P_λ denote the convex polytope defined by the convex hull of $\Pi_\lambda\cap\Lambda^+$.

Lemma 6.1. *For all λ, a we have:*

$$S_{\lambda,a}\subseteq \mathcal{L}_{\lambda,a}\cap P_{a\lambda} \tag{6.1}$$

where

$$\mathcal{L}_{\lambda,a} = \bigcup_{\sigma\in W} (a\lambda+\sigma(\rho)-\rho+a\Lambda_r) . \tag{6.2}$$

is a finite union of translates of the root lattice $a\Lambda_r$. Let

$$R_{\lambda,a} = (\mathcal{L}_{\lambda,a}\cap P_{a\lambda})\setminus S_{\lambda,a} \tag{6.3}$$

denote the set of missing points.

Proof. Recall that P_λ consists of all α that satisfy (see [12]),

$$(\alpha,\alpha_i)\geq 0, \qquad (\lambda-\alpha,\lambda_i)\geq 0 \tag{6.4}$$

for all $i=1,\dots,r$. We first prove that $S_{\lambda,a}\subseteq P_{a\lambda}$. By Lemma 3.1(b), we can write $\mu = a\nu+\sigma(\rho)-\rho\in\Lambda^+$ where $\nu\in\Pi_\lambda$. Since $\mu\in\Lambda^+$, Inequality (6.4) holds trivially. To prove the second part of Inequality (6.4), it suffices to show that $(\mu, \lambda_i) \leq (a\lambda, \lambda_i)$ for every $1 \leq i \leq r$. We have

$$\begin{aligned}(a\lambda, \lambda_i) - (\mu, \lambda_i) &= (a\lambda - \mu, \lambda_i) \\ &= (a(\lambda - \mu) + \rho - \sigma(\rho), \lambda_i) \\ &\geq 0\end{aligned}$$

since $a(\lambda - \mu) + \rho - \sigma(\rho)$ is a $\mathbb{N}$-linear combination of positive roots.

Let $\mu = a\nu + \sigma(\rho) - \rho \in S_{\lambda,a}$ where $\nu \in \Pi_\lambda$ and $\sigma \in W$. Then $\mu = a(\lambda - \alpha) + \sigma(\rho) - \rho$ where α is some positive root. It follows that $\mu \in a\Lambda_r + a\lambda + \sigma(\rho) - \rho \subset \mathcal{L}_{\lambda,a}$. This proves that $S_{\lambda,a} \subseteq \mathcal{L}_{\lambda,a}$ and completes the proof of the lemma. □

Remark 6.2. The inclusion in Equation (6.1) is not an equality in general. For example, consider $\mathfrak{g} = B_2, \lambda = \rho, a = 2$. In weight coordinates we have (see also Figure 1)

$$S_{\rho,2} = \cup_{\sigma\in W}(\sigma(\rho) - \rho + 2\Pi_\rho) \cap \Lambda^+ \tag{6.5}$$

$$= \{(2,2), (0,4), (3,0), (2,0), (0,2), (1,0), (0,0)\}\,. \tag{6.6}$$

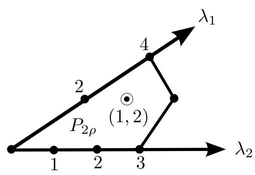


FIGURE 1. $S_{\rho,2}$.

It is clear that $(1,2) \in P_{2\rho}$. We show that $(1,2) = \lambda_1 + 2\lambda_2 \in \mathcal{L}_{\rho,2}$, hence this is a missing point. Indeed, by the definition of $\mathcal{L}_{\rho,2}$, we only need to find $\sigma \in W$ and a root α such that

$$\lambda_1 + 2\lambda_2 = 2\alpha_1 + 3\alpha_2 = 2\rho + \sigma(\rho) - \rho + 2\alpha\,. \tag{6.7}$$

In root coordinates we have

$$2\rho = 3\alpha_1 + 4\alpha_2,\ \rho - \sigma(\rho) \in \{0, \alpha_1, \alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 4\alpha_2\}\,.$$

So by choosing $\alpha = \alpha_2$ and σ such that $\rho - \sigma(\rho) = \alpha_1 + 3\alpha_2$ we have equality (6.7).

Nevertheless, equality holds when $\mathfrak{g} = A_2, a = 2, \lambda = \lambda_1$. This is the content of the next section.

6.2. **A special case: no missing points.**

Proposition 6.3. *For A_2, we have: $S_{n\lambda_1,2} = \mathcal{L}_{n\lambda_1,2} \cap P_{2n\lambda_1}$.*

Proof. Let $\mu = \mathcal{L}_{n\lambda_1,2} \cap P_{2n\lambda_1}$ of the form

$$\mu = 2n\lambda_1 + \sigma(\rho) - \rho + 2\alpha = 2\nu - (\rho - \sigma(\rho))$$

where $\nu = n\lambda_1 + \alpha$ and some $\sigma \in W$. As σ runs over the Weyl group W, $\rho - \sigma(\rho)$ is expressed in weight and root coordinates as follows

(6.8)

weight	$(0,0)$	$(2,-1)$	$(-1,2)$	$(0,3)$	$(3,0)$	$(2,2)$
root	$(0,0)$	$(0,1)$	$(1,0)$	$(1,2)$	$(2,1)$	$(2,2)$

Since $\mu \in P_{2n\lambda_1}$, from inequality (6.4) we have $(2\nu - (\rho - \sigma(\rho)), \alpha_i) \geq 0$, i.e., $2(\nu, \alpha_i) \geq (\rho - \sigma(\rho), \alpha_i)$, $i = 1, 2$. Looking at the first row of the above table we see that this forces $(\nu, \alpha_i) \geq 0$, $i = 1, 2$. Therefore we have $\nu \in \Lambda^+$.

From inequality (6.4) we have $(2n\lambda_1 - \mu, \lambda_i) \geq 0$, $i = 1, 2$. This implies that $(-2\alpha + \rho - \sigma(\rho), \lambda_i) \geq 0$ or equivalently

$$(\rho - \sigma(\rho), \lambda_i) \geq 2(\alpha, \lambda_i), \tag{6.9}$$

$i = 1, 2$. We consider the following cases.

Case 1: If $\rho - \sigma(\rho) = 0, \alpha_1$ or α_2 then inequalities (6.9) imply that $(\alpha, \lambda_i) \leq 0$ for all i so $\alpha \prec 0$. Since $\nu = n\lambda_1 + \alpha \in \Lambda^+$, it follows from [12, §13.4] that $\nu \in \Pi_{n\lambda_1}$ and hence $\mu \in S_{n\lambda_1,2}$.

Case 2: If $\rho - \sigma(\rho) = \alpha_1 + 2\alpha_2$ then from (6.9) we have $(\alpha, \lambda_1) \leq 0$ and $(\alpha, \lambda_2) \leq 1$. If we also have $(\alpha, \lambda_2) \leq 0$ then by a similar the argument to Case 1 we conclude that $\mu \in S_{n\lambda_1,2}$. If $(\alpha, \lambda_2) = 1$ we can write $\alpha = -x\alpha_1 + \alpha_2$, where $x \in \mathbb{N}$. It follows that $\mu = 2n\lambda_1 + 2\alpha - (\rho - \sigma(\rho)) = 2n\lambda_1 - 2x\alpha_1 + 2\alpha_2 - \alpha_1 - 2\alpha_2 = 2(n\lambda_1 - x\alpha_1) - \alpha_1$. Since $\nu = n\lambda_1 + \alpha \in \Lambda^+$, from inequality (6.4) we have $(n\lambda_1 - x\alpha_1 + \alpha_2, \alpha_1) \geq 0$, i.e., $n - 2x - 1 \geq 0$. We have $\langle n\lambda_1, \alpha_1 \rangle = \frac{2(n\lambda_1, \alpha_1)}{(\alpha_1, \alpha_1)} = n \geq 2x + 1 > x$, therefore $n\lambda_1 - x\alpha_1 \in \Pi_{n\lambda_1}$ (see [12, § 13.4]). Since we can choose σ' such that $\rho - \sigma'(\rho) = \alpha_1$, we have $\mu = 2(n\lambda_1 - x\alpha_1) - (\rho - \sigma'(\rho)) \in S_{n\lambda_1,2}$.

Case 3: If $\rho - \sigma(\rho) = 2\alpha_1 + \alpha_2$ then by a similar argument to the above we can write $\alpha = \alpha_1 - x\alpha_2$, $x \in \mathbb{N}$. We show that α cannot have this form. Indeed, since $\nu = n\lambda_1 + \alpha \in \Lambda^+$, we have $(n\lambda_1 + \alpha_1 - x\alpha_2, \alpha_2) \geq 0$, i.e., $-1 - 2x \geq 0$. This is in contradiction to the fact that $x \in \mathbb{N}$.

Case 4: If $\rho - \sigma(\rho) = 2\alpha_1 + 2\alpha_2 = 2\rho$ then $(\alpha, \lambda_1) \leq 1$ and $(\alpha, \lambda_2) \leq 1$. If either $(\alpha, \lambda_1) \leq 0$ or $(\alpha, \lambda_2) \leq 0$ then the same argument as in Cases 2 and 3 above apply. If $(\alpha, \lambda_1) = (\alpha, \lambda_2) = 1$ then $\alpha = \alpha_1 + \alpha_2 = \rho$ and $\mu = 2n\lambda_1 + 2\alpha - (\rho - \sigma(\rho)) = 2n\lambda_1 + 2\rho - 2\rho = 2n\lambda_1 \in \Pi_{2n\lambda_1} \subseteq S_{n\lambda_1,2}$. □

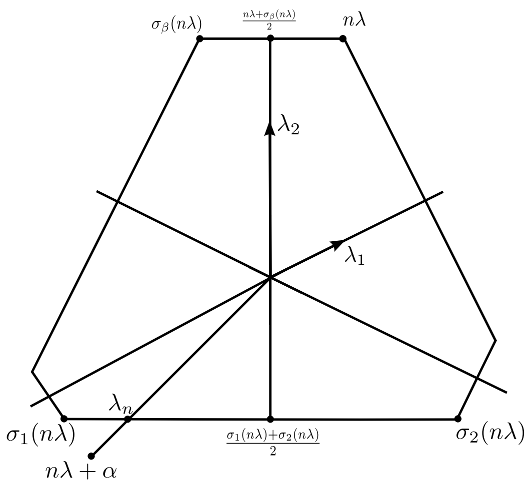
6.3. **An estimate for the missing points.** The next proposition shows that the norm of the missing points in $R_{n\lambda,a}$ is bounded below by a quadratic function of n.

Proposition 6.4. *For every $\lambda \in \Lambda^+$ there exists a simple root β such that if $\mu \in R_{n\lambda,a}$ and $n \gg 0$ then*

$$(\mu,\mu) \geq a^2 n^2 ((\lambda,\lambda) - \frac{(\lambda,\beta)^2}{(\beta,\beta)} - 1) .$$

Proof. Let $\mu = a\alpha + an\lambda + \sigma(\rho) - \rho = a(n\lambda + \alpha) + \sigma(\rho) - \rho$ for some $\alpha \in \Lambda_r$ and $\sigma \in W$. Since $\mu \notin S_{n\lambda,a}$, we have that $n\lambda + \alpha \notin \Pi_{n\lambda}$. The ray $n\lambda + \alpha$ meets one of the facets of the convex hull of $\Pi_{n\lambda}$ at some point, say λ_n. There exist $\sigma_1, \sigma_2 \in W$ such that $\sigma_1(n\lambda), \sigma_2(n\lambda)$ are the vertices of this facet, and we have

$$\begin{aligned}
(n\lambda + \alpha, n\lambda + \alpha) &\geq (\lambda_n, \lambda_n) \\
&\geq (\frac{\sigma_1(n\lambda) + \sigma_2(n\lambda)}{2}, \frac{\sigma_1(n\lambda) + \sigma_2(n\lambda)}{2}) \\
&= \frac{n^2}{4}((\sigma_1(\lambda), \sigma_1(\lambda)) + (\sigma_2(\lambda), \sigma_2(\lambda)) + 2(\sigma_1(\lambda), \sigma_2(\lambda))) \\
&= \frac{n^2}{4}(2(\lambda,\lambda) + 2(\sigma_1(\lambda), \sigma_2(\lambda))) \\
&= \frac{n^2}{2}((\lambda,\lambda) + (\sigma_1(\lambda), \sigma_2(\lambda))) .
\end{aligned}$$



Since $\sigma_1(\lambda), \sigma_2(\lambda)$ are in two nearby chambers, there exists a simple root β such that

$$(\sigma_1(\lambda), \sigma_2(\lambda)) = (\lambda, \sigma_\beta(\lambda)).$$

We have

$$(\lambda, \sigma_\beta(\lambda)) = (\lambda, \lambda - 2\frac{(\lambda,\beta)}{(\beta,\beta)}\beta) = (\lambda,\lambda) - 2\frac{(\lambda,\beta)^2}{(\beta,\beta)}.$$

So

$$(n\lambda+\alpha, n\lambda+\alpha) \geq n^2((\lambda,\lambda) - \frac{(\lambda,\beta)^2}{(\beta,\beta)}).$$

Therefore

$$\begin{aligned}(\mu,\mu) &= (a(n\lambda+\alpha) - (\rho-\sigma(\rho)), a(n\lambda+\alpha) - (\rho-\sigma(\rho)))\\ &= a^2(n\lambda+\alpha, n\lambda+\alpha) - 2a(n\lambda+\alpha, \rho-\sigma(\rho)) + (\rho-\sigma(\rho), \rho-\sigma(\rho))\\ &\geq a^2n^2((\lambda,\lambda) - \frac{(\lambda,\beta)^2}{(\beta,\beta)} - 1)\end{aligned}$$

for large enough n. □

Let us introduce some useful notation.

(6.10)
$$\hat{S}_{\lambda,a} = S_{\lambda,a} - \mu_{\lambda,a}, \qquad \hat{\mathcal{L}}_{\lambda,a} = \mathcal{L}_{\lambda,a} - \mu_{\lambda,a}$$
$$\hat{P}_{a\lambda} = P_{a\lambda} - \mu_{\lambda,a}, \qquad \hat{R}_{\lambda,a} = R_{\lambda,a} - \mu_{\lambda,a}\,.$$

Remark 6.5. From now we fix a natural number n_0 and we work with $n \equiv n_0 \bmod da$ where d is the order of the fundamental group Λ/Λ_r. Theorem 3.4 implies that for such n, we have:

- $\mu_{n\lambda,a} = n\nu^1_{\lambda,a} + \nu^0_{\lambda,a}$ for some fixed weights $\nu^1_{\lambda,a}, \nu^0_{\lambda,a} \in \Lambda^+$.
- $\hat{\mathcal{L}}_{n\lambda,a} = \hat{\mathcal{L}}_{n_0\lambda,a}$. Indeed, we have

$$\begin{aligned}\hat{\mathcal{L}}_{n\lambda,a} &= \mathcal{L}_{n\lambda,a} - \mu_{n\lambda,a}\\ &= n\lambda + \sigma(\rho) - \rho + a\Lambda_r - n\nu^1_{\lambda,a} - \nu^0_{\lambda,a}\\ &= n_0\lambda + \sigma(\rho) - \rho + a\Lambda_r - n_0\nu^1_{\lambda,a} - \nu^0_{\lambda,a} + (n-n_0)(\lambda - \nu^1_{\lambda,a})\\ &= n_0\lambda + \sigma(\rho) - \rho + a\Lambda_r - \mu_{n_0\lambda,a} + k(a.d)(\lambda-\nu^1_{\lambda,a}),\ k\in\mathbb{N}\\ &= n_0\lambda + \sigma(\rho) - \rho + a\Lambda_r - \mu_{n_0\lambda,a} \quad (\text{since } d(\lambda-\nu^1_{\lambda,a})\in\Lambda_r)\\ &= \mathcal{L}_{n_0\lambda,a} - \mu_{n_0\lambda,a} = \hat{\mathcal{L}}_{n_0\lambda,a}\,.\end{aligned}$$

Corollary 6.6. (1) $\hat{S}_{n\lambda,a} \subset \hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}$.
(2) Let $\hat{R}_{n\lambda,a} = (\hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}) \setminus \hat{S}_{n\lambda,a}$. If $\hat{\mu} \in \hat{R}_{n\lambda,a}$ then

$$(\hat{\mu},\hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) \geq a^2n^2\left((\lambda,\lambda) - \frac{(\lambda,\beta)^2}{(\beta,\beta)} - 1\right) - (\mu_{n\lambda,a}, \mu_{n\lambda,a})$$

for some simple root β.

Proof. Part (1) follows from Lemma 6.1(b) and Remark 6.5:

$$\hat{S}_{n\lambda,a} \subset \hat{\mathcal{L}}_{n\lambda,a} \cap \hat{P}_{an\lambda} = \hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}\,.$$

For part (2), recall that

$$(\mu,\mu) = (\hat{\mu} + \mu_{n\lambda,a}, \hat{\mu} + \mu_{n\lambda,a})$$

and therefore if $\hat{\mu} \in \hat{R}_{n\lambda,a}$ then

$$\begin{aligned}(\hat{\mu},\hat{\mu}) + 2(\hat{\mu},\mu_{n\lambda,a}) &= (\mu,\mu) - (\mu_{n\lambda,a},\mu_{n\lambda,a}) \\ &\geq a^2n^2\left((\lambda,\lambda) - \frac{(\lambda,\beta)^2}{(\beta,\beta)} - 1\right) - (\mu_{n\lambda,a},\mu_{n\lambda,a})\end{aligned}$$

by Proposition 6.4. □

Proposition 6.7. *If $\mathfrak{g}$ has rank 2 and $\hat{\mu} \in \hat{R}_{n\lambda,a}$ then*

$$(\hat{\mu},\hat{\mu}) + 2(\hat{\mu},\mu_{n\lambda,a}) \geq n^2\,.$$

Proof. We can prove this by a direct computation for the rank 2 simple Lie algebras A_2, B_2 and G_2 using Theorem 3.4 that gives an explicit formula for $\mu_{\lambda,a}$.

For A_2 and $m_1 \geq m_2$, from Theorem 3.4 we have

$$(\mu_{n\lambda,a},\mu_{n\lambda,a}) \leq (n(m_1-m_2)\lambda_2, n(m_1-m_2)\lambda_2) = \frac{2}{3}n^2(m_1-m_2)^2.$$

By Corollary 6.6 we have

$$\begin{aligned}(\hat{\mu},\hat{\mu}) + 2(\hat{\mu},\mu_{n\lambda,a}) &\geq a^2n^2((\lambda,\lambda) - \frac{(\lambda,\alpha_1)^2}{(\alpha_1,\alpha_1)} - 1) - (\mu_{n\lambda,a},\mu_{n\lambda,a}) \\ &\geq a^2n^2(\frac{2}{3}(m_1^2+m_1m_2+m_2^2) - \frac{m_1^2}{2} - 1) - \frac{2}{3}n^2(m_1-m_2)^2 \\ &= n^2(\frac{a^2-4}{6}m_1^2 + \frac{2}{3}(a^2+2)m_1m_2 + \frac{2}{3}(a^2-1)m_2^2 - 1) \\ &\geq n^2(4m_1m_2 + 2m_2^2 - 1) \\ &\geq n^2\end{aligned}$$

except when $a = 2$ and $m_2 = 0$. In the later case, Proposition 6.3 says that $R_{n\lambda,a} = \emptyset$ and the inequality holds trivially. The argument is similar for the case $m_1 \leq m_2$.

For B_2, $\frac{(\lambda,\beta)^2}{(\beta,\beta)}$ is either $\frac{m_1^2}{2}$ or m_2^2. We have

$$
\begin{aligned}
(\hat{\mu},\hat{\mu})+2(\hat{\mu},\mu_{n\lambda,a}) &\geq a^2n^2((\lambda,\lambda)-\frac{(\lambda,\alpha_1)^2}{(\alpha_1,\alpha_1)}-1)-(\mu_{n\lambda,a},\mu_{n\lambda,a})\\
&= a^2n^2(m_1^2+m_1m_2+\frac{m_2^2}{2}-\max\{\frac{m_1^2}{2},\frac{m_2^2}{4}\}-1)\\
&\quad-(\mu_{n\lambda,a},\mu_{n\lambda,a})\\
&\geq n^2
\end{aligned}
$$

where in the last inequality we have used the fact that $(\mu_{\lambda,a},\mu_{\lambda,a})$ is bounded for B_2, see Theorem 3.4.

For G_2, $\frac{(\lambda,\beta)^2}{(\beta,\beta)}$ is either $\frac{m_1^2}{2}$ or $\frac{m_2^2}{6}$. Therefore we have

$$
\begin{aligned}
(\hat{\mu},\hat{\mu})+2(\hat{\mu},\mu_{n\lambda,a}) &\geq a^2n^2((\lambda,\lambda)-\frac{(\lambda,\alpha_1)^2}{(\alpha_1,\alpha_1)}-1)-(\mu_{n\lambda,a},\mu_{n\lambda,a})\\
&= a^2n^2(2m_1^2+6m_1m_2+6m_2^2-\max\{\frac{m_1^2}{2},\frac{3m_2^2}{2}\}-1)\\
&\quad-(\mu_{n\lambda,a},\mu_{n\lambda,a})\\
&\geq n^2
\end{aligned}
$$

since $\mu_{\lambda,a}=0$ for G_2, see Theorem 3.4. □

7. Proof of Theorem 1.6

In this section we will prove Theorem 1.6 assuming Theorem 3.4. Corollary 3.6 implies that the shifted colored Jones polynomial defined by

$$
\hat{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q)=q^{-\delta^*_{T(a,b)}(\lambda)}J^{\mathfrak{g}}_{T(a,b),\lambda}(q)\in\mathbb{Z}[q] \tag{7.1}
$$

satisfies

$$
\hat{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q)=\frac{1}{\prod\limits_{\alpha\succ 0}(1-q^{(\lambda+\rho,\alpha)})}\check{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q)
$$

where

$$
\begin{aligned}
\check{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q)=&\sum_{\hat{\mu}\in\hat{S}_{\lambda,a}} m_{\lambda,a}^{\hat{\mu}+\mu_{\lambda,\alpha}}q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{\lambda,\alpha})}\\
&\cdot\prod_{\alpha\succ 0}(1-q^{(\hat{\mu}+\mu_{\lambda,a}+\rho,\alpha)})
\end{aligned}
\tag{7.2}
$$

with $\hat{S}_{\lambda,a}=S_{\lambda,a}-\mu_{\lambda,a}$ and $\hat{\mu}=\mu-\mu_{\lambda,a}$.

Fix a natural number n, observe that $(f_n(q))$ is c-stable if and only if $(f_{Mn+n_0}(q))$ is c-stable for all $n_0=0,1,...,M$. In what follows, we will use $M=ad$ and fix $n\equiv n_0 \bmod ad$.

Proposition 7.1. *$(\hat{J}^{\mathfrak{g}}_{T(a,b),n\lambda}(q))$ is c-stable if and only if*

$$
\frac{1}{\prod_{\alpha\succ 0}(1-q^{(n\lambda+\rho,\alpha)})} \cdot \sum_{\hat{\mu}\in\hat{\mathcal{L}}_{n_0\lambda,a}\cap\hat{P}_{an\lambda}} m_{\lambda,a}^{\hat{\mu}+\mu_{\lambda,\alpha}} q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{\lambda,\alpha})}\prod_{\alpha\succ 0}(1-q^{(\hat{\mu}+\mu_{\lambda,a}+\rho,\alpha)}) \tag{7.3}
$$

is c-stable. In that case, they have the same tails.

Proof. Fix a, b, λ and let $g_n(q)$ denote the difference between $\hat{J}_{T(a,b),n\lambda}(q)$ and Equation (7.3). Then

$$
\begin{aligned}
g_n(q) = & \frac{1}{\prod_{\alpha\succ 0}(1-q^{(n\lambda+\rho,\alpha)})} \\
& \times \sum_{\hat{\mu}\in\hat{R}_{n\lambda,a}} \Big(m_{\lambda,a}^{\hat{\mu}+\mu_{n\lambda,\alpha}} q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{n\lambda,\alpha})} \\
& \cdot \prod_{\alpha\succ 0}(1-q^{(\hat{\mu}+\mu_{n\lambda,a}+\rho,\alpha)}) \Big) .
\end{aligned} \tag{7.4}
$$

Proposition 6.7 implies that the minimum degree of the summands of Equation (7.4) is greater or equal to $\frac{b}{2a}n^2$ for $n \gg 0$. The proof then follows from Remark 4.4.

Proposition 6.7 implies that we can replace the summation set $\hat{S}_{n\lambda,a}$ by $\hat{\mathcal{L}}_{n\lambda,a} \cap \hat{P}_{an\lambda}$ without affecting the stability of $\hat{J}^{\mathfrak{g}}_{T(a,b),n\lambda}(q)$: if $\hat{\mu} \in (\hat{\mathcal{L}}_{\lambda,a} \cap \hat{P}_{an\lambda}) \setminus \hat{S}_{n\lambda,a}$ then the minimum degree of the summand of Equation (7.3) is

$$
\begin{aligned}
\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{n\lambda,\alpha}) &= \frac{b}{2a}((\hat{\mu},\hat{\mu})+2(\hat{\mu},\mu_{\lambda,a}))-(\hat{\mu},\rho) \\
&\geq \frac{b}{2a}n^2-(\hat{\mu},\rho) = \frac{b}{2a}n^2+O(n)
\end{aligned}
$$

where the last inequality follows from Proposition 6.7. By Remark 6.5 we have $\hat{\mathcal{L}}_{n\lambda,a} = \hat{\mathcal{L}}_{\lambda,a}$ and the Proposition follows. □

Let $t_{\lambda,\hat{\mu},a}(n) = m_{n\lambda,a}^{\hat{\mu}+\mu_{n\lambda,a}}$. Theorem implies that $t_{\lambda,\hat{\mu},a}$ is a quasi-polynomial. Lemma 4.2, Proposition 6.7, Proposition 7.1 together with the special case given in Section 10.1 imply the following.

Theorem 7.2. *Fix a rank 2 simple Lie algebras $\mathfrak{g}$, a dominant weight λ, and a torus knot $T(a,b)$. The colored Jones polynomial $\hat{J}^{\mathfrak{g}}_{T(a,b),n\lambda}(q)$ is* *c-stable and its (n,x,q)-tail is given by*

(7.5)

$$F_{T(a,b),\lambda}(n,x,q) = \frac{1}{\prod_{\alpha\succ 0}(1-x^{(\lambda,\alpha)}q^{(\rho,\alpha)})} \cdot \sum_{\hat{\mu}\in\hat{\mathcal{L}}_{\lambda,a}\cap\Lambda^+} \Big(t_{\lambda,\hat{\mu},a}(n) q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\nu^0_{\lambda,a})} x^{\nu^1_{\lambda,a}} \prod_{\alpha\succ 0}(1-q^{(\hat{\mu}+\nu^0_{\lambda,a}+\rho,\alpha)}x^{\nu^1_{\lambda,a}}) \Big),$$

where $\mu_{n\lambda,a} = n\nu^1_{\lambda,a} + \nu^0_{\lambda,a}$.

8. Proof of Theorem 3.3

In this section we prove Theorem 3.3. Since λ is fixed, it suffices to maximize

$$g(\mu) = \frac{b}{4}(\mu,\mu) + \left(-1+\frac{b}{2}\right)(\mu,\rho)$$

on the set $S_{\lambda,a}$.

Lemma 8.1. *Let $\mu\in\Lambda^+$ and $\alpha\succ 0$ be a positive root such that $\mu+\alpha\in\Lambda^+$. Then we have*

$$(\mu,\mu) < (\mu+\alpha,\mu+\alpha)\,.$$

Proof. We have:

$$(\mu+\alpha,\mu+\alpha)-(\mu,\mu) = 2(\mu,\alpha)+(\alpha,\alpha)\,.$$

Now $(\mu,\alpha)>0$ since μ is dominant and α is a positive root and $(\alpha,\alpha)>0$ since $(\cdot,\cdot)$ is positive definite. $\square$

If $\nu\in\Pi_\lambda$ then $\nu=\lambda-\alpha'$ where $\alpha'\succ 0$. Since $\rho-\sigma(\rho)\succ 0$, we have $\mu = a\lambda-\alpha$ where $\mu\in S_{a,\lambda}$ and $\alpha\succ 0$. It follows from the above lemma that $M_{\lambda,a}=a\lambda$ is the unique maximizer of $f(\mu)$.

Next, we compute the plethysm multiplicity $m_{\lambda,a}$. From Lemma 3.1 we have

$$\begin{aligned} m^{a\lambda}_{\lambda,a} &= \sum_{\sigma\in W}(-1)^\sigma m_\lambda^{\frac{a\lambda+\rho-\sigma(\rho)}{a}} \\ &= \sum_{\sigma\in W}(-1)^\sigma m_\lambda^{\lambda+\frac{\rho-\sigma(\rho)}{a}} \\ &= 1 \end{aligned}$$

since $\lambda+\frac{\rho-\sigma(\rho)}{a}\succ\lambda$ if $\frac{\rho-\sigma(\rho)}{a}\in\Lambda_r$, with equality only when $\sigma=1$. This concludes the proof of Theorem 3.3. $\square$

9. Proof of Theorem 3.4

This section is devoted to the proof of Theorem 3.4, done by a case-by-case analysis for a fixed simple Lie algebra $\mathfrak{g}$ of rank 2. Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ and $\mu = u_1\lambda_1 + u_2\lambda_2$ be dominant weights. Since λ is fixed, it suffices to minimize

$$g^*(\mu) = \frac{b}{4}(\mu,\mu) + \left(-1 + \frac{b}{2}\right)(\mu,\rho)$$

on the set $S_{\lambda,a}$. We use the following lemma and its consequence, Corollary 9.2, in the proof of Theorem 3.4.

Lemma 9.1. *$g^*(\mu) \geq 0$ with equality if and only if $\mu = 0$.*

Proof. $g^*(\mu)$ is non-negative since $(\cdot,\cdot)$ is a positive-definite form and $(\mu,\rho) \geq 0$ since μ is a dominant weight and ρ is a linear combination of simple roots with positive coefficients. If $g^*(\mu) = 0$ then $(\mu,\mu) = 0$ which implies that $\mu = 0$. □

Corollary 9.2. *If $m^0_{\lambda,a} \neq 0$ then $\mu_{\lambda,a} = 0$ is the unique minimizer of $g^*(\mu)$.*

We give the proof of Theorem 3.4 in Section 9.1 below.

9.1. Theorem 3.4 for A_2.

9.1.1. *Plethysm multiplicities for A_2.* There are two simple roots $\{\alpha_1, \alpha_2\}$ of A_2 and three positive roots $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$ shown in Figure 2. The Kostant function $p(u,v) = p(u\alpha_1 + v\alpha_2)$ is given by

$$p(u,v) = 1 + \min(u,v)\,.$$

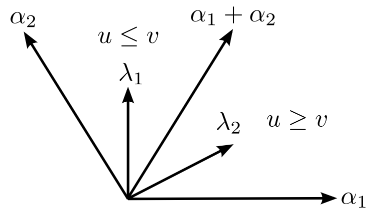


Figure 2. The two chambers of the Kostant partition function of A_2. Kostant chambers from left to right: $u \leq v$, $u \geq v$.

Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ denote a dominant weight and $m_1 \geq m_2$. Assuming $\mu = u_1\lambda_1 + u_2\lambda_2 \in \Pi_\lambda$, by Kostant's formula we have

$$\begin{aligned}
m_\lambda^\mu &= \sum_{\sigma\in W}(-1)^\sigma p(\sigma(\lambda+\rho)-\mu-\rho)\\
&= p\Big(\frac{2m_1+m_2}{3}-\frac{2u_1+u_2}{3}, \frac{m_1+2m_2}{3}-\frac{u_1+2u_2}{3}\Big)\\
&- p\Big(\frac{2m_1+m_2}{3}-\frac{2u_1+u_2}{3}, \frac{m_1-m_2}{3}-\frac{u_1+2u_2}{3}-1\Big)\\
&= \begin{cases} 1+\frac{2m_1+m_2}{3}-\frac{2u_1+u_2}{3} & \text{if } m_1-m_2 < u_1-u_2\\ 1+\frac{m_1+2m_2}{3}-\frac{u_1+2u_2}{3} & \text{if } u_1-u_2 \leq m_1-m_2 \leq u_1+2u_2+3\\ 1+m_2 & \text{if } m_1-m_2 > u_1+2u_2+3 \end{cases}.
\end{aligned}$$

Lemma 3.1 gives

$$\begin{aligned}
m_{\lambda,2}^\mu &= \sum_{\sigma\in S_3}(-1)^\sigma m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{2}}\\
&= m_\lambda^{\frac{1}{2}(u_1,u_2)} - m_\lambda^{\frac{1}{2}(u_1+2,u_2-1)} - m_\lambda^{\frac{1}{2}(u_1-1,u_2+2)}\\
&+ m_\lambda^{\frac{1}{2}(u_1,u_2+3)} + m_\lambda^{\frac{1}{2}(u_1+3,u_2)} - m_\lambda^{\frac{1}{2}(u_1+2,u_2+2)}.
\end{aligned}$$

Let us consider $\mu \in S_{\lambda,2}$. There are four cases.

Case 1: u_1, u_2 are even.

(9.1)

$$m_{\lambda,2}^\mu = m_\lambda^{(\frac{u_1}{2},\frac{u_2}{2})} - m_\lambda^{(\frac{u_1+2}{2},\frac{u_2+2}{2})} = \begin{cases} 1 & \text{if } u_1+2u_2 \geq 2(m_1-m_2)\\ 0 & \text{if } u_1+2u_2 < 2(m_1-m_2)\end{cases}.$$

Case 2: u_1 even and u_2 odd.

(9.2)

$$\begin{aligned}
m_{\lambda,2}^\mu &= m_\lambda^{(\frac{u_1}{2},\frac{u_2+3}{2})} - m_\lambda^{(\frac{u_1+2}{2},\frac{u_2-1}{2})}\\
&= \begin{cases} -1 & \text{if } u_1-u_2 \leq 2(m_1-m_2) \leq u_1+2u_2\\ 0 & \text{if } 2(m_1-m_2) < u_1-u_2 \text{ or } 2(m_1-m_2) > u_1+2u_2\end{cases}.
\end{aligned}$$

Case 3: u_1 odd and u_2 even.

$$m_{\lambda,2}^\mu = m_\lambda^{(\frac{u_1+3}{2},\frac{u_2}{2})} - m_\lambda^{(\frac{u_1-1}{2},\frac{u_2+2}{2})} = \begin{cases} -1 & \text{if } 2(m_1-m_2) < u_1-u_2\\ 0 & \text{if } 2(m_1-m_2) \geq u_1-u_2\end{cases}.$$

Case 4: u_1 and u_2 are odd.

$$m_{\lambda,2}^\mu = 0\,.$$

Corollary 9.3. *For A_2, if $m_{\lambda,2}^\mu \neq 0$ then $u_1+2u_2 \geq 2(m_1-m_2)$.*

If $m_1 \le m_2$ we have a similar corollary:

Corollary 9.4. *For A_2, if $m^{\mu}_{\lambda,2} \neq 0$ then $2u_1 + u_2 \ge 2(m_2 - m_1)$.*

9.1.2. *The minimizer for A_2.*

Case 1: $a = 2$. By Corollary 9.3 it suffices to minimize $g^*(\mu)$ over subset $\{\mu \in S_{\lambda,2} : u_1, u_2 \in \mathbb{N}, u_1 + 2u_2 \ge 2(m_1 - m_2)\}$ of $S_{\lambda,2}$. We have

$$\begin{aligned} g^*(\mu) &= \frac{b}{4}(\mu,\mu) + \left(-1 + \frac{b}{2}\right)(\mu,\rho) \\ &= \frac{b}{6}(u_1^2 + u_1u_2 + u_2^2) + (-1 + \frac{b}{2})(u_1 + u_2) \\ &= \frac{b}{6}((u_2 + \frac{u_1}{2})^2 + \frac{3u_1^2}{4}) + \frac{b-2}{4}(u_1 + u_1 + 2u_2) \\ &\ge \frac{b}{8}u_1^2 + \frac{b-2}{4}u_1 + \frac{b}{6}(m_1 - m_2)^2 + \frac{b-2}{2}(m_1 - m_2) \\ &\ge \frac{b}{6}(m_1 - m_2)^2 + \frac{b-2}{2}(m_1 - m_2) \end{aligned}$$

with equality if and only if $u_1 = 0, u_2 = m_1 - m_2$.

Next we show that $\mu_{\lambda,2} = (m_1 - m_2)\lambda_2 \in S_{\lambda,2}$. Indeed,

(1) If $m_1 - m_2 \equiv 0 \pmod 2$ then $\mu_{\lambda,2} = 2\nu - (\rho - \sigma(\rho)) \in S_{\lambda,2}$ where $\nu = \frac{m_1-m_2}{2}\lambda_2 \in \Pi_\lambda$ and $\sigma = 1$.

(2) If $m_1 - m_2 \equiv 1 \pmod 2$ then $\mu_{\lambda,2} = 2\nu - (\rho - \sigma(\rho)) \in S_{\lambda,2}$ where $\nu = \frac{m_1-m_2+3}{2}\lambda_2 \in \Pi_\lambda$ and ρ such that $\rho - \sigma(\rho) = 3\lambda_2$.

Note that from the formula for $m^{\mu}_{\lambda,2}$ in Equations (9.1) and (9.2) we have $m^{(m_1-m_2)\lambda_2}_{\lambda,2} = 1$ which proves part (a). Part (b) is obvious. The case $m_1 \le m_2$ is similar.

Case 2: $a = 3$. From Equation (3.6), we have

$$m^0_{\lambda,3} = m^0_\lambda + m^{\lambda_1}_\lambda + m^{\lambda_2}_\lambda .$$

Since the fundamental group for A_2 consists of only three elements (namely, 0, λ_1, and λ_2), at least one of the terms on the right hand side is greater than zero. Therefore $m^0_{\lambda,3} > 0$ and it follows from Lemma 9.1 that $\mu_{\lambda,3} = 0$ for all λ. Therefore part (b) follows. Part (a) follows from Corollary 9.2 and the fact that $m^0_{\lambda,3} > 0$.

Case 3: $a \ge 4$.

Claim. At most one term on the right hand side of Equation (3.5) is nonzero.

Proof. Indeed, if there are σ_1, σ_2 in the Weyl group for A_2 such that $m_\lambda^{\frac{\mu+\rho-\sigma_1(\rho)}{a}} \neq 0$ and $m_\lambda^{\frac{\mu+\rho-\sigma_2(\rho)}{a}} \neq 0$ then $\frac{\mu+\rho-\sigma_1(\rho)}{a} - \frac{\mu+\rho-\sigma_2(\rho)}{a} \in \Lambda_r$.

Equivalently, $(\rho-\sigma_1(\rho))-(\rho-\sigma_2(\rho)) \in a\Lambda_r$. This is a contradiction since $a \geq 4$ and by [13],

$$\rho - \sigma(\rho) = \sum_{\alpha\in\Delta^+:\sigma^{-1}(\alpha)\in\Delta^-} \alpha$$

which do not belong to $a\Lambda_r$ if $a \geq 4$. Here Δ^+ is the set of positive roots and $\Delta^- = -\Delta^+$. □

Case 3.1: $\lambda \in \Lambda_r$, i.e., $m_1 - m_2 \equiv 0 \bmod 3$. By the above claim we have $m^0_{\lambda,a} = m_\lambda^{\frac{\rho-\sigma(\rho)}{a}}$ for some σ. It's easy to see that the only σ for which $\frac{\rho-\sigma(\rho)}{a}$ is a weight is when $\sigma = 1$ and therefore $m^0_{\lambda,a} = m^0_\lambda > 0$. It follows from Lemma 9.1 that $\mu_{\lambda,a} = 0$. Therefore part (b) follows for this case. Part (a) follows from Corollary 9.2 and the fact that $m^0_{\lambda,3} > 0$.

Case 3.2: If $\lambda \notin \Lambda_r$, or equivalently $m_1 - m_2 \not\equiv 0 \bmod 3$ then $m^0_{\lambda,a} = m^0_\lambda = 0$ so $\mu_{\lambda,a} \neq 0$. By the above claim, we have

$$m^\mu_{\lambda,a} = (-1)^\sigma m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{a}}$$

for some σ. Furthermore, $m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{a}} \neq 0$ if and only if $\frac{\mu+\rho-\sigma(\rho)}{a} = \nu \in \Pi_\lambda$ or equivalently, $\mu = a\nu - (\rho - \sigma(\rho))$. Let $\rho - \sigma(\rho) = s\lambda_1 + t\lambda_2$, where

(9.3)

(s,t)	$(0,0)$	$(-1,2)$	$(1,-2)$	$(0,3)$	$(3,0)$	$(2,2)$
$(-1)^\sigma$	1	-1	-1	1	1	-1

So if $\nu = v_1\lambda_1 + v_2\lambda_2$ then $\mu = (av_1 - s)\lambda_1 + (av_2 - t)\lambda_2$. Since μ is a positive weight, we have we have

$$av_1 - s \geq 0$$
$$av_2 - t \geq 0$$

Since $a \geq 4$ and $|s|, |t| \leq 3$, these inequalities imply that $v_1, v_2 \geq 0$, i.e., ν is also a positive weight. There are two possibilities for λ.

Case 3.2.1: $\lambda_1 \in \Pi_\lambda$, i.e., $m_1 \equiv m_2 + 1 \bmod 3$. Then we can choose $\nu_0 = \lambda_1$ and σ_0 to be the unique element in W such that $\rho - \sigma_0(\rho) = 3\lambda_1$. We will prove that $\mu_{\lambda,a} = a\nu_0 - (\rho - \sigma_0(\rho)) = (a-3)\lambda_1$ is the minimizer. Indeed, let $\mu = a\nu - (\rho - \sigma(\rho)) \in S_{\lambda,a}$ where $\nu \in \Pi_\lambda$ as above.

Case 3.2.1.1: If $\nu = \lambda_1$ then for μ to be a dominant weight we should have, according to Table (9.3),

$$\rho - \sigma(\rho) = \begin{cases} 0 & \text{which gives } \mu = a\lambda_1 \\ \lambda_1 - 2\lambda_2 & \text{which gives } \mu = (a-1)\lambda_1 + 2\lambda_2 \\ 3\lambda_1 & \text{which gives } \mu = (a-3)\lambda_1 = \mu_{\lambda,a} \end{cases} .$$

It is easy to check that $g^*(\mu) > g^*(\mu_{\lambda,a})$ for the first two values of μ.
Case 3.2.1.2: If $\nu \neq \lambda_1$, let $\nu = v_1\lambda_1 + v_2\lambda_2$ then we have $v_1, v_2 \geq 0$ and $v_1 + v_2 \geq 3$, since the only cases where $v_1 + v_2 < 3$ are $\nu = \lambda_2$ and $\lambda_1 + \lambda_2$ but these weights donot belong in Π_λ. Let $\nu = a\nu - (\rho - \sigma(\rho)) = (av_1 - s)\lambda_1 + (av_2 - t)\lambda_2$ as before. We have

$$\begin{aligned} g^*(\mu) &= \frac{b}{2a}(\mu, \mu) + \left(-1 + \frac{b}{a}\right)(\mu, \rho) \\ &= \frac{b}{3a}(u_1^2 + u_1u_2 + u_2^2) + (-1 + \frac{b}{a})(u_1 + u_2) \\ &= \frac{b}{3a}(a^2(v_1^2 + v_1v_2 + v_2^2) - 2a(v_1 + v_2)(s + t) + s^2 + st + t^2) \\ &+ (-1 + \frac{b}{a})(a(v_1 + v_2) - s - t)\,. \end{aligned}$$

It is easy to check that for all

$$(s, t) \in \{(0, 0), (-1, 2), (1, -2), (0, 3), (3, 0), (2, 2)\}$$

and $(v_1, v_2) : v_1, v_2 \geq 0, v_1 + v_2 \geq 3$, we have

$$\begin{aligned} a^2(v_1^2 + v_1v_2 + v_2^2) - 2a(v_1 + v_2)(s + t) + s^2 + st + t^2 &> (a - 3)^2 \\ a(v_1 + v_2) - s - t &> a - 3 \end{aligned}$$

and therefore $g^*(\mu) > \frac{b}{3a}(a - 3)^2 + (-1 + \frac{b}{a})(a - 3) = g^*(\mu_{\lambda,a})$ for all $\mu \neq \lambda_1$.

The above argument showed that $\mu_{\lambda,a} = (a - 3)\lambda_1$ is the unique minimizer, and note that $m_{\lambda,a}^{(a-3)\lambda_1} = m_\lambda^{\lambda_1} \neq 0$ since $\lambda_1 \in \Pi_\lambda$. This proves parts (a) and (b) for Case 3.2.1.
Case 3.2.2: $\lambda_2 \in \Pi_\lambda$ or equivalently, $m_1 \equiv m_2 + 2 \bmod 3$. The proof for this is identical to the one above.

This completes the proof of Theorem 3.4 for A_2. □

9.2. **Theorem 3.4 for** B_2**.** There are two simple roots $\{\alpha_1, \alpha_2\}$ and four positive roots $\{a_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$ of B_2 shown in Figure 3. The Kostant partition function $p(u, v) = p(u\alpha_1 + v\alpha_2)$ is given by [18]

$$p(u, v) = \begin{cases} b(v) & \text{if } u \geq v \\ b(v) - \frac{(v-u)(v-u+1)}{2} & \text{if } u \leq v \leq 2u \\ \frac{(u+1)(v+2)}{2} & \text{if } 2u \leq v \end{cases}, \tag{9.4}$$

where

$$b(n) = \frac{n^2}{4} + n + \begin{cases} 1 & \text{if } 2|n \\ \frac{3}{4} & \text{if } 2 \not| n \end{cases}. \tag{9.5}$$

There are three Kostant chambers shown in Figure 3.

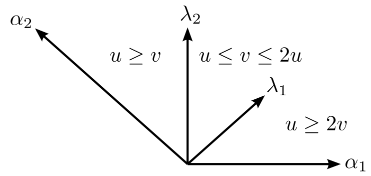


FIGURE 3. The three chambers of the Kostant partition function of B_2. Kostant chambers from left to right: $u \geq v$, $u \leq v \leq 2u$, $u \geq 2v$.

Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ denote a dominant weight. In weight coordinates we have

$$\rho - \sigma(\rho) = s\lambda_1 + t\lambda_2$$

where

(9.6)

(s,t)	$(0,0)$	$(2,-2)$	$(-1,2)$	$(-1,4)$	$(3,-2)$	$(3,0)$	$(0,4)$	$(2,2)$
$(-1)^\sigma$	1	-1	-1	1	1	-1	-1	1

Lemma 3.1 implies that

$$m^0_{\lambda,a} = m^0_\lambda + \begin{cases} -m^{\lambda_2}_\lambda - m^{2\lambda_2}_\lambda + m^{\lambda_1+\lambda_2}_\lambda & \text{if} \quad a = 2 \\ -m^{\lambda_1}_\lambda & \text{if} \quad a = 3 \\ -m^{\lambda_2}_\lambda & \text{if} \quad a = 4 \\ 0 & \text{if} \quad a \geq 5 \end{cases}. \tag{9.7}$$

Case 1: $a = 2$. Equation (9.7) implies that

$$m^0_{\lambda,2} = m^0_\lambda - m^{2\lambda_2}_\lambda - m^{\lambda_2}_\lambda + m^{\lambda_1+\lambda_2}_\lambda\,. \tag{9.8}$$

Case 1.1: $\lambda \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$. In this case, we have $\lambda_1 + \lambda_2$, $\lambda_2 \notin \Lambda_r$, and therefore $m^{\lambda_2}_\lambda = m^{\lambda_1+\lambda_2}_\lambda = 0$. Equation (9.8) becomes

$$m^0_{\lambda,2} = m^0_\lambda - m^{2\lambda_2}_\lambda = 1\,.$$

where the later equality comes from formula (9.4) and the Kostant multiplicity formula (3.4). It follows from Lemma 9.1 that $\mu_{\lambda,2} = 0$ which proves part (b). Part (a) follows from Corollary 9.2 and the fact that $m^0_{\lambda,2} = 1 \neq 0$.

Case 1.2: $\lambda \notin \Lambda_r$, i.e., $m_2 \equiv 1 \bmod 2$. Since $m^0_\lambda = m^{2\lambda_2}_\lambda = 0$ we have

$$m^0_{\lambda,2} = m^{\lambda_1+\lambda_2}_\lambda - m^{\lambda_2}_\lambda = -1\,.$$

If $m_1 > 0$ then choose $\nu = \lambda_1 + \lambda_2 \in \Pi_\lambda$ and σ such that $\rho - \sigma(\rho) = 2\lambda_1 + 2\lambda_2$ we obtain $\mu_{\lambda,2} = 2\nu - (\rho - \sigma(\rho)) = 2(\lambda_1 + \lambda_2) - (2\lambda_1 + 2\lambda_2) = 0$. If otherwise $m_1 = 0$ then we choose $\nu = \lambda_2 \in \Pi_\lambda$, σ such that $\rho - \sigma(\rho) = -\lambda_1 + 2\lambda_2$ and get $\mu_{\lambda,2} = 2\lambda_2 - (-\lambda_1 + 2\lambda_2) = \lambda_1$. This proves part (b). Part (a) follows from Corollary 9.2 and the fact that $m^0_{\lambda,2} = -1 \neq 0$.
Case 2: $a = 3$. Consider two small cases.
Case 2.1: If $\lambda = m_1\lambda_1 + m_2\lambda_2 \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$ then we have

$$m^0_{\lambda,3} = m^0_\lambda - m^{\lambda_1}_\lambda = \frac{1}{2} + \frac{(-1)^{m_1+m_2} + (-1)^{m_1+m_2+2}}{4}.$$

If $m_1 \equiv 0 \bmod 2$ then $m^0_{\lambda,3} = 1$. It follows from Lemma 9.1 that $\mu_{\lambda,3} = 0$ and this completes part (b). Part (a) follows from Corollary 9.2 and the fact that $m^0_{\lambda,3} = 1 \neq 0$.

If $m_1 \equiv 1 \bmod 2$ then $m^0_{\lambda,3} = 0$. By a similar argument to the one in Case 3 for A_2 it can be shown that $\mu_{\lambda,3} = 2\lambda_2$ is the unique minimizer and parts (a) and (b) follow.
Case 2.2: If $\lambda = m_1\lambda_1 + m_2\lambda_2 \notin \Lambda_r$, i.e., $m_2 \not\equiv 0 \bmod 2$ then by a similar argument to the one in Case 3 for A_2 we have $\mu_{\lambda,3} = \lambda_1 + \lambda_2$ is the unique minimizer and $m^{\lambda_1+\lambda_2}_{\lambda,3} \neq 0$ which completes the proof.
Case 3: $a = 4$. From Equation (3.6) we have

$$m^0_{\lambda,4} = m^0_\lambda - m^{\lambda_2}_\lambda.$$

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$ then we have $m^0_{\lambda,4} = m^0_\lambda - m^{\lambda_2}_\lambda = m^0_\lambda > 0$, since $0 \in \Pi_\lambda$.

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \notin \Lambda_r$, i.e., $m_2 \not\equiv 0 \bmod 2$ then $m^0_{\lambda,4} = m^0_\lambda - m^{\lambda_2}_\lambda = -m^{\lambda_2}_\lambda < 0$, since $\lambda_2 \in \Pi_\lambda$.

It follows from Lemma 9.1 that $\mu_{\lambda,4} = 0$, which completes part (b). Part (a) follows from Corollary 9.2.
Case 4: $a \geq 5$. The only σ for which $\mu = \frac{\rho - \sigma(\rho)}{a}$ is a weight is $\sigma = 1$ and hence $\mu = 0$. So from Equation (3.6) we have $m^0_{\lambda,a} = m^0_\lambda$.

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$ then $m^0_{\lambda,a} = m^0_\lambda > 0$. It follows from Lemma 9.1 that $\mu_{\lambda,a} = 0$, which completes part (b). Part (a) follows from Corollary 9.2.

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \notin \Lambda_r$, i.e., $m_2 \not\equiv 0 \bmod 2$ then by a similar argument to the one in Case 3 for A_2 we have that $\mu_{\lambda,a} = (a-4)\lambda_2$ is the unique minimizer and $m^{(a-4)\lambda_2}_{\lambda,a} = m^{\lambda_2}_\lambda \neq 0$. This completes both parts (a) and (b).

This completes the proof of Theorem 3.4 for B_2. $\square$

9.3. **Theorem 3.4 for G_2.** There are two simple roots $\{\alpha_1, \alpha_2\}$ and six positive roots $\{a_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$ of G_2 shown in Figure 4.

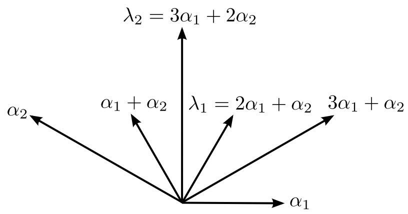


FIGURE 4. The five chambers of the Kostant partition function of G_2. Kostant chambers from left to right: $u \leq v$, $v \leq u \leq \frac{3}{2}v$, $\frac{3}{2}v \leq u \leq 2v$, $2v \leq u \leq 3v$, $3v \leq u$.

The Kostant partition function $p(u,v) = p(u\alpha_1 + v\alpha_2)$ is given by [18]

(9.9)
$$p(u,v) = \begin{cases} g(u) & \text{if} \quad u \leq v \\ g(u) - h(u-v-1) & \text{if} \quad v \leq u \leq \frac{3}{2}v \\ h(v) - g(3v-u-1) + h(2v-u-2) & \text{if} \quad \frac{3}{2}v \leq u \leq 2v \\ h(v) - g(3v-u-1) & \text{if} \quad 2v \leq u \leq 3v \\ h(v) & \text{if} \quad 3v \leq u \end{cases}$$

where

$$g(n) = \begin{cases} \frac{1}{432}(n+6)(n^3+14n^2+54n+72) & \text{if} \quad n \equiv 0 \bmod 6 \\ \frac{1}{432}(n+5)^2(n^2+10n+13) & \text{if} \quad n \equiv 1 \bmod 6 \\ \frac{1}{432}(n+4)(n^3+16n^2+74n+68) & \text{if} \quad n \equiv 2 \bmod 6 \\ \frac{1}{432}(n+3)^2(n+5)(n+9) & \text{if} \quad n \equiv 3 \bmod 6 \\ \frac{1}{432}(n+2)(n+8)(n^2+10n+22) & \text{if} \quad n \equiv 4 \bmod 6 \\ \frac{1}{432}(n+1)(n+5)(n+7)^2 & \text{if} \quad n \equiv 5 \bmod 6 \end{cases} \tag{9.10}$$

and

$$h(n) = \begin{cases} \frac{1}{48}(n+2)(n+4)(n^2+6n+6) & \text{if} \quad n \equiv 0 \bmod 2 \\ \frac{1}{48}(n+1)(n+3)^2(n+5) & \text{if} \quad n \equiv 1 \bmod 2 \end{cases} . \tag{9.11}$$

From Lemma 3.1 we have

$$m^0_{\lambda,a} = m^0_\lambda + \begin{cases} -m_\lambda^{3\alpha_1+\alpha_2} - m_\lambda^{2\alpha_1+2\alpha_2} + m_\lambda^{5\alpha_1+3\alpha_2} & \text{if} \quad a=2 \\ -m_\lambda^{3\alpha_1+2\alpha_2} & \text{if} \quad a=3 \\ -m_\lambda^{\alpha_1+\alpha_2} & \text{if} \quad a=4 \\ -m_\lambda^{2\alpha_1+\alpha_2} & \text{if} \quad a=5 \\ 0 & \text{if} \quad a\geq 6 \end{cases} . \tag{9.12}$$

From now on, let us consider $\lambda = u\alpha_1 + v\alpha_2 \in \Lambda^+$, so $\frac{3}{2}v \leq u \leq 2v$.
Case 1: $a = 2$. We have

$$m^0_{\lambda,2} = m_\lambda - m_\lambda^{3\alpha_1+\alpha_2} - m_\lambda^{2\alpha_1+2\alpha_2} + m_\lambda^{5\alpha_1+3\alpha_2} . \tag{9.13}$$

Using the Kostant multiplicity formula we can calculate the weight multiplicities on the right hand side of Equation (9.13), we have, for example

$$\begin{aligned} m^0_\lambda &= \sum_{\sigma\in W} (-1)^\sigma p(\sigma(\lambda+\rho)-\rho) \\ &= p(u,v) - p(-u+3v-1,v) - p(u,u-v-1) \\ &\quad + p(3v-u-1,2v-u-2) + p(2u-3v-4,u-v-1) \\ &= \frac{u^4}{9} - \frac{29u^3v}{36} - \frac{7u^3}{36} + \frac{17u^2v^2}{8} + \frac{2u^2v}{3} - \frac{19u^2}{24} - \frac{29uv^3}{12} - \frac{uv^2}{2} + 3uv \\ &\quad + v^4 - \frac{v^3}{12} - \frac{21v^2}{8} + c_{1,0}(u)u + c_{0,1}(v)v + c_{0,0}(u,v) \end{aligned}$$

where

$$c_{1,0}(u) = \begin{cases} \frac{1}{4} & \text{if } u\equiv 0 \bmod 3 \\ \frac{17}{36} & \text{if } u\equiv 1 \bmod 3 \\ \frac{25}{36} & \text{if } u\equiv 2 \bmod 3 \end{cases} \qquad c_{0,1}(u) = \begin{cases} \frac{1}{12} & \text{if } u\equiv 0 \bmod 3 \\ -\frac{13}{36} & \text{if } u\equiv 1 \bmod 3 \\ -\frac{29}{36} & \text{if } u\equiv 2 \bmod 3 \end{cases}$$

$$c_{0,0}(u,v) = \begin{cases} 1 & \text{if } u\equiv 0 \bmod 6,\ v\equiv 0 \bmod 2 \\ \frac{29}{72} & \text{if } u\equiv 1 \bmod 6,\ v\equiv 0 \bmod 2 \\ \frac{5}{9} & \text{if } u\equiv 2 \bmod 6,\ v\equiv 0 \bmod 2 \\ \frac{5}{8} & \text{if } u\equiv 3 \bmod 6,\ v\equiv 0 \bmod 2 \\ \frac{7}{9} & \text{if } u\equiv 4 \bmod 6,\ v\equiv 0 \bmod 2 \\ \frac{13}{72} & \text{if } u\equiv 5 \bmod 6,\ v\equiv 0 \bmod 2 \end{cases}$$

$$c_{0,0}(u,v)=\begin{cases}\frac{5}{8} & \text{if } u\equiv 0 \bmod 6,\ v\equiv 1 \bmod 2\\ \frac{5}{18} & \text{if } u\equiv 1 \bmod 6,\ v\equiv 1 \bmod 2\\ \frac{13}{72} & \text{if } u\equiv 2 \bmod 6,\ v\equiv 1 \bmod 2\\ \frac{5}{8} & \text{if } u\equiv 3 \bmod 6,\ v\equiv 1 \bmod 2\\ \frac{29}{72} & \text{if } u\equiv 4 \bmod 6,\ v\equiv 1 \bmod 2\\ \frac{13}{72} & \text{if } u\equiv 5 \bmod 6,\ v\equiv 1 \bmod 2\end{cases}.$$

$m_\lambda^{3\alpha_1+\alpha_2}, m_\lambda^{2\alpha_1+2\alpha_2}, m_\lambda^{5\alpha_1+3\alpha_2}$ can be computed similarly to show that $m^0_{\lambda,2}=1$. This confirms part (b). Part (a) follows Corollary 9.2.

Case 2: $a=3$. We have

$$\begin{aligned} m^0_{\lambda,3} =& m^0_\lambda - m_\lambda^{[3,2]}\\ =& -u^2+\frac{7uv}{2}+\frac{u}{2}-3v^2-\frac{v}{2}+c_{0,0}(u,v)\end{aligned}$$

where

$$c_{0,0}(u,v)=\begin{cases}1 & \text{if } u\equiv 0, v\equiv 0 \bmod 2\\ \frac{1}{2} & \text{if } u\equiv 1, v\equiv 0 \bmod 2\\ \frac{1}{2} & \text{if } v\equiv 1 \bmod 2\end{cases}.$$

Note that since $\frac{3v}{2}\le u\le 2v$, $-u^2+\frac{7uv}{2}+\frac{u}{2}-3v^2-\frac{v}{2}=(-u^2+\frac{7uv}{2}-3v^2)+\frac{u-v}{2}\ge 0$ and therefore $m^0_{3,\lambda}>0$ for all λ. Part (a) follows from Lemma 9.1 and part (b) follows from Corollary 9.2.

Case 3: $a=4,5$. The arguments are similar to that of Case 2.

Case 4: $a\ge 6$, can be done without computations. Indeed, we have $m^0_{\lambda,a}=m^0_\lambda>0$ since $\lambda\in\Lambda_r$; see [12, §13.4,Lem.B]. Parts (a) and (b) follow from Lemma 9.1 and Corollary 9.2.

This completes the proof of Theorem 3.4 for G_2. □

10. Examples

10.1. The tail for A_2 and the $T(2,b)$ torus knots. In this section we compute the tail of the c-stable sequence $J^{A_2}_{T(2,b),n\lambda_1}(q)$ for $b>2$ odd. From Proposition 3.4 we have $\mu_{n\lambda_1,2}=n\lambda_2$ so Equation (7.1) gives

$$\hat{J}^{A_2}_{T(2,b),n\lambda_1}(q)=\frac{1}{(1-q)(1-q^{n+1})(1-q^{n+2})}\check{J}^{A_2}_{T(2,b),n\lambda_1}(q)$$

where

$$\begin{aligned}\check{J}^{A_2}_{T(2,b),n\lambda_1}(q)=&\sum_{u_1\lambda_1+u_2\lambda_2\in S_{n\lambda_1,2}} c(u_1,u_2)q^{\frac{b}{6}(u_1^2+u_1u_2+u_2^2-n^2)+(\frac{b}{2}-1)(u_1+u_2-n)}\\ &\cdot(1-q^{u_1+1})(1-q^{u_2+1})(1-q^{u_1+u_2+2}),\end{aligned}$$

and from Cases 1-4 of Section 9.1.1,

$$c(u_1,u_2) = m_{n\lambda_1,2}^{u_1\lambda_1+u_2\lambda_2} = \begin{cases} 1 & \text{if } u_1+2u_2 \geq 2n,\ u_1,u_2 \text{ are even} \\ 0 & \text{if } u_1+2u_2 < 2n,\ u_1,u_2 \text{ are even} \\ -1 & \text{if } u_1+2u_2 \geq 2n,\ u_1 \text{ even},\ u_2 \text{ odd} \\ 0 & \text{if } u_1+2u_2 < 2n,\ u_1 \text{ even},\ u_2 \text{ odd} \\ 0 & \text{if } u_1 \text{ is odd} \end{cases}.$$

Lemma 10.1. *If $\mu = u_1\lambda_1 + u_2\lambda_2 \in S_{n\lambda_1,2}$ then $u_1 + 2u_2 \leq 2n$.*

Proof. By Lemma 6.1, we have $\mu \in S_{n\lambda_1,2} \subset P_{2n\lambda_1}$. So by Inequality (6.4) we have

$$(2n\lambda_1 - u_1\lambda_1 - u_2\lambda_2, \lambda_2) \geq 0 \quad \text{i.e.,} \quad u_1 + 2u_2 \leq 2n\,.$$

□

From Corollary 9.3 and Lemma 10.1 we have

Corollary 10.2. *$c(u_1,u_2) \neq 0$ if and only if $u_1 + 2u_2 = 2n$.*

10.2. **Proof of Theorem 1.9.** Set $s = \frac{u_1}{2} = n - u_2$, then $u_1^2 + u_1u_2 + u_2^2 - n^2 = 3s^2$ and we have

$$\check{J}^{\mathfrak{g}}_{T(2,b),n\lambda_1}(q) = \frac{\sum_{s=0}^{n} (-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}(1-q^{2s+1})(1-q^{n-s+1})(1-q^{n+s+2})}{(1-q)(1-q^{n+1})(1-q^{n+2})}.$$

Replacing q^n by x and using Lemma 4.1 it follows that $(\hat{J}^{A_2}_{T(2,b),n\lambda_1}(q))$ is c-stable and its tail $G_b(x,q)$ is given by

$$\begin{aligned} G_b(x,q) &= \frac{\sum_{s=0}^{\infty} (-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}(1-q^{2s+1})(1-xq^{1-s})(1-xq^{s+2})}{(1-q)(1-qx)(1-q^2x)} \\ &= \frac{1}{(1-q)(1-qx)(1-q^2x)} \\ &\quad \cdot \sum_{s=0}^{\infty} (-1)^s \Big((q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s} - q^{\frac{b}{2}s^2+(\frac{b}{2}+1)s+1})(1+q^3x^2) \\ &\qquad + (q^{\frac{b}{2}s^2+(\frac{b}{2}+2)s+3} - q^{\frac{b}{2}s^2+(\frac{b}{2}-2)s+1})x \Big) \end{aligned}$$

Using $s = t+1$, we have

$$\sum_{s=0}^{\infty}(-1)^{s+1}q^{\frac{b}{2}s^2+(\frac{b}{2}+1)s+1} = \sum_{t=-1}^{\infty}(-1)^{-t}q^{\frac{b}{2}(t+1)^2-(\frac{b}{2}+1)(t+1)+1}$$
$$= \sum_{s\le -1}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}\,.$$

Therefore,

$$\sum_{s=0}^{\infty}(-1)^s(q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s} - q^{\frac{b}{2}s^2+(\frac{b}{2}+1)s+1}) = \sum_{s=-\infty}^{\infty}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}$$
$$= \theta_{b,\frac{b}{2}-1}(q)\,.$$

Similarly,

$$\sum_{s=0}^{\infty}q^{\frac{b}{2}s^2+(\frac{b}{2}+2)s+3} - q^{\frac{b}{2}s^2+(\frac{b}{2}-2)s+1} = \sum_{s=-\infty}^{\infty}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}+2)s+3}$$
$$= q^3\theta_{b,\frac{b}{2}+2}(q)\,.$$

Thus,

$$G_b(x,q) = \frac{\theta_{b,\frac{b}{2}-1}(q)(1+q^3x^2) + q^3\theta_{b,\frac{b}{2}+2}(q)x}{(1-q)(1-qx)(1-q^2x)}\,.$$

Note that by replacing s with $s+1$ or s by $-s$ in Equation (1.4) it follows that

$$\theta_{b,c}(q) = -q^{\frac{b}{2}+c}\theta_{b,b+c}(q), \qquad \theta_{b,-c}(q) = \theta_{b,c}(q)\,.$$

To compute $G_3(x,q)$, use $b=3, c=\frac{1}{2}$ in the above equation and *Euler's Pentagonal Theorem* (discussed in detail in [1]) to obtain that

$$q^2\theta_{3,\frac{7}{2}}(q) = -\theta_{3,\frac{1}{2}}(q) = -(q)_\infty\,.$$

This completes the proof of Theorem 1.9. □

10.3. **The tail for A_2 and the $T(4,5)$ torus knots.** In this section we compute the tail for the c-stable sequence $(J^{A_2}_{T(4,b),n\rho}(q))$ for $b>4$ odd. This example shows that c-stability is a necessary notion for Conjecture 1.5.

Let

$$A_{b,0}(q) = \sum_{(s,t)} \sum_{(u_1,u_2)} \Big(\epsilon_{s,t} c_{s,t}(u_1,u_2) q^{\frac{b}{12}(u_1^2+u_1u_2+u_2^2)+(\frac{b}{4}-1)(u_1+u_2)}$$
$$(1-q^{u_1+1})(1-q^{u_2+1})(1-q^{u_1+u_2+2}) \Big)$$
$$A_{b,1}(q) = \sum_{(s,t)} \sum_{(u_1,u_2)} \Big(\epsilon_{s,t} q^{\frac{b}{12}(u_1^2+u_1u_2+u_2^2)+(\frac{b}{4}-1)(u_1+u_2)}$$
$$(1-q^{u_1+1})(1-q^{u_2+1})(1-q^{u_1+u_2+2}) \Big)$$

where the (s,t) summation is over the set

(10.1)

(s,t)	$(0,0)$	$(2,-1)$	$(-1,2)$	$(0,3)$	$(3,0)$	$(2,2)$
$\epsilon_{s,t}$	1	-1	-1	1	1	-1

and $(u_1,u_2) \in \mathbb{N}^2$ satisfies $u_1 \equiv -s \bmod 4$, $u_1 - u_2 \equiv t - s \bmod 12$ and

$$c_{s,t}(u_1,u_2) = \begin{cases} 1 - \frac{2u_1+u_2+2s+t}{12} & \text{if}\, u_1 + s \geq u_2 + t \\ 1 - \frac{u_1+2u_2+s+2t}{12} & \text{if}\, u_1 + s \leq u_2 + t \end{cases}.$$

Proposition 10.3. *The tail of the c-stable sequence* $(\hat{J}^{A_2}_{T(4,b),n\rho}(q))$ *is given by*

$$\frac{1}{(1-xq)^2(1-x^2q^2)} \left(A_{b,0}(q) + nA_{b,1}(q) \right).$$

Proof. We will use Theorem 7.2 and unravel its notation. To begin with, for $a = 4$, we have

$$\begin{aligned} \mathcal{L}_{n\rho,4} &= \bigcup_{\sigma \in W} 4n\rho + \sigma(\rho) - \rho + 4\Lambda_r \\ &= \bigcup_{\sigma \in W} \sigma(\rho) - \rho + 4\Lambda_r \\ &= \bigcup_{\sigma \in W} \{\mu \in \Lambda | \mu + \rho - \sigma(\rho) \in 4\Lambda_r\}. \end{aligned}$$

Since $\rho = \alpha_1 + \alpha_2 \in \Lambda_r$, we have $\mathcal{L}_{n\rho,4} = \mathcal{L}_{\rho,4}$ for all natural numbers n. Let $\mu = u_1\lambda_1 + u_2\lambda_2$ and $\rho - \sigma(\rho) = s\lambda_1 + t\lambda_2$ where (s,t) are given in (10.1) and $(-1)^\sigma = \epsilon_{s,t}$ as in (10.1). In weight coordinates we have

(10.2)
$$\mathcal{L}_{\rho,4} = \bigcup_{(s,t)} \{(u_1,u_2) \in \mathbb{Z}^2 : u_1 \equiv -s \bmod 4, u_1 - u_2 \equiv t - s \bmod 12\}.$$

Next we compute the plethysm multiplicities. Equation (3.5) implies that

$$\begin{aligned} m^{\mu}_{n\rho,4} &= \sum_{\sigma\in W} (-1)^{\sigma} m_{n\rho}^{\frac{\mu+\rho-\sigma(\rho)}{4}} \\ &= m_{n\rho}^{\frac{\mu}{4}} - m_{n\rho}^{\frac{\mu+2\lambda_1-\lambda_2}{4}} - m_{n\rho}^{\frac{\mu-\lambda_1+2\lambda_2}{4}} + m_{n\rho}^{\frac{\mu+3\lambda_1}{4}} + m_{n\rho}^{\frac{\mu+3\lambda_2}{4}} - m_{n\rho}^{\frac{\mu+2\lambda_1+2\lambda_2}{4}} . \end{aligned}$$

Since $n\rho \in \Lambda_r$, $m^{\nu}_{n\rho} \neq 0$ only if $\nu \in \Lambda_r$. Therefore at most one of the terms in the above equation is non-zero. Equation (3.4) gives

$$m^{\mu}_{n\rho} = \begin{cases} 1 + \frac{2m_1+m_2}{3} - \frac{2u_1+u_2}{3} & \text{if} \quad u_1 \geq u_2 \\ 1 + \frac{m_1+2m_2}{3} - \frac{u_1+2u_2}{3} & \text{if} \quad u_1 \leq u_2 \end{cases} .$$

Therefore

$$m^{\mu}_{n\rho,4} = \epsilon_{s,t} \begin{cases} 1 + n - \frac{2u_1+u_2+2s+t}{12} & \text{if } u_1 \equiv -s \bmod 4, \\ & u_1 - u_2 \equiv t - s \bmod 12\,, \\ & u_1 + s \geq u_2 + t \\ 1 + n - \frac{u_1+2u_2+s+2t}{12} & \text{if } u_1 \equiv -s \bmod 4, \\ & u_1 - u_2 \equiv t - s \bmod 12\,, \\ & u_1 + s \leq u_2 + t \end{cases}$$

where $\epsilon_{s,t}$ is given from (10.1). Since $\mu_{n\rho,4} = 0$, we have $\hat{\mathcal{L}}_{n\rho,4} = \mathcal{L}_{n\rho,4}$, $\hat{P}_{n\rho} = P_{n\rho}$, $\hat{S}_{n\rho,4} = S_{n\rho,4}$. Theorem 7.2 concludes the proof of Proposition 10.3. □

Exercise 10.4. Show that

$$(10.3) \qquad \begin{aligned} A_{b,1}(q) = \sum_{m_1,m_2\in\mathbb{Z}} & q^{4b(m_1^2+3m_1m_2+3m_2^2)+(b-4)(2m_1+3m_2)} \\ & \times (1-q^{4m_1+1})(1-q^{4m_1+12m_2+1})(1-q^{8m_1+12m_2+2}) . \end{aligned}$$

The above equation shows that $A_{b,0}(q)$ is a sum of theta series of rank 2, hence a modular form of weight 1; see [3].

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