A Geometric Proof of the Structure Theorem for Cyclic Splittings of Free Groups

by

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A GEOMETRIC PROOF OF
THE STRUCTURE THEOREM FOR
CYCLIC SPLITTINGS OF FREE GROUPS

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ABSTRACT. We give a geometric proof of a well-known theorem that describes splittings of a free group as an amalgamated product or HNN extension over the integers. The argument generalizes to give a similar description of splittings of a virtually free group over a virtually cyclic group.

1. INTRODUCTION

This paper describes one-edge splittings of free groups over (infinite) cyclic subgroups. Conversely, it describes when two free groups can be amalgamated along a cyclic subgroup to form a free group, or when an HNN-extension of a free group along a cyclic subgroup is free.

Theorem 1.1 (Shenitzer [17], Stallings [19], Swarup [20]). *Let A and B be finitely generated free groups, and let C be a cyclic group.*

- *$G = A *_C B$ is free if and only if one of the injections of C into A and B maps C onto a free factor of the vertex group.*
- *$G = A*_C$ is free if and only if, up to A-conjugation, the edge injections map C into independent free factors of A, and one of them is onto its factor.*

This theorem is well known. The amalgamated product case is a theorem of Abe Shenitzer [17]. The HNN case follows from a theorem of

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G. A. Swarup [20], who proves a more general theorem for splittings of free groups over free subgroups. Swarup attributes the case of cyclic splittings to Stallings. A published version of Stallings' proof appears later [19]. A simple topological proof appears in an unpublished paper of Mladen Bestvina and Mark Feighn [1, Lemma 4.1]. Generalized versions appear in work of Larsen Louder [11] and Guo-An Diao and Feighn [6].

The "if" direction of the theorem is easy. The "only if" direction we prove geometrically. We build a geometric model X for the group G. The amalgamated cyclic subgroup is a quasi-geodesic in this model. We show that this quasi-geodesic can be continuously deformed to avoid any compact set. This shows that the two distinct endpoints at infinity of the quasi-geodesic are contained in the same end of X. Therefore, the group G is not even virtually free, since then X would be a quasi-tree and every end of X would contain a single boundary point. This proof generalizes to virtually free groups.

Theorem 1.2. *Let A and B be finitely generated virtually free groups, and let C be a virtually cyclic group.*

- *$G = A *_C B$ is virtually free if and only if one of the injections of C into A and B maps C onto a factor of the vertex group.*
- *$G = A*_C$ is virtually free if and only if, up to A-conjugation, the edge injections map C into independent factors of A, and one of them is onto its factor.*

We call an infinite subgroup H of a group G a *factor* if H is a vertex group in a graph of groups decomposition of G with finite edge groups, and we call two factors *independent* if they are the vertex groups in the same graph of groups decomposition of G with finite edge groups.

Theorem 1.2 can also be derived from more general machinery for hyperbolic-elliptic splittings; see, for example, [6, Theorem 7.2]. The proof given here is different, and is elementary, other than the fact that a group is virtually free if and only if it is a quasi-tree.

Gilbert Levitt has pointed out that some virtually free groups do not have any virtually cyclic factors according to our definition. An example is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$; see Example 4.8. Consequently, no HNN extension of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ over a virtually cyclic group is ever virtually free, nor is any amalgam of two copies of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ over a virtually cyclic group.

2. Preliminaries

We first recall some preliminary material about Whitehead graphs, quasi-trees, and geometric models for graphs of groups.

2.1. Free groups and Whitehead graphs.

Let $F = F_n$ be a finite rank free group. For $f \in F$, let $\bar{f}$ denote the inverse element. A free generating set $\mathcal{B} = \{b_1, \ldots, b_n\}$ is called a *basis*. A *multiword* $\underline{w}$ is a finite list of elements in F. A multiword $\underline{w} = \{w_1, \ldots, w_k\}$ is *basic* if there exist elements $f_i \in F$ such that $\{\bar{f}_i w_i f_i\}$ is a subset of a basis. An element is *indivisible* if it is not a proper power of another element. Basic elements are often called *primitive* in the literature.

Let $|g|_\mathcal{B}$ denote the word length of an element g with respect to the basis $\mathcal{B}$. Let $|[g]|_\mathcal{B}$ denote the minimum $\mathcal{B}$-length of a conjugate of g.

The Whitehead graph $\mathrm{Wh}_\mathcal{B}(*)\{w\}$ of an indivisible, cyclically reduced word $w \in F$ with respect to a basis $\mathcal{B}$ is a graph with vertex set in bijection with the set $\mathcal{B} \cup \bar{\mathcal{B}}$ of generators and their inverses. An edge is added from vertex $\bar{x}$ to vertex y for each occurrence of xy as a subword of w written as a reduced cyclic word in the letters $\mathcal{B} \cup \bar{\mathcal{B}}$.

We can similarly define a Whitehead graph for a finite list of words $\underline{w}$. We will be interested in the conjugacy classes of maximal cyclic subgroups containing the words of $\underline{w}$. Thus, to define $\mathrm{Wh}_\mathcal{B}(*)\{\underline{w}\}$, we choose a minimal set of indivisible, cyclically reduced words $\underline{v} = \{v_i\}$ so that each $w_i \in \underline{w}$ is conjugate into some $\langle v_j \rangle$. Then we add edges as above for each v_j. The graph constructed is independent of the choice of the v_j's.

Whitehead's algorithm [21] produces a point in the $\mathrm{Aut}(F)$ orbit of w of minimal $\mathcal{B}$-length. An equivalent formulation for multiwords is that it chooses a basis $\mathcal{B}$ with respect to which $\mathrm{Wh}_\mathcal{B}(*)\{\underline{w}\}$ has the minimal number of edges.

The Whitehead graphs we deal with will not always be connected, so we make the following definitions.

Definition 2.1. A *cut point* of a graph is a point such that deleting it increases the number of connected components. A *cut vertex* is a vertex that is a cut point.

Definition 2.2. We say a graph has *2-connected components* if every connected component is 2-connected, that is, does not contain a cut point.

A special case of Menger's Theorem [14] says a graph without cut points has 2-connected components.

The next lemmas are easy exercises with Whitehead's algorithm.

Lemma 2.3. *A Whitehead graph with a cut vertex is not minimal.*

Lemma 2.4. *A Whitehead graph with a valence one vertex labeled x is not minimal unless x and $\bar{x}$ are joined by an isolated edge.*

Lemma 2.5. *Every non-trivial component of a minimal Whitehead graph is either 2-connected or an isolated edge joining a vertex to its inverse.*

Lemma 2.6. *Each word in a multiword contributes edges to only one component of a minimal Whitehead graph.*

For a fixed basis $\mathcal{B}$, the Cayley graph of F with respect to $\mathcal{B}$ is a tree $\mathcal{T}$. The Whitehead graph can be generalized to a Whitehead graph $\mathrm{Wh}_{\mathcal{B}}(\mathcal{X})\{w\}$ over a compact subtree $\mathcal{X}$ of $\mathcal{T}$. The vertex set is indexed by the elements of F that are adjacent to $\mathcal{X}$ in $\mathcal{T}$. Vertices labeled u and v are connected by an edge for each w-orbit of $\bar{u}v$ as a subword of some power of w. One way to imagine this is that there is some cyclic permutation w' of w so that if you start from the vertex u in $\mathcal{T}$ and follow the edge path that repeatedly spells out the word w', eventually you arrive at the vertex v. Thus, $\mathrm{Wh}_{\mathcal{B}}(\mathcal{X})\{w\}$ records the "line pattern" that conjugates of $\langle w \rangle$ make as they pass through $\mathcal{X}$. See [5] and [4] for more on line patterns and generalized Whitehead graphs.

The classical Whitehead graph $\mathrm{Wh}_{\mathcal{B}}(*)\{\underline{w}\}$ is the generalized Whitehead graph such that the subtree $\mathcal{X}$ is just the identity vertex $* \in \mathcal{T}$.

Jason Fox Manning [13] shows that generalized Whitehead graphs can be constructed from classical Whitehead graphs by a construction called *splicing*. It is an easy observation that splicing connected graphs with no cut vertices produces a connected graph with no cut vertices. This observation gives us the following generalization of Lemma 2.5, which will be used later as an inductive step in building detours.

Lemma 2.7. *If $\mathrm{Wh}_{\mathcal{B}}(*)\{\underline{w}\}$ has 2-connected components, then for every compact subtree $\mathcal{X} \subset \mathcal{T}$, the generalized Whitehead graph $\mathrm{Wh}_{\mathcal{B}}(\mathcal{X})\{\underline{w}\}$ has 2-connected components.*

2.2. **Quasi-trees.**

The terms in this section are standard (see, for example, [3].) The following theorem gathers together various characterizations of virtually free groups.

Theorem 2.8 (Geometric Characterization of Virtually Free Groups)**.** *Let G be a finitely generated group. Let X be a proper geodesic metric space quasi-isometric to G. The following are equivalent:*

1. *G is virtually free: It has a finite index free subgroup.*
2. *G has a finite index normal free subgroup.*
3. *G decomposes as a graph of virtually free groups with finite edge groups.*
4. *G decomposes as a graph of finite groups.*

(5) *X is a quasi-tree: There is a simplicial tree Γ and a (λ, ϵ)-quasi-isometry $\phi\colon X \to \Gamma$.*

(6) *(Bottleneck Property) There is a constant $\Delta > 0$ so that for all x and y in X, there exists a midpoint m such that $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$ and such that any path from x to y passes through $N_\Delta(m)$.*

(7) *(Bottleneck Property') For any $K \geq 1$ and any $C \geq 0$, there is a $\Delta' = \Delta'(K, C) \geq 0$ so that for any x and y in X, any (K, C)-quasi-geodesic segment γ joining x to y, and any continuous path p from x to y, we have $\gamma \subset N_{\Delta'}(p)$.*

(8) *X is hyperbolic and the natural map from ∂X onto $\mathrm{Ends}(X)$ is a bijection.*

Proof. Item (2) follows easily from (1).

The equivalence of (1) and (4) is a theorem of A. Karass, A. Pietrowski, and D. Solitar [10], using Stallings' Theorem [18].

Item (3) is a variant.

Item (4) implies (5) since G acts properly discontinuously and cocompactly on the Bass–Serre tree of the graph of groups decomposition.

The Bottleneck Property (6) is due to Manning, who shows [12, Theorem 4.6] the equivalence of (5) and (6).

Condition (7) is a different version of the bottleneck property. It is just a coarsening of the fact that for any two points x and y in a simplicial tree, there is a unique geodesic $[x, y]$ joining them, and any path p joining x to y necessarily contains $[x, y]$.

That (5) $\implies$ (7) is proven by pushing γ and p forward to Γ with ϕ, applying this fact, and then pulling back to X using a quasi-isometry inverse of ϕ.

That (7) $\implies$ (6) is proven by taking a geodesic segment γ joining x to y and taking m to be the midpoint of γ. Item (6) follows with $\Delta = \Delta'(1, 0)$.

If X is a quasi-tree, it is hyperbolic and has a well-defined boundary at infinity. Item (7) shows that no two boundary points lie in the same end; thus, (5) implies (8).

Finally, if G is finite, the theorem is trivially true, and if it is infinite and (8) holds, then ∂X and $\mathrm{Ends}(X)$ have at least two points. By Stallings' Theorem, G splits over a finite group, and by Dunwoody's Accessibility Theorem [7], there is a graph of groups decomposition of G over finite groups so that all of the vertex groups are either finite or one-ended. A one-ended vertex group would violate condition (8), however, so G satisfies condition (4). □

To show something is not a quasi-tree, we will show that it is possible to detour around some bottleneck point, violating condition (7). Formally:

Corollary 2.9. *A geodesic metric space X is not a quasi-tree if there exists a quasi-geodesic $\gamma \colon \mathbb{R} \to X$ and an increasing sequence (t_i) of positive integers such that $\gamma(-t_i)$ and $\gamma(t_i)$ can be connected by a path that does not enter $N_i(\gamma(0))$.*

2.3. GEOMETRIC MODELS.

In the torsion free case we build a Bass–Serre complex X for $G = A *_C B$ as follows (the HNN case is similar). Let $\mathcal{K}_A$ be a rose with $\pi_1(\mathcal{K}_A) = A$, and, similarly, let $\mathcal{K}_B$ be a rose for B. Let $\mathcal{K}_C = S^1 \times [0,1]$ be an annulus with $\pi_1(\mathcal{K}_C) = C$. Build a space $\mathcal{K}$ with $\pi_1(\mathcal{K}) = G$ by gluing one boundary component of $\mathcal{K}_C$ to $\mathcal{K}_A$ according to the edge injection $C \hookrightarrow A$, and, similarly, glue the other boundary component to $\mathcal{K}_B$. Let $X = \widetilde{\mathcal{K}}$. See [16] and [15] for details.

A *vertex space* is a connected component in X of the preimage of $\mathcal{K}_A$ or $\mathcal{K}_B$. In our case, these are copies of Cayley trees for A and B. An *edge strip* is a connected component of the preimage of one of the $\mathcal{K}_C$: a bi-infinite, width 1 strip. The quotient map that collapses each vertex space to a point and each edge strip to an interval gives a G-equivariant map from X to the Bass–Serre tree of the graph of groups decomposition of G, so we call X the *Bass–Serre Complex*.

The edge strips glue onto the vertex spaces along conjugates of the image $\langle w \rangle$ of the edge inclusion. Thus, the Whitehead graph for w, or for $\{w_1, w_2\}$ in the HNN case, records the intersection pattern of edge strips in a vertex space.

We will refer to paths that remain within a single vertex space as *horizontal*, and paths that go directly across an edge space as *vertical*.

In the presence of torsion, we can use the same construction to build a Bass–Serre complex, but the vertex and edge spaces may not be so nice. However, in the proof we will only need the fact that G and X are quasi-isometric, not that G acts nicely on X. Thus, we can make a trade: We will build a "nicer" space X' quasi-isometric to G, but sacrifice the G action to do so. To do this we will choose finite index normal free subgroups A' and B' of A and B, respectively. Fix bases for each of these, and replace each A-vertex space in X by a copy of the Cayley tree for A', and similarly for B. Each edge strip of X glues on to an A-vertex space and a B-vertex space along coarsely well-defined lines, and we can use quasi-isometry inverses to the inclusion maps $A' \hookrightarrow A$ and $B' \hookrightarrow B$ to give lines in the X' vertex spaces to attach edge strips to (see Equation 4.1). The resulting space X' is a coarse Bass–Serre complex (see [15, section 2.6]).

3. Proof of Theorem 1.1

We now proceed with the proof of Theorem 1.1, handling the amalgamated product and HNN extension cases separately.

3.1. Amalgamated product case.

First, consider $G = A *_{\langle w \rangle} B$. Choose a basis for the minimal free factor $\hat{w}_A$ of A containing w such that w has minimal length, and extend it arbitrarily to a basis $\mathcal{B}_A$ of A. Let $\mathcal{K}_A$ be the rose with rank(A) petals in bijection with $\mathcal{B}_A$. Repeat the construction for B.

Metrize $\mathcal{K}_A$ so that the edges have length $|[w]|_{\mathcal{B}_B}$. Metrize $\mathcal{K}_B$ so that the edges have length $|[w]|_{\mathcal{B}_A}$. Let $\mathcal{K}_C$ be a height 1 right annulus with boundary circles of length $|[w]|_{\mathcal{B}_A} \cdot |[w]|_{\mathcal{B}_B}$. With these choices, the vertex spaces and edge strips are isometrically embedded in the corresponding Bass–Serre complex X.

Choose a basepoint $\gamma(0)$ in an A-vertex space X_α. Define a map $\gamma \colon |[w]|_{\mathcal{B}_A} \mathbb{Z} \to X$ by $\gamma(|[w]|_{\mathcal{B}_A} \cdot t) = w^t . \gamma(0)$, and extend linearly to get a map from $\mathbb{R}$. To satisfy Corollary 2.9, it suffices to take the sequence $(t_i = |[w]|_{\mathcal{B}_A} \cdot i)$. To see this, for each $i > 0$, we construct a path p_i joining $\gamma(-t_i)$ to $\gamma(t_i)$ that stays outside $N_i(\gamma(0))$.

Fix any $i > 0$. Take the 0th approximation q_{i0} to p_i to be the subsegment of γ connecting $\gamma(-t_i)$ to $\gamma(t_i)$. This goes through $N_i(\gamma(0))$.

We will inductively push out the approximations of p_i until we leave $N_i(\gamma(0))$, thereby creating a detour. Depending on X, we can push vertically or horizontally.

First, suppose that w is divisible in A. In this case, for any line in X_α to which an edge strip attaches, there are at least two edge strips attached. Construct q_{i1} from q_{i0} by pushing the segment vertically across one of the edge strips that it lies on the boundary of. That is, replace the horizontal segment q_{i0} along one boundary of the edge strip by a path that goes vertically across the edge strip, horizontally across the opposite side, and then vertically back.

The vertical segments of q_{i1} lie outside $N_i(\gamma(0))$. The horizontal segment may not, but it has at least moved distance one farther away from $\gamma(0)$ than q_{i0}. This new horizontal segment lies in a B-vertex space. Now, if w is also divisible in B, then there are at least two edge strips that attach to the line we have just arrived on. Thus, we can push the horizontal segment vertically across an edge strip different from the edge strip that we used in the previous step, so that the horizontal segment gets farther from $\gamma(0)$. Continuing in this way, the vertical segments always stay outside $N_i(\gamma(0))$, and after i steps, the horizontal segment is also outside $N_i(\gamma(0))$.

If w is indivisible we must also push horizontally. Suppose w is indivisible in A. By Lemma 2.6, $\mathrm{Wh}_{\mathcal{B}_A}(*)\{w\}$ has one non-trivial connected component. If $\langle w \rangle$ is not a factor, then the non-trivial connected component is not an isolated edge, so by Lemma 2.5, $\mathrm{Wh}_{\mathcal{B}_A}(*)\{w\}$ has 2-connected components. Note that since X_α is isometrically embedded, $X_\alpha \cap N_i(\gamma(0))$ is just the i-ball $N_i^{X_\alpha}(\gamma(0))$ in X_α in its own natural metric (the one lifted from $\mathcal{K}_A$). The two vertices u and v in $\overline{N_i(0)} \cap \gamma$ are adjacent in $\mathrm{Wh}_{\mathcal{B}_A}(N_i^{X_\alpha}(\gamma(0)))\{w\}$; they are connected by an edge e corresponding to a segment of q_{i0}. By Lemma 2.7, $\mathrm{Wh}_{\mathcal{B}_A}(N_i^{X_\alpha}(\gamma(0)))\{w\}$ has 2-connected components, so there is another path connecting u and v, an edge path $e_1, \ldots, e_k$ that does not use the edge e. Each edge e_j corresponds to a geodesic segment in X_α joining vertices outside of $N_i^{X_\alpha}(\gamma(0))$. Construct $\widehat{q}_{i0}$ from q_{i0} by replacing the e-segment by the segments coming from the alternate path in the generalized Whitehead graph.

Each of the new horizontal segments has endpoints u' and v' outside of the ball $N_i(\gamma(0))$. Furthermore, each of these new segments has an edge strip attached along it. Construct q_{i1} from $\widehat{q}_{i0}$ by pushing each horizontal segment vertically across an edge strip. As in the previous case, the vertical segments of q_{i1} stay outside $N_i(\gamma(0))$, and the horizontal segments move farther from $\gamma(0)$.

The new horizontal segments lie in B-vertex spaces. We can continue the construction if it is possible to push each of these segments vertically or horizontally without pushing back across an edge strip that was already crossed. Thus, we would like to know that each of these segments is on the boundary of two edge strips or that $\mathrm{Wh}_{\mathcal{B}_B}(*)\{w\}$ has 2-connected components. If $\langle w \rangle$ is not a factor of B, then one of these is true.

Thus, if $\langle w \rangle$ is a factor in neither A nor B, we can push γ out of any $N_i(\gamma(0))$, so X is not a quasi-tree, so $A *_{\langle w \rangle} B$ is not free (not even virtually free).

3.2. HNN extension case.

Let $G = A*_C = \langle A, t \mid \bar{t}w_1 t = w_2 \rangle$, where w_1 and w_2 are words in A. The edge injections are the maps $C \xrightarrow{\cong} \langle w_i \rangle$.

If w_1 and w_2 are conjugate into a common maximal cyclic subgroup, then G contains a Baumslag-Solitar subgroup, so it is not free. Otherwise, the vertex spaces are quasi-isometrically embedded, and we may repeat the construction from the amalgamated product case.

Take t_i large enough so that $d(\gamma(\pm t_i), \gamma(0)) \geq 2i$. If there is an initial horizontal push, take the new set of vertices to also lie outside $N_{2i}(\gamma(0))$. A vertical segment from such a vertex may lead closer to $\gamma(0)$, but stays outside $N_{2i-1}(\gamma(0))$. Make sure the next round of horizontal pushing

gives vertices outside of $N_{2i-1}(\gamma(0))$, so that the next vertical segments stay outside $N_{2i-2}(\gamma(0))$, etc. $N_i(\gamma(0))$ still reaches across at most $i-1$ edge strips, so at the ith stage, all vertical and horizontal segments lie outside $N_i(\gamma(0))$.

If w_1 and w_2 are both divisible, then we only need to push vertically, as before, to avoid the bottleneck point, so G is not virtually free.

Otherwise, choose a basis $\mathcal{B}$ so that the Whitehead graph for $\underline{w} = \{w_1, w_2\}$ is minimal. Recall that by definition, $\text{Wh}_{\mathcal{B}}(*)\{\underline{w}\} = \text{Wh}_{\mathcal{B}}(*)\{\underline{v}\}$ where $\underline{v} = \{v_1, v_2\}$ such that v_1 and v_2 are indivisible and cyclically reduced with respect to $\mathcal{B}$, and so that there exists an $a_i \in A$ such that $w_i \in a_i \langle v_i \rangle \overline{a_i}$. We may assume a_1 is trivial.

There are two possibilities. Either $\text{Wh}_{\mathcal{B}}(*)\{\underline{v}\}$ has only one non-trivial connected component, or it has distinct components corresponding to v_1 and v_2. In the first case, the component has more than one edge, so by Lemma 2.5, $\text{Wh}_{\mathcal{B}}(*)\{\underline{v}\}$ has 2-connected components.

In the second case, for each i, either the component containing v_i is 2-connected or it is an isolated edge and v_i is basic.

Thus, we can repeat the construction to build a path avoiding the bottleneck point, and G is not virtually free, unless for some i, say $i = 2$, we have both

- w_2 is indivisible, and
- v_2 is basic and gives an isolated edge in $\text{Wh}_{\mathcal{B}}(*)\{\underline{v}\}$.

Now, the second condition implies there is a splitting $A = A' * \langle v_2 \rangle$ with $w_1 \in \langle v_1 \rangle \subset A'$. If w_2 is indivisible, then $w_2 = a_2 v_2 \overline{a_2}$ (after possibly exchanging v_2 and $\overline{v_2}$), so

$$A = A' * \langle v_2 \rangle = A' * \overline{a_2} \langle w_2 \rangle a_2.$$

Thus, G is not free unless, up to A-conjugation, the edge injections map C into independent factors, and one of them is onto.

4. Factors

To prove the theorem with torsion, we will need a characterization of when an infinite subgroup is a factor. Recall this means that the subgroup appears as a vertex group in a graph of groups decomposition with finite edge groups. We make use of some results about the boundaries of relatively hyperbolic groups due to B. H. Bowditch [2] and Daniel Groves and Manning [8].

A collection of subgroups $\underline{H} = \{H_1, \ldots, H_k\}$ is an *almost malnormal collection* if $|gH_i\bar{g} \cap H_j| = \infty$ implies $i = j$ and $g \in H_i$.

If G is a finitely generated hyperbolic group and $\underline{H}$ is an almost malnormal collection of infinite, finitely generated, quasi-convex subgroups, then G is hyperbolic relative to $\underline{H}$ [2, Theorem 7.11]. There is a relatively

hyperbolic boundary of $(G, \underline{H})$ that we will denote $\mathcal{D}_{\underline{H}}$. This can be seen as the boundary of the "cusped space" obtained from G by hanging a horoball off each conjugate of each of the H_i's [8]. The effect of this is to collapse the embedded image of each boundary of a conjugate of an H_i to a point. Thus, $\mathcal{D}_{\underline{H}}$ is the *decomposition space* that has one point for each distinct conjugate of each H_i and one point for each boundary point of G that is not a boundary point of some conjugate of an H_i.

We say that G splits relative to $\underline{H}$ if there is a splitting of G so that each H_i is conjugate into a vertex group of the splitting. It is easy to see that corresponding to each edge in the Bass–Serre tree of a splitting of G over a finite group relative to $\underline{H}$, there is a pair of complementary nonempty clopen sets of $\mathcal{D}_{\underline{H}}$. Moreover, there is an analogue of Stallings' Theorem: G splits over a finite group relative to $\underline{H}$ if and only if $\mathcal{D}_{\underline{H}}$ is not connected [2, Proposition 10.1].

Proposition 4.1. *Let H be an infinite subgroup of a finitely generated hyperbolic group G. Then H is a factor of G if and only if H is finitely generated, quasi-convex, almost malnormal, and the connected component of $\mathcal{D}_H$ containing the image of ∂H is a single point.*

Proof. The "only if" direction is easy. For the converse, suppose H is not a proper factor of G. We will show $H = G$.

H is infinite, so there is a unique minimal factor containing it. A factor of a factor is a factor, since finite groups act elliptically on any tree; therefore, we may assume H is not contained in a proper factor of G. This means that G does not split relative to H, so $\mathcal{D}_H$ is connected. Since the component containing the image of ∂H is a single point, all of $\mathcal{D}_H$ is a single point. This means the inclusion of H into G induces a homeomorphism between ∂H and ∂G. Since H is finitely generated, this implies that H is a finite index subgroup of G. However, H is almost malnormal, so the index must be one. $\square$

Corollary 4.2. *Let H be an infinite subgroup of a finitely generated virtually free group G. Then H is a factor of G if and only if H is finitely generated and almost malnormal, and $\mathcal{D}_H$ is totally disconnected.*

Proof. Since G is virtually free, H is a factor if and only if G has a graph of groups decomposition such that H is a vertex group and all other local groups are finite. The components of $\mathcal{D}_H$ in this case are singletons for each conjugate of H and each end of the Bass–Serre tree of the splitting. $\square$

Proposition 4.3. *Let $\underline{H} = \{H_1, H_2\}$ be an almost malnormal collection of infinite, finitely generated, quasi-convex subgroups of a hyperbolic group G. Up to conjugation, H_1 and H_2 are contained in independent factors of*

G if and only if the component of $\mathcal{D}_{\underline{H}}$ containing the image of ∂H_1 does not contain the image of the boundary of any conjugate of H_2.

Proof. The "only if" direction is easy. For the converse, for each i, let $\hat{H}_i$ be the smallest factor containing H_i. The image of $\partial \hat{H}_i$ is a connected component of $\mathcal{D}_{\underline{H}}$. The hypothesis then implies that $\{\hat{H}_1, \hat{H}_2\}$ is an almost malnormal collection whose decomposition space is not connected. Pass to a maximal graph of groups splitting of G over finite groups relative to $\{\hat{H}_1, \hat{H}_2\}$. The hypothesis implies that $\hat{H}_1$ and $\hat{H}_2$, hence H_1 and H_2, are conjugate into different vertex groups of this splitting. $\qquad\square$

4.1. Virtually cyclic factors of virtually free groups.

In this section, let H be an almost malnormal, virtually cyclic subgroup of a finitely generated virtually free group G, and let F be a finite index normal free subgroup of G. We relate connectivity of $\mathcal{D}_H$ to connectivity of Whitehead graphs.

Choose representatives g_i so that $G = \coprod F g_i$. The map $\bar{\iota} \colon G \to F \colon f g_i \mapsto f$ is a quasi-isometry inverse to the inclusion $\iota \colon F \hookrightarrow G$. Let $\langle w \rangle = F \cap H$. This is a maximal cyclic subgroup of F since H is almost malnormal. Let d_i be double coset representatives of $F \backslash G / \langle w \rangle$. Let $\underline{w} = \{d_i w \overline{d_i}\}$.

Definition 4.4. The multiword $\underline{w} = \{d_i w \overline{d_i}\}$ above is a *lift of H to F*.

For every $g \in G$, there exist $f \in F$, g_i, d_j, and $f' \in F$ such that $g = f g_i \in f' d_j \langle w \rangle$. Thus, $\bar{\iota}$ coarsely takes each G-conjugate of H to an F-conjugate of some $\langle d_j w \overline{d_j} \rangle$:

(4.1)
$$\bar{\iota}(g H \bar{g}) = \bar{\iota}(f g_i H \overline{g_i} \overline{f}) \overset{c}{=} \bar{\iota}(f g_i \langle w \rangle \overline{g_i} \overline{f})$$
$$= \bar{\iota}(f' d_j \langle w \rangle \overline{d_j} \overline{f'}) = f' \langle d_j w \overline{d_j} \rangle \overline{f'}$$

(The second equivalence is coarsely true.) It follows that $\mathcal{D}_H$ is homeomorphic to the decomposition space of the boundary of F obtained from the almost malnormal collection $\{\langle d_i w \overline{d_i} \rangle\}$, which we shall denote by $\mathcal{D}_{\underline{w}}$. Thus, to decide if $\mathcal{D}_H$ is totally disconnected, we can lift the problem to F and consider $\mathcal{D}_{\underline{w}}$.

Remark 4.5. We took F to be normal so that $\underline{w}$ would have a nice form, but lifting to any finite index subgroup gives a homeomorphism of decomposition spaces.

Lemma 4.6. *Let $\underline{w}$ be a multiword in a free group whose elements generate distinct conjugacy classes of maximal cyclic subgroups. The following are equivalent:*

(1) *$\underline{w}$ is basic.*

(2) *Some minimal Whitehead graph for $\underline{w}$ consists of isolated edges.*

(3) *Every minimal Whitehead graph for $\underline{w}$ consists of isolated edges.*

(4) $\mathcal{D}_{\underline{w}}$ *is totally disconnected.*

Proof. Using Whitehead's algorithm, the equivalence of (1), (2), and (3) is easy.

If $\underline{w}$ is basic we may take a graph of groups decomposition of F with finite edge groups whose cyclic vertex groups are generated by conjugates of the words in $\underline{w}$. The same argument as Corollary 4.2 shows that $\mathcal{D}_{\underline{w}}$ is totally disconnected. Thus, (1) implies (4).

Suppose (3) is false, so that some minimal Whitehead graph has a component containing more than one edge. By passing to a free factor we may assume that the Whitehead graph is connected. Since it has 2-connected components, this is not a rank one factor. It follows that the decomposition space of the factor is connected and not a single point (see, for example, [5, Theorem 4.1]), so $\mathcal{D}_{\underline{w}}$ is not totally disconnected. Thus, (4) implies (3). $\square$

Lemma 4.7. *Let $\underline{w} = \{d_i w \overline{d_i}\}$ as above be a lift of H to F. The following are equivalent:*

(1) *H is a factor of G.*

(2) *$\underline{w} \subset F$ is basic.*

(3) *Every minimal Whitehead graph of $\underline{w}$ consists of isolated edges.*

(4) *Some minimal Whitehead graph of $\underline{w}$ contains an isolated edge.*

The alternative is that every minimal Whitehead graph of $\underline{w}$ has 2-connected components.

Proof. The alternative follows from Lemma 2.5.

Suppose some minimal Whitehead graph for $\underline{w}$ contains an isolated edge. Such an isolated edge would mean that for some i, the point $(d_i w \overline{d_i})^\infty$ is an isolated point in $\mathcal{D}_{\underline{w}}$. Since $\mathrm{Aut}(F)$ acts transitively on $\underline{w}$ and by homeomorphisms on $\mathcal{D}_{\underline{w}}$, this would imply that $\mathcal{D}_{\underline{w}}$ is totally disconnected. By Lemma 4.6, this is equivalent to $\underline{w}$ being basic and also to every minimal Whitehead graph consisting entirely of isolated edges. Furthermore, $\mathcal{D}_H$ and $\mathcal{D}_{\underline{w}}$ are homeomorphic, and Corollary 4.2 says that H is a factor if and only if $\mathcal{D}_H$ is totally disconnected. $\square$

Example 4.8. *$G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ has no virtually cyclic factors.*

Proof. Let $G = \langle r, s \mid r^3 = s^2 = 1 \rangle$. There is a rank 2 normal free subgroup $F = \langle srsr^2, sr^2sr \rangle$, and $G/F = \langle [sr] \rangle = \mathbb{Z}/6\mathbb{Z}$. The action of sr on the abelianization of F has orbits of size 3 on lines through the origin. Thus, the words in the lift of any virtually cyclic group H to F are not contained in fewer than three distinct conjugacy classes of maximal cyclic

subgroups. A basic multiword in $\mathbb{F}_2$ has words in at most two conjugacy classes of maximal cyclic subgroup, so, by the previous lemma, H is not a factor. □

5. Proof of Theorem 1.2

Let $G = A *_C B$ be an amalgamated product of virtually free groups over a virtually cyclic group.

$\mathrm{Comm}_A(C) = \{a \in A \mid aC\bar{a} \cap C$ has finite index in both C and $aC\bar{a}\}$ is the *commensurator* of C in A. A theorem of Ilya Kapovich and Hamish Short [9, Theorem 1] says that an infinite, quasi-convex subgroup of a hyperbolic group has finite index in its commensurator. Since C is virtually cyclic, so is $\mathrm{Comm}_A(C)$, and $\mathrm{Comm}_A(C) = \{a \in A \mid |aC\bar{a} \cap C| = \infty\}$. Thus, $\mathrm{Comm}_A(C)$ is the smallest almost malnormal subgroup of A containing C.

Choose a finite index normal free subgroup A' of A. Let $\langle w \rangle = A' \cap \mathrm{Comm}_A(C)$, and let $\underline{w}_A = \{d_i w \bar{d_i}\}$ be a lift of $\mathrm{Comm}_A(C)$ to A'. Choose a basis for A' with respect to which $\underline{w}_A$ is Whitehead minimal. After making similar choices for B, let X' be the coarse Bass–Serre complex for G described in section 2.3.

The number of edge strips attaching to a given conjugate of a $\langle d_i w \bar{d_i} \rangle$ in A' is equal to the index of C in $\mathrm{Comm}_A(C)$.

X' is a tree of trees glued together along bi-infinite, width 1 edge strips just as in the torsion free case, and we repeat the previous argument to show that X' is not a quasi-tree if, for each line in A' and B' to which an edge strip attaches, either

- there is a second edge strip attached to that same line, or
- we can follow different edge strips to detour around an arbitrarily large ball centered on that line.

Now suppose C is not a factor of A. It could be that C is not almost malnormal in A, in which case the first condition above is satisfied for A. If C is almost malnormal and not a factor of A, then by Corollary 4.7, every minimal Whitehead graph for a lift of $C = \mathrm{Comm}_A(C)$ to A' has 2-connected components. This gives us the second condition.

Thus, if C is a factor of neither A nor B, then X' is not a quasi-tree, so $A *_C B$ is not virtually free.

The $G = A*_C$ case follows by making similar adjustments to the torsion free HNN case. The interesting case is when the images C_1 and C_2 of C in A form an almost malnormal collection. Proposition 4.3 shows that if C_1 and C_2 are not, up to conjugation, contained in independent factors, then the images of ∂C_1 and some $\partial g C_2 \bar{g}$ are contained in a common component of $\mathcal{D}_{\{C_1, C_2\}}$. Since $\{C_1, C_2\}$ is an almost malnormal collection,

this component is not a singleton, so $\mathcal{D}_{\{C_1,C_2\}}$ is not totally disconnected. It follows that a minimal Whitehead graph for a lift of $\{C_1, C_2\}$ to a finite index normal subgroup of A will have 2-connected components, so G is not virtually free.

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