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CLOPEN ULTRAFILTERS OF ω AND THE CARDINALITY OF THE STONE SPACE $S(\omega)$ IN ZF

KYRIAKOS KEREMEDIS

Abstract. For $X \in \{\omega, \mathbb{R}\}$ let $\mathbf{2}^X$ denote the Tychonoff product of the discrete space $\mathbf{2} = \{0, 1\}$ and, Cl_X denote the set $\{O \cap D_X : O$ is a clopen subset of $\mathbf{2}^X\}$ where, $D_X = \{d_n : n \in \omega\}$ is a dense subset of $\mathbf{2}^X$.

We show:

(i) ω has a free ultrafilter iff every Cl_ω - filter extends to an ultrafilter of D_ω iff every Cl_ω - ultrafilter extends to an ultrafilter of D_ω.

(ii) Every filter of ω extends to an ultrafilter of ω iff every $Cl_\mathbb{R}$ - filter extends to an ultrafilter of $D_\mathbb{R}$ iff every $Cl_\mathbb{R}$ - filter extends to a $Cl_\mathbb{R}$ - ultrafilter.

(iv) $\mathbf{2}^\mathbb{R}$ is the continuous image of $S(\omega)$ iff every $Cl_\mathbb{R}$ - ultrafilter extends to an ultrafilter of $D_\mathbb{R}$.

(v) If the Stone space $S(\omega)$ is countably compact then every family $\mathcal{A} = \{\{A_i, B_i\} \subseteq [\mathbb{R}]^\omega : i \in \omega\}$ has a choice set.

(vi) "Every filter of $\mathbb{R}$ extends to an ultrafilter of $\mathbb{R}$" implies $|S(\omega)| = |2^\mathbb{R}|$, "every filter of ω extends to an ultrafilter of ω" implies "$|2^{S(\omega)}| = |2^{2^\mathbb{R}}|$" and, "$\omega$ has a free ultrafilter" implies $|\mathbb{R}| \leq S(\omega)$.

1. Notation and Terminology

Let $\mathbf{X} = (X, T)$ be a topological space.

$\mathbf{X}$ is said to be *compact* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is said to be *countably compact* iff every countable open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

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$Cl(\mathbf{X})$ will denote the family of all *clopen* (= closed and open simultaneously) subsets of $\mathbf{X}$.

Let $\mathcal{E} \subseteq \mathcal{P}(X)$. A non-empty collection $\mathcal{F} \subseteq \mathcal{E}$ is an $\mathcal{E}$-*filter* iff $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under finite intersections and, for every $F \in \mathcal{F}$ and $O \in \mathcal{E}$ if $F \subseteq O$ then $O \in \mathcal{F}$. An $\mathcal{E}$-filter $\mathcal{F}$ is called *free* if $\bigcap \mathcal{F} = \emptyset$. A maximal with respect to inclusion $\mathcal{E}$-filter is called $\mathcal{E}$-*ultrafilter*. In case $\mathcal{E} = \mathcal{P}(X)$, then an $\mathcal{E}$-filter (resp. $\mathcal{E}$-ultrafilter) $\mathcal{F}$ is a *filter* on X (resp. *ultrafilter* on X).

If $\mathcal{E} = T$, then an $\mathcal{E}$-filter $\mathcal{F}$ (resp. $\mathcal{E}$-ultrafilter $\mathcal{F}$) is called *open filter* of $\mathbf{X}$ (resp. *open ultrafilter* of $\mathbf{X}$). Analogously, we define the notions of *closed filter, clopen filter, closed ultrafilter* and *clopen ultrafilter*.

$B(\mathbf{X})$ will denote the Boolean algebra of all clopen subsets of $\mathbf{X}$ under $\cup$, $\cap$ and set theoretic complementation.

We remark here that in case $\mathcal{E} = Cl(\mathbf{X})$ and $\mathcal{F} \subseteq \mathcal{E}$ then $\mathcal{F}$ is an $\mathcal{E}$-filter (resp. $\mathcal{E}$-ultrafilter) iff $\mathcal{F}$ is a filter of $B(\mathbf{X})$ (resp. $\mathcal{F}$ is an ultrafilter of $B(\mathbf{X})$). In particular, if T is the discrete topology, then $\mathcal{E} = \mathcal{P}(X)$ and the notions of $\mathcal{E}$-ultrafilter, ultrafilter, open ultrafilter and clopen ultrafilter on X all coincide with the notion of the ultrafilter of $B(\mathbf{X})$.

$\mathbf{X}$ is said to be *ultrafilter compact* iff every ultrafilter $\mathcal{F}$ on X *converges* to some point $x \in X$. i.e., for every neighborhood V of x, there exists $F \in \mathcal{F}$ with $V \supseteq F$. Given a filter $\mathcal{F}$ on X and a point $x \in X$, we say that x is a *limit point* of $\mathcal{F}$ iff $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$.

Let X be a set. $\mathbf{2}^X$ denotes the Tychonoff product of the discrete space $\mathbf{2}$ ($2 = \{0, 1\}$ is taken with the discrete topology) whose standard (clopen) base is the set $\mathcal{B}_X = \{[p] : p \in Fn(X, 2, \omega)\}$, where

$$[p] = \{f \in 2^X : p \subseteq f\},$$

and $Fn(X, 2, \omega)$ is the set of all finite partial functions from X to 2.

Let D_ω be the family of all functions from ω into 2 such that $|f^{-1}(1)| < \aleph_0$. It is easy to see that D_ω is a countable dense subset of $\mathbf{2}^\omega$.

It is known, see [2], that $\mathbf{2}^{\mathbb{R}}$ has a countable dense subset $D_{\mathbb{R}}$ and it is not hard to derive a proof of this result within the settings of the Zermelo-Fraenkel set theory **ZF** which we shall adopt in this paper. For the readers convenience we supply a proof of this result in the forthcoming Lemma 1.

Cl_ω (resp. $Cl_{\mathbb{R}}$) will denote the set $\{O \cap D_\omega : O \in Cl(\mathbf{2}^\omega)\}$ (resp. $\{O \cap D_{\mathbb{R}} : O \in Cl(\mathbf{2}^{\mathbb{R}})\}$).

Let D_ω, $D_{\mathbb{R}}$ carry the topologies generated by Cl_ω and $Cl_{\mathbb{R}}$ respectively and ω carry the discrete topology. Clearly, $B(\mathbf{D}_\omega)$ and $B(\mathbf{D}_{\mathbb{R}})$ are isomorphic to certain subalgebras of $B(\omega)$ but $B(\mathbf{D}_\omega)$ is countable and $B(\mathbf{D}_\mathbb{R})$ has cardinality $\mathfrak{c}$.

Let X be a set and $\mathcal{A} = \{A_i : i \in I\}$ be a family of subsets of X.

$\mathcal{A}$ is called *independent* if for every two disjoint finite non-empty subsets S, Q of I, $\bigcap\{A_s : s \in S\} \cap \bigcap\{A_q^c : q \in Q\} \neq \emptyset$. Clearly, the family $\mathcal{A} = \{A_x = [\{(x,1)\}] \cap D_\mathbb{R} : x \in \mathbb{R}\}$ is an independent family of subsets of $D_\mathbb{R}$.

$\mathcal{A}$ is called *almost disjoint* if for every $i, j \in I$, $|A_i \cap A_j| < \aleph_0$. As an example of an almost disjoint family of subsets of $\mathbb{Q}$ of size $|\mathbb{R}|$ consider the following: For every $x \in \mathbb{R}$, via an easy induction, define a one-to-one sequence $(q_{xn})_{n\in\omega}$ of rational numbers converging to x. Clearly, $\mathcal{A} = \{A_x : x \in \mathbb{R}\}$ where, for every $x \in \mathbb{R}$, $A_x = \{q_{xn} : n \in \omega\}$ is an almost disjoint family of $\mathbb{Q}$ of size $|\mathbb{R}|$.

Given a Boolean algebra $\mathcal{B}$, the *Stone space* of $\mathcal{B}$ is the set $U_\mathcal{B}$ of all ultrafilters of $\mathcal{B}$ together with the topology T on $U_\mathcal{B}$ generated by the family $\{[b] : b \in \mathcal{B}\}$ where, for every $b \in \mathcal{B}$, $[b] = \{\mathcal{F} \in U_\mathcal{B} : b \in \mathcal{F}\}$. In particular, given a set $X \neq \emptyset$, $S(X)$ will denote the *Stone space* of the Boolean algebra of all subsets of X. Clearly, if X, Y are any two infinite sets satisfying $|X| = |Y|$ then $S(X)$ is homeomorphic to $S(Y)$. In particular, the spaces $S(\omega)$, $S(D_\omega)$ and $S(D_\mathbb{R})$ are pairwise homeomorphic.

For every $n \in \omega$, $\mathcal{F}_n$ will denote the fixed ultrafilter on ω of all supersets of $\{n\}$. Clearly, $\mathbf{D} = \{\mathcal{F}_n : n \in \omega\}$ is a countable dense subset of $S(\omega)$ (if $A \in \mathcal{P}(\omega)$ then $[A]$ is a non-empty basic open set of $S(\omega)$ and for every $n \in A, \mathcal{F}_n \in [A]$).

Next we list the choice principles we shall deal with in this paper.

(1) $\mathbf{UF}(\omega)$: *There is a free ultrafilter on ω.*
(2) $\mathbf{BPI}(\omega)$: *Every filter on ω extends to an ultrafilter on ω.*
(3) $\mathbf{BPI}(\mathbb{R})$: *Every filter on $\mathbb{R}$ extends to an ultrafilter on $\mathbb{R}$.*
(4) $\mathbf{BPI}(Cl_\omega)$: Every Cl_ω-filter extends to a Cl_ω-ultrafilter.
(5) $\mathbf{BPI}(Cl_\omega, D_\omega)$: *Every Cl_ω-filter extends to an ultrafilter on D_ω.*
(6) $\mathbf{UBPI}(Cl_\omega, D_\omega)$: *Every Cl_ω-ultrafilter extends to an ultrafilter on D_ω.*
(7) $\mathbf{BPI}(Cl_\mathbb{R})$: *Every $Cl_\mathbb{R}$-filter extends to a $Cl_\mathbb{R}$-ultrafilter.*
(8) $\mathbf{BPI}(Cl_\mathbb{R}, D_\mathbb{R})$: *Every $Cl_\mathbb{R}$-filter extends to an ultrafilter on $D_\mathbb{R}$.*
(9) $\mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R})$: *Every $Cl_\mathbb{R}$-ultrafilter extends to an ultrafilter on $D_\mathbb{R}$.*
(10) $\mathbf{CI}(\omega)$: $S(\omega)$ can be mapped continuously onto $\mathbf{2}^{\mathcal{P}(\omega)}$.
(11) $\mathbf{CC}S(\omega)$: $S(\omega)$ is countably compact.

2. The size of $S(\omega)$ in ZF

The famous Hewitt-Marczewski-Pondiczery theorem in general topology is concerned with the density of product spaces and has some significant consequences not only in topology but in set theory also. The fact that every infinite set X has an independent family of size $|\mathcal{P}(X)|$ is such a consequence. As expected, its proof depends on the axiom of choice. However, some special cases which we are going to use in the sequel are choice free as the following well known lemma indicates.

Lemma 1. *(**ZF**) (i) The Tychonoff product $S(\omega)^{2^{\mathbb{R}}}$ has a dense subset G of size $|\mathbb{R}|$.*
(ii) The Tychonoff product $\mathbf{2}^{\mathbb{R}}$ is separable.
(iii) $\mathbb{R}$ has an independent family of size $|2^{\mathbb{R}}|$.
(iv) ω has an independent family of size $|\mathbb{R}|$.

Proof. (i) Let $\mathcal{B}_{\mathbb{R}} = \{[p] : p \in Fn(\mathbb{R}, 2, \omega)\}$ be the canonical base for $\mathbf{2}^{\mathbb{R}}$ and let $D = \{\mathcal{F}_n : n \in \omega\}$. Clearly $|\mathcal{B}_{\mathbb{R}}| = |Fn(\mathbb{R}, 2, \omega)|$ and the latter cardinal number is easily seen to be equal to $|\mathbb{R}|$. Now define a subset $F \subseteq D^{2^{\mathbb{R}}}$ by requiring:

$(**)$ $f \in F$ iff there exists a finite non-empty subset $B_f \subseteq \mathcal{B}_{\mathbb{R}}$ of pairwise disjoint sets such that f takes on the value $\mathcal{F}_0$ on $2^{\mathbb{R}} \setminus \bigcup B_f$ and for every $U \in B_f$, f is constant on U.

For every pairwise disjoint $B \in [\mathcal{B}_{\mathbb{R}}]^{<\omega}$ let

$$G_B = \{f \in D^{2^{\mathbb{R}}} : f \text{ satisfies } (**) \text{ with } B \text{ in place of } B_f\}.$$

Clearly, $|G_B| = \aleph_0$ and we can construct a one-to-one and onto function $h_B : G_B \to \aleph_0$. Since $|[\mathcal{B}_{\mathbb{R}}]^{<\omega}| = |\mathbb{R}|$ it follows that $|F| = |\mathbb{R}|$.

It remains to show that F is dense in $S(\omega)^{2^{\mathbb{R}}}$. Since $D^{2^{\mathbb{R}}}$ is clearly a dense subspace of $S(\omega)^{2^{\mathbb{R}}}$, it suffices to show that F is dense in $D^{2^{\mathbb{R}}}$. Let g be a finite partial function from $2^{\mathbb{R}}$ to D; we need to show $[g] \cap F$ is non-empty. Let $\{x_0, \ldots, x_n\}$ be the domain of g, and let $\{p_i \in Fn(\mathbb{R}, 2, \omega) : i \leq n\}$ be a pairwise incompatible family such that $x_i \in [p_i]$ for all $i \leq n$. Let $f \in F$ be the element satisfying: $f([p_i]) = g(x_i)$ for all $i \leq n$. Clearly, $f \in D \cap [g]$ and F is dense as required.

(ii) This can be proved along the established lines of reasoning of part (i).

(iii) Working as in the proof of part (i) we can show that the product $2^{2^{\mathbb{R}}}$ has a dense set $D = \{d_i : i \in \mathbb{R}\}$. It is easy to see that $\mathcal{A} = \{A_i : i \in 2^{\mathbb{R}}\}$ where for every $i \in 2^{\mathbb{R}}$, $A_i = \pi_i^{-1}(1) \cap D$ is an independent family of subsets of D.

(iv) See the example following the definition of an independent family. $\square$

Remark 1. The referee has pointed out to us that Lemma 1 has a known more general form. The same arguments for assertions (i) and (ii) can be found in [1], Theorem 2.3.15, where it is proved without the use **AC** that for every separable space $\mathbf{X}$, the products $\mathbf{X}^{\mathbb{R}}$ and $\mathbf{X}^{2^{\mathbb{R}}}$ have dense sets of size ω and $\mathfrak{c}$ respectively. Likewise, regarding the assertions (iii) and (iv), another and more general form is the Hausdorff's construction of an independent family [Chapter VIII, Exercise A6, in [10]].

Clearly, for every $f \in 2^{\mathbb{R}}$ the $Cl_{\mathbb{R}}$-filter $\mathcal{F}_f$ generated by the family

$$\mathcal{H}_f = \{[\{(x, f(x))\}] \cap D_{\mathbb{R}} : x \in \mathbb{R}\} \tag{1}$$

is a $Cl_{\mathbb{R}}$-ultrafilter. Conversely, for every $Cl_{\mathbb{R}}$-ultrafilter $\mathcal{F}$ (resp. every ultrafilter $\mathcal{F}$ on ω) there exists a unique $f \in 2^{\mathbb{R}}$ such that $\mathcal{F}_f = \mathcal{F}$ (resp. there exists a unique $f \in 2^{\mathbb{R}}$ such that $\mathcal{F}_f \subseteq \mathcal{F}$). Indeed, for every $x \in \mathbb{R}$, exactly one of the sets $[\{(x,1)\}] \cap D_{\mathbb{R}}$, $[\{(x,0)\}] \cap D_{\mathbb{R}}$ is a member of $\mathcal{F}$. Hence the mapping $f : \mathbb{R} \to 2$ given by the rule: For every $x \in \mathbb{R}$,

$$f(x) = \begin{cases} 1 \text{ if } [\{(x,1)\}] \cap D_{\mathbb{R}} \in \mathcal{F} \\ 0 \text{ if } [\{(x,0)\}] \cap D_{\mathbb{R}} \in \mathcal{F} \end{cases}$$

is a function such that $\mathcal{F} = \mathcal{F}_f$. Thus, the function

$$H : 2^{\mathbb{R}} \to U_{B(\mathbf{D}_{\mathbb{R}})},\ H(f) = \mathcal{F}_f \tag{2}$$

is one-to-one and onto meaning that $|U_{B(\mathbf{D}_{\mathbb{R}})}| = |2^{\mathbb{R}}|$. Next we show that H is more than one-to-one and onto function.

Proposition 2. *(i)* $\mathbf{2}^{\mathbb{R}}$ *is homeomorphic with the Stone space of* $B(\mathbf{D}_{\mathbb{R}})$.
(ii) $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ *iff for every* $f \in 2^{\mathbb{R}}$, $\mathcal{F}_f$ *extends to an ultrafilter on* $D_{\mathbb{R}}$ *where,* $\mathcal{F}_f$ *is the filter generated by* $\mathcal{H}_f$ *given by (1).*

Proof. (i) It suffices, in view of the discussion before the statement of the proposition, to show that the function H given by (2) maps basic open sets of $\mathbf{2}^{\mathbb{R}}$ to basic open sets of $U_{B(\mathbf{D}_{\mathbb{R}})}$. To this end, fix $p \in Fn(\mathbb{R}, 2, \omega)$. We have: $H([p]) = \{H(f) : p \subseteq f\} = \{\mathcal{F} \in U_{B(\mathbf{D}_{\mathbb{R}})} : \forall x \in Dom(p), [\{(x, f(x))\}] \cap D_{\mathbb{R}} \in \mathcal{F}\} = \{\mathcal{F} \in U_{B(\mathbf{D}_{\mathbb{R}})} : \bigcap\{[\{(x, f(x))\}] \cap D_{\mathbb{R}} : x \in Dom(p)\} \in \mathcal{F}\} = [\bigcap\{[\{(x, f(x))\}] \cap D_{\mathbb{R}} : x \in Dom(p)\}]$. Hence, H is as required.

(ii) This follows at once from the discussion preceding the statement of the proposition. □

Regarding $\mathfrak{u} = |S(\omega)|$, unlike $U_{B(\mathbf{D}_{\mathbb{R}})}$, the situation might be very complicated. The reason is that given $f \in 2^{\mathbb{R}}$ there might be more that one members of $S(\omega)$ including $\mathcal{F}_f$ or, there might be no member of $S(\omega)$

including $\mathcal{F}_f$. Let us scrutinize a little bit more on the last comment. Working in **ZF** one can easily verify that:

$$\aleph_0 \leq \mathfrak{u} \leq |2^{\mathbb{R}}|. \tag{3}$$

Indeed, since $S(\omega) \subseteq \mathcal{P}(\mathcal{P}(\omega))$ it follows that $\mathfrak{u} \leq |\mathcal{P}(\mathcal{P}(\omega))| = |2^{\mathbb{R}}|$. On the other hand, for every $n \in \omega$, $\mathcal{F}_n \in S(\omega)$ meaning that $\aleph_0 \leq \mathfrak{u}$.

It is well known that in **ZFC** (= **ZF** plus the axiom of choice **AC**) $\mathfrak{u} = |2^{\mathbb{R}}|$. However, in **ZF**, $\mathfrak{u}$ may not be equal to $|2^{\mathbb{R}}|$. Indeed, if $\mathbf{UF}(\omega)$ fails then $\mathfrak{u} = \aleph_0$. Hence, "$\mathfrak{u} = |2^{\mathbb{R}}|$" and "$\mathfrak{u} \neq |2^{\mathbb{R}}|$" are both consistent with **ZF**. So, it is plausible to ask about the set theoretical strength of "$\mathfrak{u} = |2^{\mathbb{R}}|$". Clearly, "$\mathfrak{u} = |2^{\mathbb{R}}|$" implies $\mathbf{UF}(\omega)$ and is a consequence of the proposition "$\mathbb{R}$ is well-orderable". To see the latter implication let, for every $i \in 2^{\mathbb{R}}$, $\mathcal{H}_i$ be given by (1). Clearly,

(4) for every $i, j \in 2^{\mathbb{R}}, i \neq j$ there exist $A \in \mathcal{H}_i, B \in \mathcal{H}_j$ with $A \cap B = \emptyset$.

Since, "$\mathbb{R}$ is well-orderable" implies for every $i \in 2^{\mathbb{R}}$, $\mathcal{H}_i$ extends, via a straightforward transfinite induction, to an ultrafilter $\mathcal{F}_i$ on $D_{\mathbb{R}}$, it follows by (4), that $\{\mathcal{F}_i : i \in 2^{\mathbb{R}}\}$ is a list of distinct ultrafilters on $D_{\mathbb{R}}$. Since, $|\omega| = |D_{\mathbb{R}}|$ implies $|S(\omega)| = |S(D_{\mathbb{R}})|$, we see that "$\mathfrak{u} = |2^{\mathbb{R}}|$" holds true as required.

More generally, given any independent family, $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ of ω and $S \in \mathcal{P}(\mathbb{R})$, the family $\mathcal{H}_S = \{A_s : s \in S\} \cup \{A_s^c : s \in S^c\}$ has the finite intersection property (fip for abbreviation). As before, "$\mathbb{R}$ is well-orderable" implies $\mathcal{H}_S$ extends constructively to an ultrafilter $\mathcal{F}_S$ on ω. Clearly, for every $S, Q \in \mathcal{P}(\mathbb{R})$ with $S \neq Q$, $\mathcal{H}_S \neq \mathcal{H}_Q$. Hence, $\{\mathcal{F}_S : S \in \mathcal{P}(\mathbb{R})\}$ is a family of distinct ultrafilters of ω and "$\mathfrak{u} = |2^{\mathbb{R}}|$" holds true.

Let

$$\mathcal{C} = \{C_i : i \in 2^{\mathbb{R}}\} \tag{5}$$

where, for every $i \in 2^{\mathbb{R}}$,

$$C_i = \{\mathcal{F} \in S(D_{\mathbb{R}}) : \mathcal{H}_i \subseteq \mathcal{F}\} \tag{6}$$

and $\mathcal{H}_i$ is given by (1). Clearly, in view of (1) and (4), $\mathcal{C}$ is partition of $S(D_{\mathbb{R}})$ such that:

- For every $i \in 2^{\mathbb{R}}$ and $\mathcal{F} \in \mathbf{D}_{\mathbb{R}}, \mathcal{F} \in C_i$ iff $\lim \mathcal{F} = i$.

We observe that any weak choice principle implying $|2^{\mathbb{R}}| \leq |\mathcal{C}| \leq |S(\omega)|$, also implies "$\mathfrak{u} = |2^{\mathbb{R}}|$". Such a weak choice form for example is easily seen to be the conjunction $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}}) \wedge$ "$\mathcal{C}$ has a choice set" or, by the argument following the proof of Lemma 1, the statement "$\mathbb{R}$ is well-orderable". We would like draw attention here that if X is a set and $\mathcal{C}$ is a partition of X then $|\mathcal{C}|$ need not be less than or equal to $|X|$. Indeed, in [11] Monro constructs a **ZF** model $N[F]$ extending the Halpern-Levy model $\mathcal{N}$ containing an amorphous partition $\mathcal{C}$ of $\mathbb{R}$. i.e., $\mathcal{C}$ cannot be partitioned into two infinite sets. Since infinite linearly ordered sets are not amorphous, see e.g., [7], it follows that in $N[F]$, $|\mathcal{C}| \nleq |\mathbb{R}|$.

As pointed out implicitly in the previous paragraph, $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ hence $\mathbf{BPI}(\omega)$ also, implies for every $i \in 2^{\mathbb{R}}$, $C_i \neq \emptyset$. However, it is unknown to us whether $\mathbf{BPI}(\omega)$ implies any one of the statements "$\mathcal{C}$ has a choice set" and "$|\mathcal{C}| \leq |S(\omega)|$" in order to infer "$\mathfrak{u} = |2^{\mathbb{R}}|$".

Next we show that the higher cardinal analogue $\mathbf{BPI}(\mathbb{R})$ of $\mathbf{BPI}(\omega)$ which, by the way, is independent from the statement "$\mathbb{R}$ is well-orderable" (in the basic Cohen model $\mathcal{M}1$ in [7] $\mathbf{BPI}(\mathbb{R})$ holds true but $\mathbb{R}$ is not well-orderable and, in [8] there is a **ZF** model $\mathcal{N}$ where "$\mathbb{R}$ is well-orderable" holds true but $\mathbf{BPI}(\mathbb{R})$ fails), implies both statements "$|2^{\mathbb{R}}| \leq |\mathcal{C}| \leq |S(\omega)|$" and "$\mathcal{C}$ has a choice set".

Theorem 3. *(i)* $\mathbf{BPI}(\mathbb{R}) \to$ *"*$\mathfrak{u} = |2^{\mathbb{R}}|$*". The converse fails in* **ZF**.
(ii) $\mathbf{BPI}(\omega) \to$ "$|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}|$" $\to \mathbf{UF}(\omega)$.
(iii) $\mathbf{UF}(\omega) \to$ "$\mathfrak{u} \geq |\mathbb{R}|$" *and* $\mathbf{BPI}(\omega) \to$ "$\mathfrak{u} > |\mathbb{R}|$".
(iv) The conjunction $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ *and "every partition of* $S(\omega)$ *has size* $\leq |S(\omega)|$*" implies "*$\mathfrak{u} = |2^{\mathbb{R}}|$*".*

Proof. (i) It suffices, in view of (3) to show that $\mathfrak{u} \geq |2^{\mathbb{R}}|$. Let $\mathcal{C}$ be given by (5). Since $\mathbf{BPI}(\mathbb{R})$ clearly implies $\mathbf{BPI}(\omega)$ we see that $\mathcal{C}$ is a disjoint family of non-empty sets. We claim that for every $i \in 2^{\mathbb{R}}$, C_i is closed. To see this fix $\mathcal{W} \in S(D_{\mathbb{R}}) \backslash C_i$. Clearly, there exist $H \in \mathcal{H}_i$, $W \in \mathcal{W}$ with $H \cap W = \emptyset$. Hence, $[W]$ is a neighborhood of $\mathcal{W}$ such that $[W] \cap C_i = \emptyset$ meaning that C_i is closed.

It is easy to see that $\mathcal{G} = \{G_i : i \in 2^{\mathbb{R}}\}$ where, for every $i \in 2^{\mathbb{R}}$, $G_i = \pi_i^{-1}(C_i)$ is a family of closed sets of $S(D_{\mathbb{R}})^{2^{\mathbb{R}}}$ with the fip. Hence, by the following claim

Claim 1. $\mathbf{BPI}(\mathbb{R})$ implies the Tychonoff product $S(\omega)^{2^{\mathbb{R}}}$ is compact.

Proof of Claim 1. This follows from Lemma 1 and the following results:

(A) [4] $S(D_\omega)$ is ultrafilter compact.

(B) [3] Products of T_2 ultrafilter compact spaces are ultrafilter compact.

(C) [5] $\mathbf{BPI}(\mathbb{R})$ implies every ultrafilter compact T_3 space $\mathbf{Y}$ with a dense subset D of size $|\mathbb{R}|$ is compact.

we have $\bigcap \mathcal{G} \neq \emptyset$. Clearly, any member g of the latter intersection is a choice set of $\mathcal{C}$ and the function $H : 2^{\mathbb{R}} \to S(D_{\mathbb{R}}), H(i) = g(i)$ is one-to-one. Thus, $|2^{\mathbb{R}}| \leq |S(D_{\mathbb{R}})| = |S(\omega)|$ as required.

The second assertion is known. The **ZF** model $\mathcal{N}$ in [8] satisfies the negation of $\mathbf{BPI}(\mathbb{R})$ and "$\mathbb{R}$ is well-orderable". Hence, it satisfies "$\mathfrak{u} = |2^{\mathbb{R}}|$" also.

(ii) $\mathbf{BPI}(\omega) \rightarrow$ "$|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}|$". By (3) $\mathfrak{u} \leq |2^{\mathbb{R}}|$. Hence, $|2^{\mathfrak{u}}| \leq |2^{2^{\mathbb{R}}}|$. To see the other direction of the latter inequality we note that the function $H : \mathcal{P}(2^{\mathbb{R}}) \rightarrow \mathcal{P}(S(D_{\mathbb{R}}))$ given by

$$H(X) = \bigcup\{C_i : i \in X\}$$

where, for every $i \in 2^{\mathbb{R}}$, C_i is given by (6) is one-to-one. Indeed, if $X, Y \in \mathcal{P}(2^{\mathbb{R}})$, $X \neq Y$ then $X \backslash Y \neq \emptyset$ or $Y \backslash X \neq \emptyset$. Assume that $X \backslash Y \neq \emptyset$ and fix $i_0 \in X \backslash Y$. Since $\mathbf{BPI}(\omega)$ implies $\mathcal{C} = \{C_i : i \in 2^{\mathbb{R}}\}$ is a partition of $S(D_{\mathbb{R}})$ into non-empty sets it follows that $\emptyset \neq C_{i_0} \subseteq H(X)$ and $C_{i_0} \cap H(Y) = \emptyset$. Hence $H(X) \neq H(Y)$ and H is one-to-one as required. Thus, $|2^{2^{\mathbb{R}}}| = |\mathcal{P}(2^{\mathbb{R}})| \leq |\mathcal{P}(S(D_{\mathbb{R}}))| = |\mathcal{P}(\mathfrak{u})| = |2^{\mathfrak{u}}|$.

"$|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}|$" $\rightarrow \mathbf{UF}(\omega)$. $|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}|$ clearly implies $\mathfrak{u} > \aleph_0$. Hence, ω has uncountably many ultrafilters and $\mathbf{UF}(\omega)$ holds true.

(iii) Fix $\mathcal{A} = \{A_i : i \in \mathbb{R}\}$ an almost disjoint family of infinite subsets of ω and let, by $\mathbf{UF}(\omega)$, $\mathcal{F}$ be a free ultrafilter on ω. For every $i \in \mathbb{R}$, let $f_i : \omega \rightarrow A_i$ be the function given by:

$$f(n) = \begin{cases} \min(A_i) \text{ if } n = 0 \\ \min(A_i \backslash \{f(j) : j \in n\}) \text{ if } n > 0 \end{cases}.$$

Clearly, f_i is a one-to-one and onto function. Hence, for every $i \in \mathbb{R}, \mathcal{H}_i = \{f_i(F) : F \in \mathcal{F}\}$ is a free ultrafilter on A_i and the filter $\mathcal{F}_i$ on ω generated by $\mathcal{H}_i$ is a free ultrafilter on ω such that $A_i \in \mathcal{F}_i$.

Let $f : \mathbb{R} \rightarrow S(\omega)$ be the function given by the rule: $f(i) = \mathcal{F}_i, i \in \mathbb{R}$. We claim that f is one-to-one. Assume on the contrary and fix $i_1, i_2 \in \mathbb{R}$, $i_1 \neq i_2$ such that $f(i_1) = f(i_2)$. Since, $A_{i_1} \in \mathcal{F}_{i_1}$, $A_{i_2} \in \mathcal{F}_{i_2}, \mathcal{F}_{i_1} = \mathcal{F}_{i_2}, A_{i_1} \cap A_{i_2} \in \mathcal{F}_{i_1}$ is finite and $\mathcal{F}_{i_1}$ is ultrafilter on ω, it follows that $\mathcal{F}_{i_1}$ is fixed. Contradiction! Thus f is one-to-one and $|\mathbb{R}| \leq |S(\omega)| = \mathfrak{u}$ as required.

The second assertion is straightforward. From the equality $|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}|$ it follows that $2^{\mathfrak{u}} > 2^{\mathbb{R}}$. Hence, $\mathfrak{u} \not\leq |\mathbb{R}|$.

(iv) See the discussion before the statement of the theorem. □

As a corollary to the proof of Theorem 3 (i) we get that $\mathbf{BPI}(\mathbb{R})$ implies that $S(\omega)$ is a *Loeb space.* i.e., the family of all non-empty closed subsets of $S(\omega)$ has a choice function.

Corollary 4. *(i)* $\mathbf{BPI}(\mathbb{R})$ *implies that the family* $\mathcal{G}$ *of all non-empty closed subsets of* $S(\omega)$ *has a choice function.*
(ii) $\mathbf{BPI}(\omega)$ *and "*$S(\omega)$ *is a Loeb space" together imply "*$\mathfrak{u} = |2^{\mathbb{R}}|$*".*

Proof. (i) Let $\mathcal{G}$ be as in the statement of the corollary. It suffices, in view of the proof of Theorem 3, to show that $|\mathcal{G}| \leq |2^{\mathbb{R}}|$. The latter inequality follows at once from the observation that $\mathcal{B} = \{[A] : A \in \mathcal{P}(\omega)\}$ is a base for the closed sets of $S(\omega)$.

(ii) This follows from the discussion before Theorem 3. $\square$

3. Clopen ultrafilters of ω

Proposition 5. *(i) $Cl(\mathbf{2}^\omega)$ is countable. In particular, Cl_ω is countable.*
(ii) $|Cl(\mathbf{2}^{\mathbb{R}})| = |\mathbb{R}|$. In particular, $|Cl_{\mathbb{R}}| = |\mathbb{R}|$.

Proof. (i) Let $\mathcal{B} = \{B_n : n \in \omega\}$ be an enumeration of the standard clopen base of $\mathbf{2}^\omega$. Clearly, $|[\mathcal{B}]^{<\omega}| = \aleph_0$. Since $\mathbf{2}^\omega$ is compact, it follows that every clopen set of $\mathbf{2}^\omega$, being compact, can be expressed as a finite union of members of $\mathcal{B}$. Hence, $Cl(\mathbf{2}^\omega)$ is countable. Indeed, if $\{Q_n : n \in \omega\}$ is an enumeration of $[\mathcal{B}]^{<\omega}$, then the function $H : Cl(\mathbf{2}^\omega)\backslash\{\emptyset\} \to \omega$ given by: $H(O) = Q_{n_O}$ where, $n_O = \min\{n \in \omega : O = \bigcup Q_n\}$ is easily seen to be one-to-one.

The second assertion follows at once from the first part of (i) and the fact that for every $U, V \in Cl(\mathbf{2}^\omega)$ and every dense subset $D = \{D_n : n \in \omega\}$ of $\mathbf{2}^\omega$ if $U \neq V$ then $U \cap D \neq V \cap D$.

(ii) Let $\mathcal{B} = \{[p] : p \in Fn(\mathbb{R}, 2, \omega)\}$ denote the standard clopen base of $\mathbf{2}^{\mathbb{R}}$. Since, $|\mathbb{R}| \leq |\mathcal{B}| \leq |Cl(\mathbf{2}^{\mathbb{R}})|$ is clear, it suffices to show that $|Cl(\mathbf{2}^{\mathbb{R}})| \leq |\mathbb{R}|$. Let $H : Cl(\mathbf{2}^{\mathbb{R}}) \to \mathcal{P}(D_{\mathbb{R}})$ be the function given by: $H(O) = O \cap D_{\mathbb{R}}$. Clearly, if U, $V \in Cl(\mathbf{2}^{\mathbb{R}})$, $U \neq V$ then $U \cap D_{\mathbb{R}} \neq V \cap D_{\mathbb{R}}$. Hence, H is one-to-one and $|Cl(\mathbf{2}^{\mathbb{R}})| \leq |\mathcal{P}(D_{\mathbb{R}})| \leq |\mathbb{R}|$. Thus, $|Cl(\mathbf{2}^{\mathbb{R}})| = |\mathbb{R}|$ and $|Cl_{\mathbb{R}}| = |\mathbb{R}|$ as required. $\square$

For every filter $\mathcal{H}$ of $B(\mathbf{2}^{\mathbb{R}})$ (resp. every filter $\mathcal{F}$ of $B(\mathbf{D}_{\mathbb{R}})$) let $\mathcal{F}_{\mathcal{H}} = H(\mathcal{H}) = \{O \cap D_{\mathbb{R}} : O \in \mathcal{H}\}$ (resp. $\mathcal{H}_{\mathcal{F}} = H^{-1}(F) = \{O \in Cl(\mathbf{2}^{\mathbb{R}}) : O \cap D_{\mathbb{R}} \in \mathcal{F}\}$), where $H : B(\mathbf{2}^{\mathbb{R}}) \to B(\mathbf{D}_{\mathbb{R}})$ is the mapping given by the rule: $H(O) = O \cap D_{\mathbb{R}}$. Likewise, for every $f \in 2^{\mathbb{R}}$, let $\mathcal{H}_f = \{O \in Cl(\mathbf{2}^{\mathbb{R}}) : f \in O\}$ and $\mathcal{F}_f = \mathcal{F}_{\mathcal{H}_f}$.

Proposition 6. *Let $H, \mathcal{F}_{\mathcal{H}}, \mathcal{H}_{\mathcal{F}}, \mathcal{H}_f$ and $\mathcal{F}_f$ be as in the last paragraph preceding this proposition. Then:*
(i) H is an isomorphism.
(ii) For every filter $\mathcal{H}$ of $B(\mathbf{2}^{\mathbb{R}})$ (resp. ultrafilter $\mathcal{H}$ of $B(\mathbf{2}^{\mathbb{R}})$), $\mathcal{F}_{\mathcal{H}}$ is a filter of $B(\mathbf{D}_{\mathbb{R}})$ (resp. ultrafilter of $B(\mathbf{D}_{\mathbb{R}})$). In particular, for every $f \in 2^{\mathbb{R}}$, $\mathcal{F}_f = \mathcal{F}_{\mathcal{H}_f}$ is a free ultrafilter of $B(\mathbf{D}_{\mathbb{R}})$.
(iii) For every filter $\mathcal{F}$ of $B(\mathbf{D}_{\mathbb{R}})$ (resp. ultrafilter $\mathcal{F}$ of $B(\mathbf{D}_{\mathbb{R}})$), $\mathcal{H}_{\mathcal{F}}$ is a filter of $B(\mathbf{2}^{\mathbb{R}})$ (resp. ultrafilter of $B(\mathbf{2}^{\mathbb{R}})$).
(iv) Every $Cl_{\mathbb{R}}$-ultrafilter has a limit point in $\mathbf{2}^{\mathbb{R}}$.

Proof. (i) Clearly, $H(\emptyset) = \emptyset$, $H(\mathbf{2}^{\mathbb{R}}) = D_{\mathbb{R}}$ and H is onto.

H is one-to-one. Indeed, for every, $O, Q \in B(\mathbf{2}^{\mathbb{R}})$ with $O \neq Q$, $O \backslash Q \neq \emptyset$ or, $Q \backslash O \neq \emptyset$. Assume that $O \backslash Q \neq \emptyset$. Then $\emptyset \neq (O \backslash Q) \cap D_{\mathbb{R}} \subseteq (O \cap D_{\mathbb{R}}) \backslash (Q \cap D_{\mathbb{R}})$. Hence, $H(O) \neq H(Q)$.

To see the rest of the requirements for an isomorphism we note that:
$H(O \cup Q) = (O \cup Q) \cap D_{\mathbb{R}} = O \cap D_{\mathbb{R}} \cup Q \cap D_{\mathbb{R}} = H(O) \cup H(Q)$,
$H(O \cap Q) = O \cap Q \cap D_{\mathbb{R}} = O \cap D_{\mathbb{R}} \cap Q \cap D_{\mathbb{R}} = H(O) \cap H(Q)$,
$H(O^c) = D_{\mathbb{R}} \backslash O = D_{\mathbb{R}} \backslash (O \cap D_{\mathbb{R}}) = D_{\mathbb{R}} \backslash H(O) = H(O)^c$.

(ii) Since, by part (i), H is an isomorphism, it follows easily that H maps filters (resp. ultrafilters) of $B(\mathbf{2}^{\mathbb{R}})$ onto filters (resp. ultrafilters of $B(\mathbf{D}_{\mathbb{R}})$).

The second assertion of (ii) follows from the first part and fact that for every $f \in 2^{\mathbb{R}}$, $\mathcal{H}_f = \{O \in Cl(\mathbf{2}^{\mathbb{R}}) : f \in O\}$ is a clopen ultrafilter of $\mathbf{2}^{\mathbb{R}}$ such that $\bigcap \mathcal{F}_f = \bigcap \{O \cap D_{\mathbb{R}} : O \in \mathcal{H}_f\} = \emptyset$.

(iii) This is follows at once from the observation that $H^{-1} : B(\mathbf{D}_{\mathbb{R}}) \to B(\mathbf{2}^{\mathbb{R}}), H^{-1}(O \cap D_{\mathbb{R}}) = O$ is also an isomorphism.

(iv) Fix a $Cl_{\mathbb{R}}$-ultrafilter $\mathcal{W}$. By part (iii)

$$\mathcal{R} = \{O \in Cl(\mathbf{2}^{\mathbb{R}}) : O \cap D_{\mathbb{R}} \in \mathcal{W}\}. \tag{7}$$

is an ultrafilter of $B(\mathbf{2}^{\mathbb{R}})$. Since for every $x \in \mathbb{R}$, $[\{(x,1)\}]$, $[\{(x,0)\}] \in Cl(\mathbf{2}^{\mathbb{R}})$, $[\{(x,1)\}] \cap [\{(x,0)\}] = \emptyset$, $[\{(x,1)\}] \cup [\{(x,0)\}] = 2^{\mathbb{R}}$, it follows that $[\{(x,1)\}] \in \mathcal{R}$ or $[\{(x,0)\}] \in \mathcal{R}$ but not both. Hence, $\mathcal{R}$ converges to the element $f \in 2^{\mathbb{R}}$ given by the rule:

$$f(x) = \begin{cases} 1 \text{ if } [\{(x,1)\}] \in \mathcal{R} \\ 0 \text{ if } [\{(x,0)\}] \in \mathcal{R} \end{cases}, \ x \in \mathbb{R}. \tag{8}$$

Clearly, $\{f\} = \bigcap \{\overline{W} : W \in \mathcal{W}\}$ and consequently f is the unique limit point of $\mathcal{W}$ as required. □

The following implications are obvious:

- $\mathbf{BPI}(\omega) \to \mathbf{BPI}(Cl_\omega, D_\omega) \to \mathbf{UBPI}(Cl_\omega, D_\omega) \to \mathbf{BPI}(Cl_\omega)$;
- $\mathbf{BPI}(\omega) \to \mathbf{BPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}}) \to \mathbf{BPI}(Cl_{\mathbb{R}}) \to \mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$.

Regarding the remaining implications or, non-implications, in Proposition 7 we show:

- $\mathbf{BPI}(Cl_\omega, D_\omega)$ and $\mathbf{UBPI}(Cl_\omega, D_\omega)$ are equivalent to $\mathbf{UF}(\omega)$,
- $\mathbf{BPI}(Cl_\omega)$ is provable in $\mathbf{ZF}$ and,
- $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}}) \to \mathbf{UBPI}(Cl_\omega, D_\omega)$.

In Theorem 8 we show that:

- Both $\mathbf{BPI}(Cl_{\mathbb{R}})$ and $\mathbf{BPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ are equivalent to $\mathbf{BPI}(\omega)$.

Since $\mathbf{UF}(\omega) \nrightarrow \mathbf{BPI}(\omega)$, (in the model $\mathcal{N}[\Gamma]$ of Theorem 5.2 in [6], $\mathbf{UF}(\omega)$ holds true but $\mathbf{BPI}(\omega)$ fails) and, $\mathbf{UF}(\omega)$ is unprovable in $\mathbf{ZF}$, it follows that:

- $\mathbf{BPI}(Cl_\omega, D_\omega) \nrightarrow \mathbf{BPI}(Cl_\mathbb{R}, D_\mathbb{R})$ and $\mathbf{BPI}(Cl_\omega) \nrightarrow \mathbf{UBPI}(Cl_\omega, D_\omega)$.

In Theorem 9 we show that:

- $\mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R})$ is equivalent to $\mathbf{CI}(\omega)$.

Proposition 7. *The following holds:*
(i) $\mathbf{BPI}(Cl_\omega)$.
(ii) $\mathbf{BPI}(\omega) \to \mathbf{BPI}(Cl_\mathbb{R}, D_\mathbb{R}) \to \mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R}) \to \mathbf{UF}(\omega)$.
(iii) $\mathbf{UF}(\omega) \leftrightarrow \mathbf{BPI}(Cl_\omega, D_\omega) \leftrightarrow \mathbf{UBPI}(Cl_\omega, D_\omega)$.

Proof. (i) By Proposition 5, Cl_ω is countable. Therefore, given any Cl_ω-filter, we can extend it via a straightforward induction to a Cl_ω-ultrafilter.

(ii) The implications $\mathbf{BPI}(\omega) \to \mathbf{BPI}(Cl_\mathbb{R}, D_\mathbb{R}) \to \mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R})$ are straightforward.

$\mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R}) \to \mathbf{UF}(\omega)$ It suffices to show that there exists a free ultrafilter on $D_\mathbb{R}$. By Proposition 6, for every $f \in 2^\mathbb{R}$, $\mathcal{F}_f = \{O \cap D_\mathbb{R} : f \in O, O$ is a clopen subset of $\mathbf{2}^\mathbb{R}\}$ is a free $Cl_\mathbb{R}$-ultrafilter on $D_\mathbb{R}$. Hence, by $\mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R})$, $\mathcal{F}_f$ extends to a free ultrafilter $\mathcal{F}$ on $D_\mathbb{R}$.

(iii) $\mathbf{UBPI}(Cl_\omega, D_\omega) \to \mathbf{UF}(\omega)$ This can be proved as in $\mathbf{UBPI}(Cl_\mathbb{R}, D_\mathbb{R}) \to \mathbf{UF}(\omega)$.

$\mathbf{UF}(\omega) \to \mathbf{BPI}(Cl_\omega, D_\omega)$ Fix $\mathcal{H}$ a Cl_ω-filter. Since, by Proposition 5, $Cl(\mathbf{2}^\omega)$ is countable, it follows that $\mathcal{H}$ is countable. It is straightforward to verify that there exists an infinite subset S of ω such that $|S \backslash H| < \aleph_0$ for every $H \in \mathcal{H}$. By our hypothesis S has a free ultrafilter $\mathcal{W}$. Hence, the filter $\mathcal{F}$ on ω generated by $\mathcal{W}$ is the required ultrafilter on ω extending $\mathcal{H}$.

$\mathbf{BPI}(Cl_\omega, D_\omega) \to \mathbf{UBPI}(Cl_\omega, D_\omega)$. This is obvious. $\square$

Theorem 8. *The following are equivalent: (i)* $\mathbf{BPI}(Cl_\mathbb{R}, D_\mathbb{R})$.
(ii) $\mathbf{BPI}(Cl_\mathbb{R})$ *(: Every $Cl_\mathbb{R}$-filter extends to a $Cl_\mathbb{R}$-ultrafilter).*
(iii) $\mathbf{BPI}(\omega)$.
(iv) The Tychonoff product $\mathbf{2}^\mathbb{R}$ is compact.
(v) Every closed filter of $\mathbf{2}^\mathbb{R}$ is included in a closed ultrafilter.
(vi) Every filter of $B(\mathbf{2}^\mathbb{R})$ is included in an ultrafilter of $B(\mathbf{2}^\mathbb{R})$.
(vii) Every open filter of $\mathbf{2}^\mathbb{R}$ is included in an open ultrafilter.
(viii) $S(\omega)$ is compact.
(ix) For every countable dense set D of $\mathbf{2}^\mathbb{R}$ every D-filter has a limit point.
(x) Every $Cl_\mathbb{R}$-filter has a limit point.
Consequently, $\mathbf{UF}(\omega)$ *does not imply* $\mathbf{BPI}(Cl_\mathbb{R}, D_\mathbb{R})$.

Proof. (iv) $\leftrightarrow$ (v) $\leftrightarrow$ (vi) $\leftrightarrow$ (vii). These have been established in [9].

(iii) $\leftrightarrow$ (iv) has been established in [8].

(iii) $\leftrightarrow$ (viii) is well known. See, e.g., [5].

(iii) $\rightarrow$ (i), (iv) $\rightarrow$ (ix), (ix) $\rightarrow$ (x) These are straightforward.

(i) $\rightarrow$ (ii) Fix $\mathcal{H}$ an $Cl_{\mathbb{R}}$-filter and let, by our hypothesis, $\mathcal{F}$ be an ultrafilter on $D_{\mathbb{R}}$ extending $\mathcal{H}$. It is easy to see that $\mathcal{W} = \mathcal{F} \cap Cl_{\mathbb{R}}$ is a $Cl_{\mathbb{R}}$-ultrafilter extending $\mathcal{H}$.

(ii) $\rightarrow$ (iii) It suffices in view of (iii) $\rightarrow$ (i) and (iii) $\leftrightarrow$ (iv) to show that (ii) $\rightarrow$ (iv). Let $\mathcal{U} = \{[p_i] : i \in I, p_i \in Fn(\mathbb{R}, 2, \omega)\}$ be a basic open cover of $\mathbf{2}^{\mathbb{R}}$. Assume, aiming for a contradiction, that $\mathcal{U}$ has no finite subcover. Clearly, the family $\mathcal{G} = \{[p_i]^c \cap D_{\mathbb{R}} : i \in I\} \subseteq Cl_{\mathbb{R}}$ has the fip. Let, by our hypothesis, $\mathcal{F}$ be a $Cl_{\mathbb{R}}$-ultrafilter extending the $Cl_{\mathbb{R}}$-filter generated by $\mathcal{G}$. Since

$$[p_i]^c = \bigcup\{[\{(x, 1 - p_i(x))\}] : x \in Dom(p_i)\},$$

it follows easily that for every $i \in I$ there exists $x \in Dom(p_i)$ such that $[\{(x, 1 - p_i(x))\}] \cap D_{\mathbb{R}} \in \mathcal{F}$. Let $h \in 2^{\mathbb{R}}$ be the element given by the rule: $h(x) = j \in 2$ provided $[\{(x, j)\}] \cap D_{\mathbb{R}} \in \mathcal{F}$. It is straightforward to see that $h \in \bigcap\{[p_i]^c : i \in I\}$. Hence, $\mathcal{U}$ is not a cover of $\mathbf{2}^{\mathbb{R}}$. Contradiction!

(x) $\rightarrow$ (iv) Fix $\mathcal{U}$ as in the proof of (ii) $\rightarrow$ (iii). If $\mathcal{U}$ has no finite subcover then the $Cl_{\mathbb{R}}$-filter $\mathcal{F}$ generated by $\mathcal{G} = \{[p_i]^c \cap D_{\mathbb{R}} : i \in I\}$ has, by our hypothesis, a limit point f. Since for all $i \in I$, $f \in \overline{[p_i]^c \cap D_{\mathbb{R}}} = \overline{[p_i]^c \cap D_{\mathbb{R}}} = [p_i]^c$ it follows that $f \in \bigcap\{[p_i]^c : i \in I\}$. Hence, $f \notin \bigcup\{[p_i] : i \in I\}$ and $\mathcal{U}$ is not a cover of $2^{\mathbb{R}}$. Contradiction!

The second assertion follows from the fact that in the Model $\mathcal{N}[\Gamma]$ in [6] there exists a free ultrafilter on ω but the product $\mathbf{2}^{\mathbb{R}}$ is not compact. $\square$

Theorem 9. *The following are equivalent: (i)* $\mathbf{CI}(\omega)$ *(:* $\mathbf{2}^{\mathbb{R}}$ *is the continuous image of* $S(\omega)$*).*
(ii) Every $Cl_{\mathbb{R}}$*-filter with a limit point extends to an ultrafilter on* $D_{\mathbb{R}}$*.*
(iii) $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$*.*

Proof. (i) $\rightarrow$ (ii) Fix a continuous onto function $H : S(\omega) \rightarrow \mathbf{2}^{\mathbb{R}}$ and let $D = \{h_n : n \in \omega\}$ where for every $n \in \omega$, $h_n = H(\mathcal{F}_n)$. Since $\mathbf{D} = \{\mathcal{F}_n : n \in \omega\}$ is dense in $S(\omega)$ and H is continuous, it follows that D is a countably infinite dense subset of $\mathbf{2}^{\mathbb{R}}$. Without loss of generality we may assume that $D = D_{\mathbb{R}}$.

Fix a $Cl_{\mathbb{R}}$-filter $\mathcal{W}$ with a limit point f. Clearly, $\mathcal{W}$ extends to a $Cl_{\mathbb{R}}$-ultrafilter. So, we may assume that $\mathcal{W}$ is a $Cl_{\mathbb{R}}$-ultrafilter. Let $\mathcal{R}$ be given by (7). We show that $\mathcal{W}$ extends to an ultrafilter $\mathcal{U}$ on $D_{\mathbb{R}}$.

Clearly, for every $O \in \mathcal{R}$, $f \in O$. Since H is onto $2^{\mathbb{R}}$, there is $\mathcal{F} \in S(\omega)$ with $H(\mathcal{F}) = f$. It is easy to see that

$$\mathcal{Q} = \{\{\mathcal{F}_n : n \in F\} : F \in \mathcal{F}\}$$

is an ultrafilter of $\mathbf{D}$. Let $\mathcal{U}_{\mathcal{F}}$ be the ultrafilter of $S(\omega)$ generated by $\mathcal{Q}$. We claim that $\mathcal{U}_{\mathcal{F}}$ converges to $\mathcal{F}$. Indeed, if $[F]$, $F \in \mathcal{F}$ is a basic neigborhood of $\mathcal{F}$ then $\{\mathcal{F}_n : n \in F\} \in \mathcal{U}_{\mathcal{F}}$. Since, $\{\mathcal{F}_n : n \in F\} \subseteq [F]$ it follows that $[F] \in \mathcal{U}_{\mathcal{F}}$. Hence, $\lim \mathcal{U}_{\mathcal{F}} = \mathcal{F}$ as required. Since H is onto we have that $H(\mathcal{U}_{\mathcal{F}})$ is an ultrafilter of $2^{\mathbb{R}}$. Furthermore, by the continuity of H we infer that

$$\lim H(\mathcal{U}_{\mathcal{F}}) = H(\lim \mathcal{U}_{\mathcal{F}}) = H(\mathcal{F}) = f.$$

Hence, the neighborhood base $\mathcal{V}_f$ of all open neighborhoods of f is included in $H(\mathcal{U}_{\mathcal{F}})$. In particular, $\mathcal{R} \subseteq H(\mathcal{U}_{\mathcal{F}})$. Since $D_{\mathbb{R}} \in H(\mathcal{U}_{\mathcal{F}})$ ($\omega \in \mathcal{F}$ implies $\mathbf{D} \in \mathcal{U}_{\mathcal{F}}$ and consequently $D_{\mathbb{R}} = H(\mathbf{D}) \in H(\mathcal{U}_{\mathcal{F}})$), it follows that $\mathcal{U} = \mathcal{P}(D_{\mathbb{R}}) \cap H(\mathcal{U}_{\mathcal{F}})$ is an ultrafilter on $D_{\mathbb{R}}$. Since, $\mathcal{W} = \{O \cap D_{\mathbb{R}} : O \in \mathcal{R}\}$ and $\mathcal{R} \subseteq H(\mathcal{U}_{\mathcal{F}})$, we have that for every $O \in \mathcal{R}$, $O \cap D_{\mathbb{R}} \in H(\mathcal{U}_{\mathcal{F}}) \cap \mathcal{P}(D_{\mathbb{R}})$. Hence, $\mathcal{W} \subseteq \mathcal{U}$ and $\mathbf{UBPI}(Cl_\omega, \omega)$ holds true as required.

(ii) $\rightarrow$ (iii) This is straightforward. (Every $Cl_{\mathbb{R}}$-ultrafilter $\mathcal{W}$ converges to a point f of $\mathbf{2}^{\mathbb{R}}$. Hence, f is a limit point of $\mathcal{W}$. Therefore, by (ii) $\mathcal{W}$ extends to an ultrafilter on $D_{\mathbb{R}}$).

(iii) $\rightarrow$ (i) Fix $\{d_n : n \in \omega\}$ an enumeration of $D_{\mathbb{R}}$. Since $\mathbf{2}^{\mathbb{R}}$ is ultrafilter compact and T_2 it follows easily that every ultrafilter $\mathcal{H}$ on $D_{\mathbb{R}}$ converges to a unique point $g_{\mathcal{H}} \in 2^{\mathbb{R}}$. Since, for every $\mathcal{F} \in S(\omega)$, $\mathcal{H}_{\mathcal{F}} = \{\{d_n : n \in F\} : F \in \mathcal{F}\}$ is an ultrafilter on $D_{\mathbb{R}}$, $\mathcal{H}_{\mathcal{F}}$ converges to a unique point $g_{\mathcal{F}} \in 2^{\mathbb{R}}$. Let $f : S(\omega) \rightarrow \mathbf{2}^{\mathbb{R}}$ be the function given by the rule: $f(\mathcal{F}) = g_{\mathcal{F}}$ where,

$$\{g_{\mathcal{F}}\} = \bigcap \{\overline{\{d_n : n \in F\}} : F \in \mathcal{F}\}. \tag{9}$$

Clearly, for every $n \in \omega$,

$$f(\mathcal{F}_n) = d_n. \tag{10}$$

We claim that f is continuous. To see this, fix $\mathcal{W}$ a filter of $S(\omega)$ such that $lim \mathcal{W} = \mathcal{F}_{\mathcal{W}}$. Clearly,

$$\mathcal{Q} = \{[F] : F \in \mathcal{F}_{\mathcal{W}}\} \subseteq \mathcal{W} \tag{11}$$

is a filterbase of $S(\omega)$. Since, for every $F \in \mathcal{F}_{\mathcal{W}}$, $\{\mathcal{F}_n : n \in F\} \subseteq [F]$ we conclude in view of (10) that

$$\overline{\{d_n : n \in F\}} = \overline{\{f(\mathcal{F}_n) : n \in F\}} \subseteq \overline{f([F])}. \tag{12}$$

On the other hand if $h \in f([F])$ then, by (9), $\{h\} = \bigcap \{\overline{\{d_n : n \in H\}} : H \in \mathcal{H}\}$ for some $\mathcal{H} \in [F]$. Since $F \in \mathcal{H}$ we see that $h \in \overline{\{d_n : n \in F\}}$

and consequently

$$\overline{f([F])} \subseteq \overline{\{d_n : n \in F\}}. \tag{13}$$

From (12) and (13) we conclude that for every $F \in \mathcal{F}_{\mathcal{W}}$,

$$\overline{f([F])} = \overline{\{d_n : n \in F\}}. \tag{14}$$

From (9) and (14), it follows that $\bigcap\{\overline{f([F])} : F \in \mathcal{F}_{\mathcal{W}}\} = \bigcap\{\overline{\{d_n : n \in F\}} : F \in \mathcal{F}_{\mathcal{W}}\} = \{g_{\mathcal{F}_{\mathcal{W}}}\}$. Thus, $\lim f(\mathcal{Q}) = g_{\mathcal{F}_{\mathcal{W}}} = f(\mathcal{F}_w)$. From (11) it follows that $\lim f(\mathcal{W}) = \lim f(\mathcal{Q}) = f(\mathcal{F}_w) = f(\lim \mathcal{W})$ meaning that f is continuous.

To complete the proof we need to show that f is onto. Fix $g \in \mathbf{2}^{\mathbb{R}}$ and let, by our hypothesis, $\mathcal{F}$ be an ultrafilter on $D_{\mathbb{R}}$ extending the $Cl_{\mathbb{R}}$-ultrafilter

$$\mathcal{W} = \{O \cap D_{\mathbb{R}} : O \in cl(\mathbf{2}^{\mathbb{R}}) \text{ and } g \in O\}.$$

Since $\mathcal{F}$ is an ultrafilter on $D_{\mathbb{R}}$, it follows easily that

$$\mathcal{U}_g = \{U_F : F \in \mathcal{F}\} \text{ where for every } F \in \mathcal{F},\ U_F = \{n \in \omega : d_n \in F\}$$

is an ultrafilter on ω. Since $\{[p] \cap D_{\mathbb{R}} : p \in [g]^{<\omega}\} \subseteq \mathcal{W} \subseteq \mathcal{F}$ and $\overline{[p] \cap D_{\mathbb{R}}} = [p]$ we have:

$f(\mathcal{U}_g) = \bigcap\{\overline{\{d_n : n \in U_F\}} : F \in \mathcal{F}\} = \bigcap\{\overline{F} : F \in \mathcal{F}\} \subseteq \bigcap\{\overline{[p] \cap D_{\mathbb{R}}} : p \in [g]^{<\omega}\} = \bigcap\{[p] : p \in [g]^{<\omega}\} = \{g\}$. Thus, $f(\mathcal{F}) = g$ and f is onto as required. □

Corollary 10. *(i)* $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ *implies "$S(\omega)$ is countably compact with respect to the base $\mathcal{C}_\omega = \{[A] : A \in \mathcal{P}(\omega)\}$ of closed subsets of $S(\omega)$".*
(ii) $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ *and* $\mathbf{CC}S(\omega)$ *together imply "$\mathbf{2}^{\mathbb{R}}$ is countably compact".*

Proof. (i) Fix $\mathcal{A} = \{[A_n] : n \in \omega\}$ a family of members of $\mathcal{C}_\omega$ with the fip. Without loss of generality we may assume that $\mathcal{A}$ is strictly descending. We show that $\bigcap\{[A_n] : n \in \omega\} \neq \emptyset$. Clearly, for every $n \in \omega, A_n \supseteq A_{n+1}$ and $A_n \neq A_{n+1}$. If not then there exists $m \in A_{n+1} \backslash A_n$ and the fixed ultrafilter $\mathcal{F}_m$ of all supersets of $\{m\}$ satisfies $\mathcal{F}_m \in [A_{n+1}] \backslash [A_n]$ contradicting the fact that $[A_n] \supseteq [A_{n+1}]$. Let $A = \{a_n : n \in \omega\}$ where for all $n \in \omega$, $a_n \in A_n \backslash A_{n+1}$. Fix, by our hypothesis and Proposition 7, a free ultrafilter $\mathcal{F}$ on A. Clearly, for every $n \in \omega$, $A_n \in \mathcal{F}$ and consequently $\mathcal{F} \in \bigcap\{[A_n] : n \in \omega\} \neq \emptyset$ as required.

(ii) Since, $S(\omega)$ is countably compact and, by $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ and Theorem 9, $\mathbf{2}^{\mathbb{R}}$ is a continuous image of $S(\omega)$, it follows that $\mathbf{2}^{\mathbb{R}}$ is countably compact as well. □

Remark 2. (i) In the model $\mathcal{N}[\Gamma]$ of Theorem 5.2 in [6], ω has a free ultrafilter but there is a family $\mathcal{A} = \{\{A_i, B_i\} : A_i, B_i \in \mathcal{P}(\mathbb{R}), i \in \omega\}$ without a choice set. Hence, by Corollary 10 and the following result from [9],

Theorem 11. [9] *"$\mathbf{2}^{\mathbb{R}}$ is countably compact" implies "every family $\mathcal{A} = \{\{A_i, B_i\} \subseteq \mathcal{P}(\mathbb{R})\backslash\{\emptyset\} : i \in \omega\}$ has a choice set".*

in $\mathcal{N}[\Gamma]$ at least one of the statements: $\mathbf{CC}S(\omega)$ and $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ fails.

(ii) Unlike the statement $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ its analogue "every ultrafilter $\mathcal{H}$ of $\mathcal{E} = \{[X] \cap \mathbf{D} : X \in \mathcal{P}(\omega)\}$ extends to an ultrafilter of $\mathbf{D}$" holds true in $\mathbf{ZF}$. Indeed, if $\mathcal{H}$ is an $\mathcal{E}$-ultrafilter, then for every $X \in \mathcal{P}(\omega)$, either $[X] \cap \mathbf{D} \in \mathcal{H}$ or $[X^c] \cap \mathbf{D} = [X]^c \cap \mathbf{D} \in \mathcal{H}$. Hence, $\mathcal{W} = \{X \in \mathcal{P}(\omega) : [X] \cap \mathbf{D} \in \mathcal{H}\}$ is an ultrafilter on ω and consequently $\mathcal{H} = \{[X] \cap \mathbf{D} : X \in \mathcal{W}\}$ is an ultrafilter of $\mathbf{D}$.

(iii) It is known, see e.g., [5] that $S(\omega)$ embeds as a closed subspace of $\mathbf{2}^{\mathbb{R}}$. The function $T : S(\omega) \to \mathbf{2}^{\mathcal{P}(\omega)}$ given by: $T(\mathcal{F}) = \mathcal{X}_{\mathcal{F}}$ where, $\mathcal{X}_{\mathcal{F}}$ denotes the characteristic function of $\mathcal{F}$ is such an embedding. Hence, "$\mathbf{2}^{\mathbb{R}}$ is countably compact" implies $\mathbf{CC}S(\omega)$.

Next we show that the conjunction $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ and $\mathbf{CC}S(\omega)$ implies "every family $\mathcal{A} = \{\{A_i, B_i\} : A_i, B_i \in \mathcal{P}(\mathbb{R}), i \in \omega\}$ has a choice set" and, the second part of the latter conjunction implies the weaker statement "every family $\mathcal{A} = \{\{A_i, B_i\} \subseteq [\mathbb{R}]^{\omega} : i \in \omega\}$ has a choice set".

Theorem 12. *(i) $\mathbf{CC}S(\omega)$ implies "every family $\mathcal{A} = \{\{A_i, B_i\} \subseteq [\mathbb{R}]^{\omega} : i \in \omega\}$ has a choice set".*
(ii) The conjunction $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ and $\mathbf{CC}S(\omega)$ implies "every family $\mathcal{A} = \{\{A_i, B_i\} : A_i, B_i \in \mathcal{P}(\mathbb{R}), i \in \omega\}$ has a choice set".

Proof. (i) Fix a family $\mathcal{A} = \{\{A_i, B_i\} \subseteq [\mathbb{R}]^{\omega} : i \in \omega\}$. Without loss of generality we may assume that for all $i \in \omega$, $A_i \cap B_i = \emptyset$ and, for all $i, j \in \omega$, $i \neq j$, $(A_i \cup B_i) \cap (A_j \cup B_j) = \emptyset$. For every $i \in \omega$ let $S_i = \{\mathcal{F} \in S(D_{\mathbb{R}}) : (\forall x \in A_i)([\{(x,1)\}] \cap D_{\mathbb{R}}) \in \mathcal{F} \wedge (\forall x \in B_i)([\{(x,0)\}] \cap D_{\mathbb{R}}) \in \mathcal{F}\}$, $Q_i = \{\mathcal{F} \in S(D_{\mathbb{R}}) : (\forall x \in A_i)([\{(x,0)\}] \cap D_{\mathbb{R}}) \in \mathcal{F} \wedge (\forall x \in B_i)([\{(x,1)\}] \cap D_{\mathbb{R}}) \in \mathcal{F}\}$. Since $\{[\{(x,1)\}] \cap D_{\mathbb{R}} : x \in A_i\} \cup \{[\{(x,0)\}] \cap D_{\mathbb{R}} : x \in B_i\}$ has the fip it follows that $\{[[\{(x,1)\}] \cap D_{\mathbb{R}}] : x \in A_i\} \cup \{[[\{(x,0)\}] \cap D_{\mathbb{R}}] : x \in B_i\}$ has the fip. Since $A_i \cup B_i$ is countable it follows by $\mathbf{CC}S(\omega)$ that $S_i = \bigcap\{[[\{(x,1)\}] \cap D_{\mathbb{R}}] : x \in A_i\} \cap \bigcap\{[[\{(x,0)\}] \cap D_{\mathbb{R}}] : x \in B_i\} \neq \emptyset$.

Similarly Q_i, is a non-empty subset of $S(D_{\mathbb{R}})$. We show that S_i is a closed subset of $S(\omega)$. To see this, fix $\mathcal{H} \in S(\omega)\backslash S_i$. By the definition of S_i, there exists $x \in A_i$ such that $[\{(x,1)\}] \cap D_{\mathbb{R}}) \notin \mathcal{H}$ or, there exists $x \in B_i$ with $[\{(x,0)\}] \cap D_{\mathbb{R}}) \notin \mathcal{H}$. Assume that the former is the case. Thus, $(D_{\mathbb{R}} \cap [\{(x,0)\}]) \in \mathcal{H}$ and $[D_{\mathbb{R}} \cap [\{(x,0)\}]]$ is a neigborhood of $\mathcal{H}$ avoiding S_i. Hence, S_i is closed as required. Similarly, for every $i \in \omega$, Q_i is non-empty and closed. Hence, for every $i \in \omega$, $G_i = S_i \cup Q_i$ is a closed non-empty subset of $S(\omega)$. Put $\mathcal{G} = \{G_i : i \in \omega\}$. We claim that $\mathcal{G}$ has the fip. To this end, it suffices to show that for every $n \in \mathbb{N}$, $\bigcap\{S_i : i \leq n\} \neq \emptyset$. Fix $n \in \mathbb{N}$ and let $A = \bigcup\{A_i : i \leq n\}$ and $B = \bigcup\{B_i : i \leq n\}$. Since $A \cup B$ is countable it follows by $\mathbf{CC}S(\omega)$ and the above argument that $\bigcap\{S_i : i \leq n\} \supseteq \{\mathcal{F} \in S(D_{\mathbb{R}}) : (\forall x \in A)([\{(x,1)\}] \cap D_{\mathbb{R}}) \in \mathcal{F} \wedge (\forall x \in B)([\{(x,0)\}] \cap D_{\mathbb{R}}) \in \mathcal{F}\} \neq \emptyset$. By $\mathbf{CC}S(\omega)$ again it follows that $\bigcap \mathcal{G} \neq \emptyset$. Fix $\mathcal{F} \in \bigcap \mathcal{G}$. Clearly, the function f given by the rule:

$$f(i) = \begin{cases} A_i \text{ if } \mathcal{F} \in S_i \\ B_i \text{ if } \mathcal{F} \in Q_i \end{cases}$$

is a choice function for the family $\mathcal{A}$ finishing the proof of (i).

(ii) Fix a family $\mathcal{A} = \{\{A_i, B_i\} \subseteq \mathcal{P}(\mathbb{R}) : i \in \omega\}$. Mimic the proof of part (i) and use the extra hypothesis $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$ to show that the sets S_i, Q_i are non-empty. We leave the details as an easy exercise for the reader.

For an indirect proof, combine part (ii) of Corollary 10 with the following result from [9]: "$\mathbf{2}^{\mathbb{R}}$ is countably compact" implies "every family $\mathcal{A} = \{\{A_i, B_i\} \subseteq \mathcal{P}(\mathbb{R})\backslash\{\emptyset\} : i \in \omega\}$ has a choice set". □

4. Summary

- $\mathbf{UF}(\omega) \leftrightarrow \mathbf{BPI}(Cl_\omega, D_\omega) \leftrightarrow \mathbf{UBPI}(Cl_\omega, D_\omega)$ (Proposition 7).
- $\mathbf{BPI}(\omega) \leftrightarrow \mathbf{BPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}}) \leftrightarrow \mathbf{BPI}(Cl_{\mathbb{R}}) \leftrightarrow S(\omega)$ is compact $\leftrightarrow$ $\mathbf{2}^{\mathbb{R}}$ is compact (Theorem 8).
- $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}}) \leftrightarrow \mathbf{CI}(\omega)$ (Theorem 9).

The following diagram records some of the implications and non-implications between the statements $\mathbf{UF}(\omega)$, $\mathbf{BPI}(\omega)$, $\mathbf{BPI}(\mathbb{R})$, $\mathbf{BPI}(Cl_\omega)$, $\mathbf{BPI}(Cl_\omega, D_\omega)$, $\mathbf{UBPI}(Cl_\omega, D_\omega)$, $\mathbf{BPI}(Cl_{\mathbb{R}})$, $\mathbf{BPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$, $\mathbf{UBPI}(Cl_{\mathbb{R}}, D_{\mathbb{R}})$, "$\mathfrak{u} = |2^{\mathbb{R}}|$", "$|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}|$", $\mathbf{CI}(\omega)$ and $\mathbf{CC}S(\omega)$ as well as, the implications which remain open.

$$
\begin{array}{ccccc}
 & & \mathbf{UF}(\omega) & & \\
 & \begin{array}{c}\nearrow\\ (?\swarrow)\end{array} & \uparrow \quad \not\downarrow & \begin{array}{c}\nwarrow\\ (\searrow ?)\end{array} & \\
|2^{\mathfrak{u}}| = |2^{2^{\mathbb{R}}}| & \begin{array}{c}\leftarrow\\ (\rightarrow ?)\end{array} & \mathbf{BPI}(\omega) & \begin{array}{c}\rightarrow\\ (?\leftarrow)\end{array} & \mathbf{CI}(\omega) \\
\uparrow \quad (\downarrow ?) & (\nearrow ?)(\swarrow ?) & \uparrow \quad \not\downarrow & \searrow (\nwarrow ?) & (\downarrow ?)(\uparrow ?) \\
\mathfrak{u} = |2^{\mathbb{R}}| & \begin{array}{c}\leftarrow\\ \not\rightarrow\end{array} & \mathbf{BPI}(\mathbb{R}) & \begin{array}{c}\rightarrow\\ (?\leftarrow)\end{array} & \mathbf{CC}S(\omega)
\end{array}
$$

Diagram 1

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