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DENSITY OF THE OPEN-POINT, BI-POINT-OPEN, AND BI-COMPACT-OPEN TOPOLOGIES ON $C(X)$

ANUBHA JINDAL, R. A. McCOY, AND S. KUNDU

Abstract. This paper studies the density of the space $C(X)$, the space of all real-valued continuous function on a Tychonoff space X, equipped with the open-point, bi-point-open, and bi-compact-open topologies introduced by Anubha Jindal, R. A. McCoy, and S. Kundu in *The open-point and bi-point-open topologies on C(X)* (Topology Appl. **18** (2015), 62–74) and in *The bi-compact-open topology on C(X)* (Boll. Unione Mat. Ital. (2016). doi:10.1007/s40574-016-0095-8).

1. Introduction

The set $C(X)$ of all real-valued continuous functions on a Tychonoff space X has a number of natural topologies. One important type of topology on $C(X)$ is the set-open topology, introduced by Richard Arens and James Dugundji [1]. In the definition of a set-open topology on $C(X)$, we use a certain family of subsets of X and open subsets of $\mathbb{R}$. Two important set-open topologies on $C(X)$ are the point-open topology p and the compact-open topology k. In [3] and [5], by adopting a radically different approach, we have defined three new kinds of topologies on $C(X)$: the open-point, bi-point-open, and bi-compact-open topologies. One main reason for adopting such a different approach is to ensure that both X and $\mathbb{R}$ play equally significant roles in the construction of topologies on $C(X)$. This gives a function space where the functions get more involved in the behavior of the topology defined on $C(X)$.

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The open-point topology on $C(X)$ has a subbase consisting of sets of the form

$$[U, r]^- = \{f \in C(X) : f^{-1}(r) \cap U \neq \emptyset\},$$

where U is an open subset of X and $r \in \mathbb{R}$. The open-point topology on $C(X)$ is denoted by h and the space $C(X)$ equipped with the open-point topology h is denoted by $C_h(X)$. The term "h" comes from the word "horizontal" because a subbasic open set in $C_h(\mathbb{R})$ can be viewed as the set of functions in $C(\mathbb{R})$ whose graphs pass through some given horizontal open segment in $\mathbb{R} \times \mathbb{R}$, as opposed to a subbasic open set in $C_p(\mathbb{R})$ which consists of the set of functions in $C(\mathbb{R})$ whose graphs pass through some given vertical open segment in $\mathbb{R} \times \mathbb{R}$.

The *bi-point-open topology on* $C(X)$ is the join of the point-open topology p and the open-point topology h. In other words, it is the topology having subbasic open sets of both kinds: $[x, V]^+ = \{f \in C(X) : f(x) \in V\}$ and $[U, r]^-$, where $x \in X$ and V is an open subset of $\mathbb{R}$, while U is an open subset of X and $r \in \mathbb{R}$. The bi-point-open topology on the space $C(X)$ is denoted by ph and the space $C(X)$ equipped with the bi-point-open topology ph is denoted by $C_{ph}(X)$. One can also view the bi-point-open topology on $C(X)$ as the weak topology on $C(X)$ generated by the identity maps $id_1 : C(X) \to C_p(X)$ and $id_2 : C(X) \to C_h(X)$.

Similarly, the *bi-compact-open topology on* $C(X)$ is defined as the join of the compact-open topology k and the open-point topology h. In other words, it is the topology having subbasic open sets of both kinds: $[A, V]^+ = \{f \in C(X) : f(A) \subseteq V\}$ and $[U, r]^-$, where A is a compact subset of X and V is open in $\mathbb{R}$, while U is an open subset of X and $r \in \mathbb{R}$. The bi-compact-open topology on $C(X)$ is denoted by kh and the space $C(X)$ equipped with the bi-compact-open topology kh is denoted by $C_{kh}(X)$. One can also view the bi-compact-open topology on $C(X)$ as the weak topology on $C(X)$ generated by the identity maps $id_1 : C(X) \to C_k(X)$ and $id_2 : C(X) \to C_h(X)$.

The separability of the spaces $C_h(X)$ and $C_{ph}(X)$ has been studied in [3], [8], and [9]. In [5], the authors studied the separability of the space $C_{kh}(X)$. One necessary condition for the separability of the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ is that X must be uncountable without having any isolated point.

The separability of the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ is not well understood except in some particular cases. In order to understand the separability of these spaces in a broader perspective, in this paper, we look at the density of these spaces.

Throughout this paper the following conventions are used. The symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the space of real numbers, rational numbers, integers, and natural numbers, respectively. For a space X, the symbol

X^0 denotes the set of all isolated points in X, $|X|$ denotes the cardinality of the space X, $\overline{A}$ denotes the closure of A in X, A^c denotes the complement of A in X, and 0_X denotes the constant zero-function in $C(X)$. Also, for any two topological spaces X and Y that have the same underlying set, $X = Y$ means that the topology of X is the same as the topology of Y, $X \leq Y$ means that the topology of X is weaker than or equal to the topology of Y, and $X < Y$ means that the topology of X is strictly weaker than the topology of Y. For other basic topological notions, refer to [2].

2. Density of $C_\tau(X)$, $\tau = h, ph, kh$

In order to study the density of the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$, we first need to study the concepts of R-set, R-dense collection, and R-density. In [3, Theorem 5.2], the authors prove that if $\{f_n : n \in \mathbb{N}\}$ is a countable dense subset of $C_h(X)$, then for any nonempty open set U in X, we have $\cup_{n\in\mathbb{N}} f_n(U) = \mathbb{R}$.

Definition 2.1. A subset B of a space X is said to be an *R-set* if there exists a countable collection $T = \{f_n : n \in \mathbb{N}\}$ of real-valued continuous functions on X such that $\bigcup_{n\in\mathbb{N}} f_n(B) = \mathbb{R}$.

We define a collection $\mathcal{C}$ of nonempty subsets of X to be *R-dense* if every member of $\mathcal{C}$ is an R-set in X and every nonempty open set in X contains some member of the collection $\mathcal{C}$. So an R-dense collection is a π-network consisting of R-sets.

A space may not have any R-sets. An R-set, by definition, is uncountable. Hence, a countable space cannot have any R-sets. Now we define the R-density of a space X, provided it has an R-dense collection.

If X has an R-dense collection, then we define the *R-density* of X, denoted by $Rd(X)$, as follows:

$$Rd(X) = \aleph_0 + \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } R\text{-dense collection in } X\}.$$

If $Rd(X) = \aleph_0$, then we call X an *R-separable* space.

In [8], the author uses the concept of an $\mathcal{I}$-set to study the separability of the spaces $C_h(X)$ and $C_{ph}(X)$. A subset A of a space X is called an *$\mathcal{I}$-set* if there is a continuous function $f \in C(X)$ such that $f(A)$ contains an interval $I = [a, b] \subseteq \mathbb{R}$.

In our next result we prove that the concept of an $\mathcal{I}$-set is equivalent to that of an R-set. But first we need the following definition. A subset B of $\mathbb{R}$ is called a *Bernstein set* if no uncountable closed subset of $\mathbb{R}$ is contained in either B or $\mathbb{R} \setminus B$.

Proposition 2.2. *A subset A of a space X is an R-set if and only if it is an $\mathcal{I}$-set.*

Proof. Let $\{f_n : n \in \mathbb{N}\}$ be a countable collection in $C(X)$ such that $\cup_{n\in\mathbb{N}} f_n(A) = \mathbb{R}$. Suppose that A is not an $\mathcal{I}$-set. Therefore, for all $f \in C(X)$, $f(A)$ does not contains an interval. If, for every Cantor set C, C is not a subset of $\mathbb{R} \setminus f_1(A)$, then $f_1(A)$ is a Bernstein set. By [8, Lemma 2.2], there exists a continuous function $g \in C(f_1(A))$ such that $g(f_1(A))$ contains an interval. This contradicts our assumption. Thus, there exists a Cantor set C_1 such that $C_1 \cap f_1(A) = \emptyset$. Similarly, for the set $f_2(A)$, we have a Cantor set C_2 such that $C_2 \subseteq C_1$ and $C_2 \cap f_2(A) = \emptyset$. By proceeding inductively, we obtain a countable family of Cantor sets $\{C_i\}_i$ such that $C_{i+1} \subseteq C_i$ for each $i \in \mathbb{N}$. Choose $r \in \cap_{n\in\mathbb{N}} C_n$; we have $r \notin \cup_{n\in\mathbb{N}} f_n(A)$, which contradicts our assumption.

Conversely, suppose there exists $f \in C(X)$ such that $[a, b] \subseteq f(A)$. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to [-n, n]$ be a continuous function such that f_n, when restricted to $[a, b]$, is a homeomorphism. Then $\cup_{n\in\mathbb{N}} f_n \circ f(A) = \mathbb{R}$. Hence, A is an R-set. □

It is easy to see that any set containing an R-set is an R-set. Consequently, if X has an R-dense collection, then any base or π-base of X forms an R-dense collection.

Also recall that a space X is called *perfect* if it has no isolated point. The following propositions are immediate.

Proposition 2.3. *If X has an R-dense collection, then every nonempty open subset of X is uncountable, and hence X is a perfect space.*

Proposition 2.4. *If X has an R-dense collection, then $d(X) \leq Rd(X) \leq \pi w(X)$.*

Note that a countable, second-countable space X does not have any R-dense collection, and hence X is not R-separable. So the second countability of a space does not ensure that it will be R-separable.

Recall that a space is said to be a *perfect Polish space* if it is a separable completely metrizable space without isolated points. A *Cantor subset* of X is a subset of X which is homeomorphic to the Cantor set. It is easy to see that every Cantor set is an R-set.

Proposition 2.5 ([5, Proposition 4.1])**.** *Every perfect Polish space is R-separable.*

Proposition 2.6. *Every nontrivial connected subset of X is an R-set.*

Proof. Let S be a nontrivial connected subset of X and let x_S and y_S be distinct elements of S. Since X is a Tychonoff space and S is a connected subset of X, for each $n \in \mathbb{N}$, there exists $f_n \in C(X)$ such that $[-n, n] \subseteq f_n(S)$. Therefore, $\cup_{n\in\mathbb{N}} f_n(S) = \mathbb{R}$, and hence S is an R-set. □

Corollary 2.7. *If X is a locally connected space without isolated points, then X has an R-dense collection.*

Corollary 2.8. *If X is a space having a countable π-base consisting of nontrivial connected sets, then X is R-separable.*

The following theorem gives a necessary condition for the separability of the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$.

Theorem 2.9. *If $C_\tau(X)$ is separable, where $\tau = h, ph, kh$, then every nonempty open set in X is an R-set, and hence X has an R-dense collection.*

Proof. Let $D = \{f_n : n \in \mathbb{N}\}$ be a countable dense set in $C_\tau(X)$ and U be any nonempty open set in X. Suppose that there exists a $y \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} f_n(U)$. So $[U, y]^-$ is a nonempty open set in $C_\tau(X)$ which does not intersect D. Therefore, U must be an R-set, and hence X has an R-dense collection. □

Corollary 2.10. *If X has a countable π-base and the space $C_\tau(X)$ is separable, where $\tau = h, ph, kh$, then X is R-separable.*

Now we relate the density of the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ with the R-density of X.

Theorem 2.11. *If X has an R-dense collection, then $d(C_h(X)) \leq Rd(X)$.*

Proof. Let $\mathcal{K}$ be an R-dense collection in X such that $|\mathcal{K}| = Rd(X)$, and let $\mathcal{S} = \{(S_1, \ldots, S_m) : m \in \mathbb{N}$, for each $1 \leq i \leq m, S_i \in \mathcal{K}$, and S_i, S_j are completely separated sets for $1 \leq i \neq j \leq m\}$. Since X is a Tychonoff space, the collection $\mathcal{S}$ is nonempty. For each $S \in \mathcal{K}$, there exists a countable set F_S in $C(X)$ such that $\cup\{f(S) : f \in F_S\} = \mathbb{R}$. Consider $\mathcal{F} = \{(f_{S_1}, \ldots, f_{S_n}) : n \in \mathbb{N}, (S_1, \ldots, S_n) \in \mathcal{S}, f_{S_i} \in F_{S_i}\}$.

For each $T = (f_{S_1}, \ldots, f_{S_n}) \in \mathcal{F}$, define a continuous function $f_T : X \to \mathbb{R}$ such that $f_T(x) = f_{S_i}(x)$ for $x \in S_i$.

Finally, we prove that the collection $\mathcal{F}' = \{f_T : T \in \mathcal{F}\}$ is dense in $C_h(X)$. Clearly, $|\mathcal{F}'| = |\mathcal{F}| = |\mathcal{S}| \leq |\mathcal{K}| = Rd(X)$.

Let $[U_1, t_1]^- \cap [U_2, t_2]^- \cap \ldots \cap [U_n, t_n]^-$ be any basic open set in $C_h(X)$, where U_i is an open set in X and $t_i \in \mathbb{R}$, for each $i \in \{1, 2, \ldots, n\}$, and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Since X is Tychonoff and has an R-dense collection, for each $i \in \{1, \ldots, n\}$, there exists $S_i \in \mathcal{K}$ such that $S_i \subseteq U_i$ and $(S_1, \ldots, S_n) \in \mathcal{S}$. For each $i = \{1, \ldots, n\}$, there exists $f_{S_i} \in F_{S_i}$ such that $t_i \in f_{S_i}(S_i)$. So $T = (f_{S_1}, \ldots, f_{S_n}) \in \mathcal{F}$ and $f_T \in [U_1, t_1]^- \cap \ldots \cap [U_n, t_n]^-$. Hence, $\mathcal{F}'$ is a dense subset of $C_h(X)$. □

Corollary 2.12 ([3, Theorem 5.5])**.** *If X has a countable π-base consisting of nontrivial connected sets, then $C_h(X)$ is separable.*

Corollary 2.13 ([8, Corollary 2.5]). *Let X be a space with a countable π-base. Then the following are equivalent:*

(a) *$C_h(X)$ is separable.*
(b) *X is R-separable.*

If $X = \oplus\{X_\alpha : \alpha \in \Gamma\}$, where for each $\alpha \in \Gamma$, X_α is R-separable, then Theorem 2.11 implies that for each $\alpha \in \Gamma$, $C_h(X_\alpha)$ is separable. If the cardinality of $\Gamma \leq \mathfrak{c}$ (the cardinality of $\mathbb{R}$), then $\Pi\{C_h(X_\alpha) : \alpha \in \Gamma\}$ is also separable. Consequently, [4, Proposition 2.11] implies that $C_h(\oplus\{X_\alpha : \alpha \in \Gamma\})$ is separable. So we have the following result.

Corollary 2.14. *If X is the topological sum of $\mathfrak{c}$ or fewer R-separable spaces, then $C_h(X)$ is separable.*

In order to study the density of the space $C_{ph}(X)$, recall that the *i-weight* of a space X is defined by

$$iw(X) = \aleph_0 +$$

$\min\{w(Y) : \exists$ a continuous bijection from X to a Tychonoff space $Y\}$.

Theorem 2.15. *If X has an R-dense collection, then $iw(X) \leq d(C_{ph}(X)) \leq Rd(X) \cdot iw(X)$.*

Proof. Since $d(C_p(X)) \leq d(C_{ph}(X)$ and $d(C_p(X)) = iw(X)$ (see [7]), we have $iw(X) \leq d(C_{ph}(X))$. Now we prove that $d(C_{ph}(X)) \leq Rd(X) \cdot iw(X)$. Let $\mathcal{K}$ be an R-dense collection in X such that $|\mathcal{K}| = Rd(X)$, and let $\mathcal{S} = \{\{S_1, \ldots, S_m\} : m \in \mathbb{N}$, for each $1 \leq i \leq m, S_i \in \mathcal{K}\}$. Also, for each $S \in \mathcal{K}$, there exists a countable subset F_S of $C(X)$ such that $\cup\{f(S) : f \in F_S\} = \mathbb{R}$. Let $\mathcal{V}$ be a countable base for $\mathbb{R}$ and $\mathcal{U}$ be a base for a coarser topology on X such that $|\mathcal{U}| = iw(X)$. Let $\mathcal{G}$ be the collection of finite families of members of $\mathcal{U} \times \mathcal{V}$. For each $G = \{(W_1, V_1), \ldots, (W_n, V_n)\}$ in $\mathcal{G}$, let $G_1 = \{W_1, \ldots, W_n\}$. Let $\mathcal{H} = \{(S, G) \in \mathcal{S} \times \mathcal{G}$: members of $S \cup G_1$ are pairwise completely separated sets in $X\}$. Since X is a Tychonoff space, $\mathcal{H}$ is nonempty. For each $S = \{S_1, \ldots, S_m\} \in \mathcal{S}$, let $\mathcal{T}_S = \{(f_{S_1}, \ldots, f_{S_m}) : f_{S_i} \in F_{S_i}$ for $1 \leq i \leq m\}$. Clearly, $\mathcal{T}_S$ is countable for each $S \in \mathcal{S}$ and $|\mathcal{H}| \leq Rd(X) \cdot iw(X)$.

For each $H = (S, G) \in \mathcal{H}$ and $T \in \mathcal{T}_S$, where $G = \{(W_1, V_1), \ldots, (W_n, V_n)\}$, $S = \{S_1, \ldots, S_m\}$ and $T = (f_{S_1}, \ldots, f_{S_m})$. Fix $v_j \in V_j$ for each $j \in \{1, \ldots, n\}$ and define a continuous function $f_{H,T} : X \to \mathbb{R}$ such that

$$f_{H,T}(x) = \begin{cases} f_{S_i}(x) & x \in S_i \text{ and } 1 \leq i \leq m \\ v_j & x \in W_j \text{ and } 1 \leq j \leq n. \end{cases}$$

Take $\mathcal{D}$ to be the collection $\{f_{H,T} : H = (S, G) \in \mathcal{H}, T \in \mathcal{T}_S\}$. Clearly, $|\mathcal{D}| \leq Rd(X) \cdot iw(X)$, and we now prove that $\mathcal{D}$ is dense in $C_{ph}(X)$.

Let Z be any nonempty open set in $C_{ph}(X)$. Then [3, Proposition 2.2] implies that there is a basic open set $B = [x_1, V_1]^+ \cap \ldots \cap [x_m, V_m]^+ \cap [U_1, t_1]^- \cap [U_2, t_2]^- \cap \ldots \cap [U_n, t_n]^-$ in $C_{ph}(X)$ contained in Z, where $x_i \neq x_j$ for $i \neq j$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Since X contains no isolated point, we can choose $y_j \in U_j$ for $1 \leq j \leq n$ such that $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is a collection of distinct points in X. Also, for $1 \leq i \leq m$ and $1 \leq j \leq n$, there exist $W_i, K_j \in \mathcal{U}$ such that $x_i \in W_i$, $y_j \in K_j$, and $\{W_1, \ldots, W_m, K_1, \ldots, K_n\}$ is a collection of pairwise completely separated sets in X. For each $j \in \{1, \ldots, n\}$, $U_j \cap K_j$ is an open set in X containing y_j and there exists $S_{K_j} \in \mathcal{K}$ such that $S_{K_j} \subseteq U_j \cap K_j$. So $\{W_1, \ldots, W_m, S_{K_1}, \ldots, S_{K_n}\}$ is a collection of pairwise completely separated sets in X. Take $S = \{S_{K_1}, \ldots, S_{K_n}\}$ and $G = ((W_1, V_1), \ldots, (W_m, V_m))$; then $H = (S, G)$ is in $\mathcal{H}$. For each $j \in \{1, \ldots, n\}$, there exists $f_{S_{K_j}} \in F_{S_{K_j}}$ such that $t_j \in f_{S_{K_j}}(S_{K_j})$. So $T = (f_{S_{K_1}}, \ldots, f_{S_{K_n}}) \in \mathcal{T}_S$ and $f_{H,T} \in \mathcal{D} \cap B$. Hence, $\mathcal{D}$ is a dense subset of $C_{ph}(X)$. □

Corollary 2.16. *If X is the topological sum of $\mathfrak{c}$ or fewer R-separable submetrizable spaces, then $C_{ph}(X)$ is separable.*

Corollary 2.17 ([3, Theorem 5.10])**.** *If X has a countable π-base consisting of nontrivial connected sets and a coarser metrizable topology, then $C_{ph}(X)$ is separable.*

Corollary 2.18 ([8, Theorem 2.4])**.** *Let X be a space with a countable π-base. Then the following are equivalent:*

(a) *$C_{ph}(X)$ is separable.*
(b) *X is an R-separable submetrizable space.*

If the space $C_{ph}(X)$ is separable, then obviously $C_h(X)$ and $C_p(X)$ are also separable. Now we give an example of a space for which $C_h(X)$ is separable, but neither $C_p(X)$ nor $C_{ph}(X)$ is separable.

Example 2.19 ([10, Example 107])**.** Let X be the Helly space; that is, X is a subspace of I^I (where I is the closed unit interval in $\mathbb{R}$) consisting of all nondecreasing functions. Then X is a locally connected space without isolated points. It is a separable first countable space, so it has a countable π-base. This space is pseudocompact but not metrizable, so it is not submetrizable. Therefore, the space $C_h(X)$ is separable by Corollary 2.8 and Theorem 2.11, but neither $C_p(X)$ nor $C_{ph}(X)$ is separable (see [6, Corollary 4.2.2] and Corollary 2.18).

In Theorem 2.11, we have given a bound on the density of the space $C_h(X)$, when the space X has an R-dense collection. By Proposition 2.7, if a space X has a π-base consisting of nontrivial connected sets, then X

has an R-dense collection. Now we study the density of the space $C_h(X)$ whenever X has a π-base consisting of nontrivial connected sets. We first need the following definition from [8].

Definition 2.20. For a space X, define a *cardinal function* ξ on X by $\xi(X) = \aleph_0 + \min\{|\gamma| :$ for every finite family of pairwise disjoint nonempty open subsets $\{V_i\}_{i=1}^k$ of X, there is a family of pairwise disjoint nonempty zero sets $\gamma' = \{Z_i\}_{i=1}^k \subseteq \gamma$ such that $V_i \cap Z_i \neq \emptyset$ for $i = 1, \ldots, k\}$.

Theorem 2.21. *If X has a π-base consisting of nontrivial connected sets, then*

$$d(C_h(X)) = \xi(X).$$

Proof. In order to show that $\xi(X) \leq d(C_h(X))$, let $\mathcal{A}$ be a dense subset of $C_h(X)$ such that $|\mathcal{A}| = d(C_h(X))$, and let $\mathcal{V}$ be a countable base for $\mathbb{R}$ consisting of bounded open intervals. Consider $\gamma = \{f^{-1}(\overline{B}) : f \in \mathcal{A}, B \in \mathcal{V}\}$. Let $\{V_i\}_{i=1}^k$ be a finite family of pairwise disjoint nonempty open subsets of X. Since X has no isolated point, $W = [V_1, 1]^- \cap \ldots \cap [V_k, k]^-$ is a nonempty open set in $C_h(X)$. Then there exist $f \in \mathcal{A} \cap W$ and the family $\{B_{j_s} : s \in B_{j_s}$ for $s = 1, \ldots, k$ and $\overline{B_{j_{s'}}} \cap \overline{B_{j_{s''}}} = \emptyset$ for $1 \leq s' \neq s'' \leq k\}$ such that $\gamma' = \{f^{-1}(\overline{B_{j_s}})\}_{s=1}^k$ is a required subfamily of γ. Hence, $\xi(X) \leq d(C_h(X))$.

Now we prove the inequality $d(C_h(X)) \leq \xi(X)$. Let γ be the family of zero sets in X such that $|\gamma| = \xi(X)$. We can assume that γ is closed under the finite union of its elements. Consider the collection $\mathcal{D} = \{f_{i,j,p,q} \in C(X) : f_{i,j,p,q}(F_i) = p$ and $f_{i,j,p,q}(F_j) = q$ for $F_i, F_j \in \gamma$ such that $F_i \cap F_j = \emptyset$ and $p, q \in \mathbb{Q}\}$. We prove that $\mathcal{D}$ is dense in $C_h(X)$. Clearly, $|\mathcal{D}| \leq |\gamma|$.

Let $W = [U_1, r_1]^- \cap \ldots \cap [U_n, r_n]^-$ be any nonempty basic open set in $C_h(X)$ and let $\mathcal{B}$ be a π-base for X consisting of nontrivial open connected sets. Then there exists a nontrivial open connected set B_j in $\mathcal{B}$ such that $B_j \subseteq U_j$ for each $1 \leq j \leq n$. Choose the different points $a_j, b_j \in B_j$ for each $1 \leq j \leq n$. Let $\{O_j\}_{j=1}^n$ and $\{P_j\}_{j=1}^n$ be families of pairwise disjoint open subsets of X such that $a_j \in O_j \subseteq B_j$, $b_j \in P_j \subseteq B_j$, and $(\cup_{j=1}^n O_j) \cap (\cup_{j=1}^n P_j) = \emptyset$. There exists a family $\gamma' = \{F_k\}_{k=1}^{2n} \subseteq \gamma$ of pairwise disjoint nonempty zero sets such that $F_j \cap O_j \neq \emptyset$ and $F_{j+n} \cap P_j \neq \emptyset$ for $j = 1, \ldots, n$. Take $p, q \in \mathbb{Q}$ such that $p \leq \min\{r_j : j = 1, \ldots, n\}$ and $q \geq \max\{r_j : j = 1, \ldots, n\}$. Consider $f = f_{i',j',p,q} \in C(X)$ such that $f(F_{i'}) = p$ and $f(F_{j'}) = q$, where $F_{i'} = \cup_{k=1}^n F_k$ and $F_{j'} = \cup_{k=n+1}^{2n} F_k$. Then $f \in \mathcal{D} \cap W$. □

Corollary 2.22 ([8, Theorem 2.10])**.** *If X is a locally connected space without isolated points, then the following are equivalent:*

(a) *$C_h(X)$ is a separable space.*

(b) $\xi(X) = \aleph_0$.

In Theorem 2.15, we have given a bound on the density of the space $C_{ph}(X)$, when the space X has an R-dense collection. By Proposition 2.7, if a space X has a π-network consisting of nontrivial connected sets, then X has an R-dense collection. Now we study the density of the space $C_{ph}(X)$ whenever X has a π-network consisting of nontrivial connected sets. Our next result is a generalization of [9, Theorem 2.2].

Theorem 2.23. *If X has a π-network consisting of nontrivial connected sets, then*

$$d(C_{ph}(X)) = iw(X).$$

Proof. Since $d(C_p(X)) \leq d(C_{ph}(X)$ and $d(C_p(X)) = iw(X)$, we have $iw(X) \leq d(C_{ph}(X))$. Now we prove that $d(C_{ph}(X)) \leq iw(X)$. Let $\mathcal{W}$ be a base for the coarser topology on X such that $|\mathcal{W}| = iw(X)$, and let $\mathcal{G}'$ be the collection of all finite subfamilies of subsets of $\mathcal{W}$. Consider $\mathcal{G} = \{G \in \mathcal{G}' : \text{members of } G \text{ are pairwise completely separated sets}\}$. Then for each $G \in \mathcal{G}$, where $G = \{W_1, \ldots, W_m\}$ and $p = (p_1, \ldots, p_m) \in \mathbb{Q}^m$, define a continuous function $f_{G,p} : X \to \mathbb{R}$ such that $f_{G,p}(x) = p_i$ if $x \in W_i$ for $1 \leq i \leq m$.

Take $\mathcal{D} = \{f_{G,p} : G \in \mathcal{G}, p \in \mathbb{Q}^m, m \in \mathbb{N}\}$. We now prove that $\mathcal{D}$ is dense in $C_{ph}(X)$. Clearly, $|\mathcal{D}| \leq iw(X)$.

Let Z be any nonempty open set in $C_{ph}(X)$. Then [3, Proposition 2.3] implies that there exists an open set $B = [x_1, V_1]^+ \cap \ldots \cap [x_m, V_m]^+ \cap [U_1, t_1]^- \cap \ldots \cap [U_n, t_n]^-$ in $C_{ph}(X)$ contained in Z, where $m, n \in \mathbb{N}$, and for $1 \leq i \leq m$, each $x_i \in X$ and V_i is open in $\mathbb{R}$, and for $1 \leq j \leq n$, each U_j is open in X and $t_j \in \mathbb{R}$, and for $i \neq j$, $x_i \neq x_j$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$, and $x_i \notin \overline{U_j}$. Since X has a π-network consisting of nontrivial connected sets, there exists a nontrivial connected set B_j in X such that $B_j \subseteq U_j$ for each $1 \leq j \leq n$, and we can choose $y_j, z_j \in B_j$ such that $\{x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_n\}$ is a collection of distinct points in X. So for each $1 \leq i \leq m$ and $1 \leq j \leq n$, there exist $W_i, G_j, H_j \in \mathcal{W}$ such that $x_i \in W_i$, $y_j \in G_j$, $z_j \in H_j$, and $\{W_1, \ldots, W_m, G_1, \ldots, G_n, H_1, \ldots, H_n\}$ is a collection of pairwise completely separated sets in X. For each $1 \leq i \leq m$, fix $v_i \in V_i \cap \mathbb{Q}$, and for each $1 \leq j \leq n$, choose $p_j, q_j \in \mathbb{Q}$ such that $p_j \leq t_j$ and $q_j \geq t_j$. Consider $G = (W_1, \ldots, W_m, G_1, \ldots, G_n, H_1, \ldots, H_n)$ and $p = (v_1, \ldots, v_m, p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{Q}^{m+2n}$. Since for each $j \in \{1, \ldots, n\}$, B_j is connected, $f_{G,p} \in B \subseteq Z$. $\square$

Corollary 2.24 ([8, Theorem 2.8])**.** *If X is a locally connected space without isolated points, then the following are equivalent:*

(a) $C_{ph}(X)$ *is a separable space.*

(b) *X has a coarser separable and metrizable topology.*

The next theorem gives bounds on the density of the space $C_{kh}(X)$.

Theorem 2.25. *If X has an R-dense collection, then*

$$iw(X) \leq d(C_{kh}(X)) \leq iw(X) \cdot \pi w(X).$$

Proof. Since $d(C_k(X)) \leq d(C_{kh}(X))$ and $d(C_k(X)) = iw(X)$ (see [6, Theorem 4.4.1]), we have $iw(X) \leq d(C_{kh}(X))$. Now we prove that $d(C_{kh}(X)) \leq iw(X) \cdot \pi w(X)$. Suppose that $\mathcal{K}$ is an R-dense collection in X such that $|\mathcal{K}| = Rd(X)$. Let $\mathcal{U}$ be a π-base for X such that $|\mathcal{U}| = \pi w(X)$ and $\mathcal{V}$ be a countable base for $\mathbb{R}$ consisting of bounded open intervals. For convenience of notation, for each $n \in \mathbb{N}$, let $\hat{\mathcal{U}}^n$ be the set of $(U_1, \ldots, U_n) \in \mathcal{U}^n$ such that $\{U_1, \ldots, U_n\}$ is a collection of pairwise disjoint sets. Let $\mathcal{F}$ be a dense subset of $C_k(X)$ such that $|\mathcal{F}| = d(C_k(X)) = iw(X)$.

For each $f \in \mathcal{F}$ and $n \in \mathbb{N}$, let

$$\mathcal{T}_f^n = \{((U_1, \ldots, U_n), (V_1, \ldots, V_n)) \in \hat{\mathcal{U}}^n \times \mathcal{V}^n : f(\overline{U_i}) \subseteq V_i \text{ for } 1 \leq i \leq n\}.$$

We prove that for each $n \in \mathbb{N}$, the set $\mathcal{T}_f^n \neq \emptyset$. Let $\{x_1, \ldots, x_n\}$ be a collection of distinct points in X, and $f(x_i) \in V_i \in \mathcal{V}$. Since f is continuous and X is Hausdorff, there exists $U_i \in \mathcal{U}$ such that $f(\overline{U_i}) \subseteq V_i$ for $1 \leq i \leq n$, and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. So $((U_1, \ldots, U_n), (V_1, \ldots, V_n)) \in \mathcal{T}_f^n$. Take $\mathcal{T}_f = \cup_{n \in \mathbb{N}} \mathcal{T}_f^n$ and, clearly, $|\mathcal{T}_f| \leq \pi w(X)$.

Let $T \in \mathcal{T}_f$, say $T = ((U_1, \ldots, U_n), (V_1, \ldots, V_n))$. Since $\mathcal{K}$ is an R-dense collection in X and X is Tychonoff, for each $i = 1, \ldots, n$, there exist $E_i \in \mathcal{U}$, $S_i \in \mathcal{K}$, and a countable collection F_{S_i} in $C(X)$ such that $S_i \subseteq E_i \subseteq U_i$, $\overline{U_i} \setminus E_i$, and S_i are completely separated sets in X and $\cup\{g(S_i) : g \in F_{S_i}\} = \mathbb{R}$. For each $i = 1, \ldots, n$, take $\mathcal{L}_i = \{f_{S_i} \in F_{S_i} : (f_{S_i})^{-1}(V_i) \cap S_i \neq \emptyset\}$; then for each $f_{S_i} \in \mathcal{L}_i$, there exists a continuous function $f_{S_i,i} : X \to \overline{V_i}$ such that $f_{S_i,i}(x) = f_{S_i}(x)$ for $x \in (f_{S_i})^{-1}(V_i) \cap S_i$ and $f_{S_i,i}(x) = f(x)$ for each $x \in \overline{U_i} \setminus E_i$. Take $S_T = \{S_1, \ldots, S_n\} \in \mathcal{S}$ and let $\mathcal{H}_{S_T} = \{(f_{S_1,1}, \ldots, f_{S_n,n}) : f_{S_i} \in \mathcal{L}_i, 1 \leq i \leq n\}$.

Then for each $T \in \mathcal{T}_f$ and $H \in \mathcal{H}_{S_T}$, where

$$T = ((U_1, \ldots, U_m), (V_1, \ldots, V_m)) \text{ and } H = \{f_{S_1,1}, \ldots, f_{S_m,m}\},$$

define a continuous function $f_{T,H} : X \to \mathbb{R}$ as follows:

$$f_{T,H}(x) = \begin{cases} f_{S_i,i}(x) & x \in U_i \text{ and } 1 \leq i \leq m \\ f(x) & x \in X \setminus\ U_i \text{ and } 1 \leq i \leq m. \end{cases}$$

Take $\mathcal{F}'$ to be the collection $\{f_{T,H} : f \in \mathcal{F}, T \in \mathcal{T}_f, H \in \mathcal{H}_{S_T}\}$. We now prove that $\mathcal{F}'$ is dense in $C_{kh}(X)$. Since $Rd(X) \leq \pi w(X)$, we have $|\mathcal{F}'| \leq iw(X) \cdot \pi w(X)$.

Let Z be any nonempty open set in $C_{kh}(X)$, then [5, Proposition 2.2] implies that Z contains a nonempty open set $B = [A_1, G_1]^+ \cap \ldots \cap [A_m, G_m]^+ \cap [U_1, r_1]^- \cap \ldots \cap [U_n, r_n]^-$ such that, for each $j = 1, \ldots, n$, either $U_j \cap A_i = \emptyset$ for all $i = 1, \ldots, m$, or $U_j \subseteq \cup\{A_i : r_j \in G_i\}$, and $U_j \cap (\cup\{A_i : r_j \notin G_i\}) = \emptyset$; and A_i is a compact subset of X, $G_i \in \mathcal{V}$ whenever $i \in \{1, \ldots, m\}$, and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Let $f \in \mathcal{F} \cap [A_1, G_1]^+ \cap \ldots \cap [A_m, G_m]^+$ and let $I_j = \{1 \leq i \leq m : r_j \in G_i, U_j \cap A_i \neq \emptyset\}$. We can assume that $I_j \neq \emptyset$ and $|I_j| = t_j$ for $1 \leq j \leq k$, where $k \leq n$ and $I_j = \emptyset$ for $k+1 \leq j \leq n$.

If $I_j \neq \emptyset$, then, for each $i \in I_j$, $r_j \in G_i$ and $U_j \cap A_i \cap f^{-1}(G_i) \neq \emptyset$. Since X is a perfect space, for each $i \in I_j$, we can choose $x_i^j \in f^{-1}(G_i) \cap U_j$ such that $\{x_i^j : i \in I_j\}$, is a collection of distinct points. Also there exists $V_i^j \in \mathcal{V}$ such that $f(x_i^j), r_j \in V_i^j \subseteq \overline{V_i^j} \subseteq G_i$. Since X is a Hausdorff space, there exists $\{B_i^j : i \in I_j\}$ a collection of disjoint open sets in X such that $x_i^j \in B_i^j \subseteq \overline{B_i^j} \subseteq f^{-1}(V_i^j) \cap U_j$. Then this implies that there exists $(U_1^j, \ldots, U_{t_j}^j) \in \hat{\mathcal{U}}^{t_j}$ such that $U_i^j \subseteq \overline{U_i^j} \subseteq \overline{B_i^j} \subseteq f^{-1}(V_i^j) \cap U_j$ for each $i \in I_j$. Now for each $i \in I_j$, $f(\overline{U_i^j}) \subseteq V_i^j$. Also, there exist $S_i^j \in \mathcal{K}$, $f_{S_i^j} \in F_{S_i^j}$, and $E_i^j \in \mathcal{U}$ such that $S_i^j \subseteq E_i^j \subseteq U_i^j$, $\overline{U_i^j} \setminus E_i^j$, and S_i^j are completely separated in X, and $r_j \in f_{S_i^j}(S_i^j)$ for each $i \in I_j$.

If $I_j = \emptyset$, take any $x \in U_j$ and let V_j be any neighborhood of $f(x)$ containing r_j. Since f is continuous, there exists $U_j^{'} \in \mathcal{U}$ such that $U_j^{'} \subseteq \overline{U_j^{'}} \subseteq U_j$ and $f(\overline{U_j^{'}}) \subseteq V_j$. And also there exist $S_j \in \mathcal{K}$, $f_{S_j} \in F_{S_j}$, and $E_j^{'} \in \mathcal{U}$ such that $S_j \subseteq E_j^{'} \subseteq U_j^{'}$, $\overline{U_j^{'}} \setminus E_j^{'}$, and S_j are completely separated in X, and $r_j \in f_{S_j}(S_j)$. Consider

$$U = (U_1^1, \ldots, U_{t_1}^1, U_1^2, \ldots, U_{t_2}^2, \ldots, U_1^k, \ldots, U_{t_k}^k, U_{k+1}^{'}, \ldots, U_n^{'}),$$
$$V = (V_1^1, \ldots, V_{t_1}^1, V_1^2, \ldots, V_{t_2}^2, \ldots, V_1^k, \ldots, V_{t_k}^k, V_{k+1}, \ldots, V_n).$$

Since $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$, we have $T = (U, V) \in \mathcal{T}_f$. Take $H = (f_{S_1^1,1}, \ldots, f_{S_{t_1}^1,t_1}, f_{S_1^2,1}, \ldots, f_{S_{t_2}^2,t_2}, \ldots, f_{S_1^k,1}, \ldots, f_{S_{t_k}^k,t_k}, f_{S_{k+1},k+1}, \ldots, f_{S_n,n})$. Then it is easy to see that $f_{T,H} \in \mathcal{F}' \cap B$. □

The next result can be proved by using Corollary 2.18 and the same technique as in the proof of Theorem 2.25.

Corollary 2.26 ([5, Theorem 4.3])**.** *Let X be a space with a countable π-base. Then the following are equivalent:*

(a) *$C_{kh}(X)$ is separable.*
(b) *$C_{ph}(X)$ is separable.*
(c) *X is an R-separable submetrizable space.*

Corollary 2.27. *If X is the topological sum of $\mathfrak{c}$ or fewer R-separable submetrizable spaces each having a countable π-base, then $C_{kh}(X)$ is separable.*

Example 2.28. Let $X = S$ be the Sorgenfrey line. Then X is an R-separable submetrizable space having a countable π-base, and hence, by Corollary 2.26, the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ are separable.

Example 2.29. Let X be the Niemytzki plane. Then X is an R-separable submetrizable space having a countable π-base, and hence, Corollary 2.26 implies that the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ are separable.

Now Proposition 2.5 and Corollary 2.26 imply the following result.

Corollary 2.30 ([5, Corollary 4.6])**.** *If X is a perfect Polish space, then the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ are separable.*

Remark 2.31. Since the set of irrationals with the usual topology is a perfect Polish space, [3, Theorem 5.1] is an immediate consequence of Corollary 2.30.

Example 2.32. If $X = \mathbb{N}^{\mathbb{N}}$ (so X is homeomorphic to the space of irrationals), then the spaces $C_h(X)$, $C_{ph}(X)$, and $C_{kh}(X)$ are separable.

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