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by

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# **REMARKS ON POINTED DIGITAL HOMOTOPY**

# LAURENCE BOXER AND P. CHRISTOPHER STAECKER

ABSTRACT. We present and explore in detail a pair of digital images with  $c_u$ -adjacencies that are homotopic but not pointed homotopic. For two digital loops  $f, g: [0, m]_Z \to X$  with the same basepoint, we introduce the notion of *tight at the basepoint (TAB)* pointed homotopy, which is more restrictive than ordinary pointed homotopy and yields some different results.

We present a variant form of the digital fundamental group. Based on what we call *eventually constant* loops, this version of the fundamental group is equivalent to that of [2], but offers advantages that we discuss.

We show that homotopy equivalent digital images have isomorphic fundamental groups, even when the homotopy equivalence does not preserve the basepoint. This assertion appeared in [3], but there was an error in the proof; here, we correct the error.

# 1. INTRODUCTION

Digital topology adapts tools from geometric and algebraic topology to the study of digital images. Fundamental questions concerning the form and motion of a digital image are considered using tools of this field. The following appears in the abstract of [18].

> Digital topology deals with the topological properties of digital images.... It provides the theoretical foundations for important image processing operations such as connected component labeling and counting, border following, contour filling, and thinning - and their generalizations to three- (or higher-) dimensional "images."

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 $Key\ words\ and\ phrases.$  Digital topology, digital homotopy, homotopy equivalence, fundamental group.

Concerning the digital fundamental group (although it's a somewhat different version of the fundamental group than used in the current paper), we find in [17]:

... the digital fundamental group has an immediate application to the theory of 3-d image thinning algorithms. For in order to preserve "tunnels" a 3-d thinning algorithm must preserve the digital fundamental groups of the input binary picture.

Relations between other image processing operations and digital topology are explored in papers such as [11, 12, 6, 8].

In this paper, we consider questions of pointed homotopy in digital topology. Homotopy can be thought of as a body of mathematics underlying continuous motion and continuous deformation. Thus, digital homotopy underlies a great deal of digital animation and can help answer questions in object recognition of the form *could object A be a match to object B*? Pointed homotopy is concerned with questions of continuous deformations in which some point must be held fixed.

We give an example showing that homotopy equivalence between digital images  $(X, c_u)$  and  $(Y, c_v)$  does not imply pointed homotopy equivalence between these images. This can be interpreted as showing that there are images X and Y that have the same form such that X cannot be continuously deformed to match Y unless every point of X is disturbed by the deforming transformation. This example is then used to illustrate a new variant on the pointed homotopy of digital loops. We present an alternate version of the digital fundamental group that is equivalent to, but has advantages over, the version introduced in [2]. We correct the argument of [3] for the assertion that homotopy equivalent connected digital images  $(X, \kappa)$  and  $(Y, \lambda)$  have isomorphic fundamental groups  $\Pi_1^{\kappa}(X, x_0)$ and  $\Pi_1^{\lambda}(Y, y_0)$ .

#### 2. Preliminaries

Much of the material in this section is quoted or paraphrased from [7].

2.1. General properties. Let  $\mathbb{Z}$  be the set of integers. A *(binary) digital image* is a pair  $(X, \kappa)$ , where  $X \subset \mathbb{Z}^n$  for some positive integer n, and  $\kappa$  is some adjacency relation for the members of X.

Adjacency relations commonly used in the study of digital images in  $\mathbb{Z}^n$  include the following [14]. For an integer u such that  $1 \leq u \leq n$ , we define an adjacency relation as follows. Points

$$p = (p_1, p_2, \dots, p_n), q = (q_1, q_2, \dots, q_n)$$

are  $c_u$ -adjacent [4] if

- $p \neq q$ , and
- there are at most u distinct indices i for which  $|p_i q_i| = 1$ , and
- for all indices i, if  $|p_i q_i| \neq 1$  then  $p_i = q_i$ .

We often denote a  $c_u$ -adjacency in  $\mathbb{Z}^n$  by the number of points that are  $c_u$ -adjacent to a given point in  $\mathbb{Z}^n$ . E.g.,

- in  $\mathbb{Z}^1$ ,  $c_1$ -adjacency is 2-adjacency;
- in  $\mathbb{Z}^2$ ,  $c_1$ -adjacency is 4-adjacency and  $c_2$ -adjacency is 8-adjacency.
- in Z<sup>3</sup>, c<sub>1</sub>-adjacency is 6-adjacency, c<sub>2</sub>-adjacency is 18-adjacency, and c<sub>3</sub>-adjacency is 26-adjacency.

More general adjacency relations appear in [15]. The work in [13] treats digital images as abstract sets of points with arbitrary adjacencies without regard for their embeddings in  $\mathbb{Z}^n$ .

**Definition 2.1.** [1] Let  $a, b \in \mathbb{Z}$ , a < b. A *digital interval* is a set of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$$

in which  $c_1$ -adjacency is assumed.

The following generalizes an earlier definition of [21].

**Definition 2.2.** [2] Let  $(X, \kappa)$  and  $(Y, \lambda)$  be digital images. Then the function  $f: X \to Y$  is  $(\kappa, \lambda)$ -continuous if and only if for every pair of  $\kappa$ -adjacent points  $x_0, x_1 \in X$ , either  $f(x_0) = f(x_1)$ , or  $f(x_0)$  and  $f(x_1)$  are  $\lambda$ -adjacent.

See also [9, 10], where similar concepts are named *immersion*, gradually varied operator, or gradually varied mapping.

A path from p to q in  $(X, \kappa)$  is a  $(2, \kappa)$ -continuous function  $F : [0, m]_{\mathbb{Z}} \to X$  such that F(0) = p and F(m) = q. For a given path F, we define the reverse path,  $F^{-1} : [0, m]_{\mathbb{Z}} \to X$  defined by  $F^{-1}(t) = F(m-t)$ . A loop is a path  $F : [0, m]_{\mathbb{Z}} \to X$  such that F(0) = F(m).

2.2. **Digital homotopy.** Intuitively, a homotopy between continuous functions  $f, g : X \to Y$  is a continuous deformation of, say, f over a time period until the result of the deformation coincides with g.

**Definition 2.3.** ([2]; see also [16]) Let X and Y be digital images. Let  $f, g : X \to Y$  be  $(\kappa, \lambda)$ -continuous functions and suppose there is a positive integer m and a function

$$F: X \times [0, m]_{\mathbb{Z}} \to Y$$

such that

- for all  $x \in X$ , F(x, 0) = f(x) and F(x, m) = g(x);
- for all  $x \in X$ , the induced function  $F_x : [0,m]_{\mathbb{Z}} \to Y$  defined by

 $F_x(t) = F(x,t)$  for all  $t \in [0,m]_{\mathbb{Z}}$ ,

is  $(c_1, \lambda)$ -continuous;

• for all  $t \in [0, m]_{\mathbb{Z}}$ , the induced function  $F_t : X \to Y$  defined by

$$F_t(x) = F(x,t)$$
 for all  $x \in X$ ,

is  $(\kappa, \lambda)$ -continuous.

Then F is a digital  $(\kappa, \lambda)$ -homotopy between f and g, and f and g are  $(\kappa, \lambda)$ -homotopic in Y. If m = 1, then f and g are homotopic in 1 step.

If, further, there exists  $x_0 \in X$  such that  $F(x_0, t) = F(x_0, 0)$  for all  $t \in [0, m]_{\mathbb{Z}}$ , we say F is a *pointed homotopy*. If g is a constant function, we say F is a *nullhomotopy*, and f is *nullhomotopic*.

The notation  $f \simeq_{(\kappa,\lambda)} g$  indicates that functions f and g are digitally  $(\kappa,\lambda)$ -homotopic in Y. If  $\kappa = \lambda$ , we abbreviate this as  $f \simeq_{\kappa} g$ . When the adjacencies are understood we simply write  $f \simeq g$ .

Digital homotopy is an equivalence relation among digitally continuous functions [16, 2].

Let  $H : [0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}} \to X$  be a homotopy between paths  $f,g : [0,m]_{\mathbb{Z}} \to X$ . We say H holds the endpoints fixed if f(0) = H(0,t) = g(0) and f(m) = H(m,t) = g(m) for all  $t \in [0,n]_{\mathbb{Z}}$ . If f and g are loops, we say H is loop preserving if H(0,t) = H(m,t) for all  $t \in [0,n]_{\mathbb{Z}}$ . Notice that if f and g are loops and H holds the endpoints fixed, then H is a loop preserving pointed homotopy between f and g.

As in classical topology, we say two digital images  $(X, \kappa)$  and  $(Y, \lambda)$ are *homotopy equivalent* when there are continuous functions  $f: X \to Y$ and  $g: Y \to X$  such that  $g \circ f \simeq_{(\kappa,\lambda)} 1_X$  and  $f \circ g \simeq_{(\lambda,\kappa)} 1_Y$ .

2.3. Digital fundamental group. The fundamental group is an invariant of the (unpointed) homotopy type of a digital image (Theorem 5.3), and it is often easier to compute the fundamental groups of images X and Y than to decide directly whether these images have the same homotopy type. Thus, the fundamental group is a useful tool in studying the form of a digital image.

If f and g are paths in X such that g starts where f ends, the product (see [16]) of f and g, written f \* g, is, intuitively, the path obtained by following f, then following g. Formally, if  $f : [0, m_1]_{\mathbb{Z}} \to X$ ,  $g : [0, m_2]_{\mathbb{Z}} \to X$ , and  $f(m_1) = g(0)$ , then  $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$  is defined by

$$(f * g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathbb{Z}}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

Restriction of loop classes to loops defined on the same digital interval would be undesirable. The following notion of *trivial extension* to permit a loop to "stretch" within the same pointed homotopy class. In section 4, we will introduce a different method of "stretching" a loop within its pointed homotopy class. Intuitively, f' is a *trivial extension of* f if f' follows the same path as f, but more slowly, with pauses for rest (subintervals of the domain on which f' is constant).

**Definition 2.4.** [2] Let f and f' be loops in a pointed digital image  $(X, x_0)$ . We say f' is a *trivial extension of* f if there are sets of paths  $\{f_1, f_2, \ldots, f_k\}$  and  $\{F_1, F_2, \ldots, F_p\}$  in X such that

(1)  $0 < k \le p;$ (2)  $f = f_1 * f_2 * \dots * f_k;$ (3)  $f' = F_1 * F_2 * \dots * F_p;$ (4) there are indices  $1 \le i_1 < i_2 < \dots < i_k \le p$  such that •  $F_{i_j} = f_j, 1 \le j \le k$ , and •  $i \notin \{i_1, i_2, \dots, i_k\}$  implies  $F_i$  is a trivial loop.

This notion lets us compare the digital homotopy properties of loops whose domains may have differing cardinality, since if  $m_1 \leq m_2$ , we can obtain [2] a trivial extension of a loop  $f : [0, m_1]_{\mathbb{Z}} \to X$  to  $f' : [0, m_2]_{\mathbb{Z}} \to X$  via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \le t \le m_1; \\ f(m_1) & \text{if } m_1 \le t \le m_2. \end{cases}$$

Observe that every digital loop f is a trivial extension of itself.

**Definition 2.5.** ([14], correcting an earlier definition in [3]). Two loops  $f_0, f_1$  with the same base point  $p \in X$  belong to the same loop class  $[f]_X$  if they have trivial extensions that can be joined by a homotopy H that keeps the endpoints fixed.

When X is understood, we will often use the notation [f] for  $[f]_X$ .

It was incorrectly asserted as Proposition 3.1 of [3] that the assumption in Definition 2.5, that the homotopy keeps the endpoints fixed, could be replaced by the weaker assumption that the homotopy is loop-preserving; the error was pointed out in [5].

Membership in the same loop class in  $(X, x_0)$  is an equivalence relation among loops [2].

The digital fundamental group is derived from a classical notion of algebraic topology (see [19, 20, 22]). The version discussed in this section is that developed in [2]. The next result is used in [2] to show the product operation of our digital fundamental group is well defined.

**Proposition 2.6.** [2, 16] Let  $f_1, f_2, g_1, g_2$  be digital loops based at  $x_0$  in a pointed digital image  $(X, x_0)$ , with  $f_2 \in [f_1]_X$  and  $g_2 \in [g_1]_X$ . Then  $f_2 * g_2 \in [f_1 * g_1]_X$ .

Let  $(X, x_0)$  be a pointed digital image; *i.e.*, X is a digital image, and  $x_0 \in X$ . Define  $\Pi_1(X, x_0)$  to be the set of loop classes  $[f]_X$  in X with base point  $x_0$ . When we wish to emphasize an adjacency relation  $\kappa$ , we denote this set by  $\Pi_1^{\kappa}(X, x_0)$ . By Proposition 2.6, the *product* operation

$$[f]_X \cdot [g]_X = [f * g]_X$$

is well defined on  $\Pi_1(X, x_0)$ ; further, the operation  $\cdot$  is associative on  $\Pi_1(X, x_0)$  [16].

**Lemma 2.7.** [2] Let  $(X, x_0)$  be a pointed digital image. Let  $\overline{x_0} : [0, m]_{\mathbb{Z}} \to X$  be a constant loop with image  $\{x_0\}$ . Then  $[\overline{x_0}]_X$  is an identity element for  $\Pi_1(X, x_0)$ .

**Lemma 2.8.** [2] If  $f : [0,m]_{\mathbb{Z}} \to X$  represents an element of  $\Pi_1(X,x_0)$ , then the reverse loop  $f^{-1}$  is an element of  $[f]_X^{-1}$  in  $\Pi_1(X,x_0)$ .

**Theorem 2.9.** [2]  $\Pi_1(X, x_0)$  is a group under the  $\cdot$  product operation, the fundamental group of  $(X, x_0)$ .

**Theorem 2.10.** [2] Suppose  $F : (X, \kappa, x_0) \to (Y, \lambda, y_0)$  is a pointed continuous function. Then F induces a homomorphism  $F_* : \Pi_1^{\kappa}(X, x_0) \to \Pi_1^{\lambda}(Y, y_0)$  defined by  $F_*([f]) = [F \circ f]$ .

# 3. Homotopy Equivalent Images that Aren't Pointed Homotopy Equivalent

In [3], it was asked if, given digital images  $(X, \kappa)$  and  $(Y, \lambda)$  that are homotopy equivalent, must  $(X, x_0, \kappa)$  and  $(Y, y_0, \lambda)$  be pointed homotopy equivalent for arbitrary base points  $x_0 \in X$ ,  $y_0 \in Y$ ? The paper [13] gives an example, not using any of the  $c_u$ -adjacencies, that answers this question in the negative. It is desirable to have an example that uses  $c_u$ -adjacencies. In this section, we give such an example by modifying that of [13].

**Example 3.1.** Let  $X = \{x_i\}_{i=0}^{10} \subset \mathbb{Z}^2$  where  $x_0 = (2,0), x_1 = (1,1), x_2 = (0,2), x_3 = (-1,2), x_4 = (-2,1), x_5 = (-2,0), x_6 = (-2,-1), x_7 = (-1,-2), x_8 = (0,-2), x_9 = (1,-1), x_{10} = (0,0).$  Let  $Y = X \setminus \{x_0\} = \{x_i\}_{i=1}^{10}$ . We consider both X and Y as digital images with  $c_2$ -adjacency. See Figure 1.



FIGURE 1. A figure  $X = \{x_i\}_{i=0}^{10}$  and its subset  $Y = X \setminus \{x_0\}$  that are homotopy equivalent but not pointed homotopy equivalent as images in  $\mathbb{Z}^2$  with  $c_2$ -adjacency

**Proposition 3.2.** Let X and Y be the images of Example 3.1. Then X and Y are  $(c_2, c_2)$ -homotopy equivalent.

*Proof.* Let  $f: X \to Y$  be defined by

$$f(x_i) = \begin{cases} x_{i+1} & \text{if } 0 \le i \le 9; \\ x_1 & \text{if } i = 10. \end{cases}$$

Let  $g: Y \to X$  be the inclusion map. Clearly, both f and g are  $(c_2, c_2)$ continuous. The function  $H: X \times [0, 1]_Z \to X$  defined by

$$H(x_i, t) = \begin{cases} f(x_i) = g \circ f(x_i) & \text{if } t = 0; \\ x_i & \text{if } t = 1, \end{cases}$$

is clearly a  $(c_2, c_2)$ -homotopy between  $g \circ f$  and  $1_X$ . The function  $K : Y \times [0, 1]_Z \to Y$  defined by

$$K(x_i, t) = \begin{cases} f(x_i) = f \circ g(x_i) & \text{if } t = 0 \text{ and } 1 \le i \le 10; \\ x_i & \text{if } t = 1 \text{ and } 1 \le i \le 10, \end{cases}$$

is clearly a  $(c_2, c_2)$ -homotopy between  $f \circ g$  and  $1_Y$ . Thus,  $(X, c_2)$  and  $(Y, c_2)$  are homotopy equivalent.

**Proposition 3.3.** Let  $Y = \{x_i\}_{i=1}^{10}$  be as above. Let  $h : (Y, c_2) \to (Y, c_2)$  be a continuous map such that h(x) = x for some  $x \in Y$  and h is  $(c_2, c_2)$ -homotopic to  $1_Y$  in 1 step. Then  $h = 1_Y$ .

*Proof.* For convenience, we prove the statement in the case where  $x = x_1$ . Since  $(Y, c_2)$  is a simple cycle of 10 points, the same argument will work for any other value of x. Since h is  $(c_2, c_2)$ -homotopic to  $1_Y$  in 1 step,  $h(x_i)$  and  $x_i$  are  $c_2$ adjacent or equal for all i. Suppose  $h \neq 1_Y$ . Since  $h(x_1) = x_1$ , by  $c_2$ -continuity,  $h(x_i) \in \{x_{i-1}, x_i\}$  for  $2 \leq i \leq 10$ , and since  $h \neq 1_Y$ , there is a  $j_0$  such that  $2 \leq j_0 \leq 10$  and  $h(x_j) = x_{j-1}$  for  $j_0 \leq j \leq 10$ . In particular,  $h(x_{10}) = x_9$ , so we have a discontinuity since the  $c_2$ -adjacent points  $x_1$  and  $x_{10}$  do not have  $c_2$ -adjacent images under h. Since h was assumed continuous, the contradiction leads us to conclude that  $h = 1_Y$ .  $\Box$ 

A similar argument shows the following.

**Corollary 3.4.** Let  $X = \{x_i\}_{i=0}^{10}$  be as above. Let  $h : (X, c_2) \to (X, c_2)$  be a continuous map such that  $h(x_0) = x_0$  and h is homotopic in 1 step to  $1_X$ . Then  $h = 1_X$ .

**Proposition 3.5.** Let  $X = \{x_i\}_{i=0}^{10}$  and  $Y = X \setminus \{x_0\}$  be as above. Then for any  $x \in X$  and  $y \in Y$ , (X, x) and (Y, y) are not pointed  $(c_2, c_2)$ -homotopy equivalent.

*Proof.* Suppose otherwise. Then for some  $x \in X$  and  $y \in Y$ , there are  $(c_2, c_2)$ -continuous pointed maps  $f : (X, x) \to (Y, y)$  and  $g : (Y, y) \to (X, x)$  such that  $f \circ g$  is pointed homotopic to  $1_X$  and  $g \circ f$  is pointed homotopic to  $1_Y$ .

First we argue that  $g \circ f$  must in fact equal  $1_X$ . Since f and g are pointed maps we have  $g \circ f(x) = x$ , and our pointed homotopy from  $g \circ f$ to  $1_X$  will fix x at all stages. If  $g \circ f$  were not  $1_X$ , then there would be some final stage h of the pointed homotopy from  $g \circ f$  to  $1_X$  for which  $h \neq 1_X$  but h is pointed homotopic to  $1_X$  in one step. This is impossible by Proposition 3.3, and so we conclude that  $g \circ f = 1_X$ . Similarly, using Corollary 3.4, we have  $f \circ g = 1_Y$ .

Since  $f \circ g = 1_Y$  and  $g \circ f = 1_X$ , it follows that X and Y are  $(c_2, c_2)$ isomorphic images, which is impossible, as X and Y have different cardinalities. The assertion follows.

Example 3.1 is an image in  $\mathbf{Z}^2$  with  $c_2$ -adjacency that exhibits interesting pointed homotopy properties. We remark that images exist in  $\mathbf{Z}^2$ with  $c_1$ -adjacency with similar properties. The image in Figure 2 exhibits the same behavior as that of Example 3.1.

Let X be the digital image in Example 3.1, and define two loops f, g:  $[0, 10]_Z \to X$  as follows:

$$f = (x_1, x_2, \dots, x_9, x_{10}, x_1)$$
  
$$g = (x_1, x_2, \dots, x_9, x_0, x_1).$$

FIGURE 2. An image in  $Z^2$  with  $c_1$ -adjacency having the same properties as in Example 3.1.

These loops are equivalent in  $\Pi_1(X, x_1)$ : consider the following trivial extensions

$$f' = (x_1, x_2, x_3, \dots, x_9, x_{10}, x_1, x_1)$$
  
$$g' = (x_1, x_1, x_2, \dots, x_8, x_9, x_0, x_1).$$

These loops f' and g' are homotopic in one step, and so f and g are equivalent in  $\Pi_1(X, x_1)$ . Notice that the one-step equivalence above uses trivial extensions at the base point  $x_1$ . That is, there is some t with  $f'(t) = f'(t+1) = x_1$ , and likewise for g'. In fact this is necessary for any equivalence between f and g, as the following proposition shows.

**Proposition 3.6.** Let X be as in Example 3.1. Let f and g be the loops described above. Let  $f', g' : [0, k]_Z \to X$  be trivial extensions of f and g that are homotopic by  $H(t,s) : [0,k]_Z \times [0,n]_Z \to X$ . Then there is some time  $p \in [0,n]_Z$  and intermediate stage of the homotopy H, i.e.,  $h: [0,k]_Z \to X$  defined by h(t) = H(t,p), such that  $h(k-1) = h(k) = x_1$ . Similarly there is some  $q \in [0,n]_Z$  and intermediate stage of the homotopy H, i.e.,  $l: [0,k]_Z \to X$  defined by l(t) = H(t,q), such that  $l(0) = l(1) = x_1$ .

*Proof.* We will prove the first statement; the second follows similarly. Suppose that no intermediate loop h obeys  $h(k-1) = h(k) = x_1$ . Then we have  $H(k-1,s) \neq x_1$  for all s. We must in particular have  $f'(k-1) \neq x_1$ , and so  $f'(k-1) = x_{10}$  since f' is a trivial extension of f.

Thus, considering H(k - 1, s) for various s gives a path from  $H(k - 1, 0) = f'(k - 1) = x_{10}$  to  $g'(k - 1) = x_0$  which never passes through  $x_1$ . Because of the structure of our image X, this path must at some point pass through  $x_9$ . Thus there is some r with  $H(k - 1, r) = x_9$ . But  $H(k, r) = x_1$  since all stages of H are loops at  $x_1$ . This contradicts continuity of H from H(k - 1, r) to H(k, r) since  $x_9$  is not adjacent to  $x_1$  in X.

Thus we see that f and g are equivalent as loops in  $\Pi_1(X, x_1)$ , but this equivalence requires trivial extensions at the base point. This suggests a finer equivalence relation than the one used for the fundamental group, one in which loops are equivalent only by homotopies that do not extend the base point. Specifically, we call a loop f tight at the basepoint (TAB)  $x_0$  when there is no t with  $f(t) = f(t+1) = x_0$ . Two TAB loops are called TAB equivalent when there are TAB trivial extensions that are homotopic by a homotopy that is TAB in each stage.

Thus our example loops f and g above are equivalent in  $\Pi_1(X, x_1)$ , but not TAB equivalent, because any homotopy of trivial extensions must have a non-TAB intermediate stage. The equivalence classes using the TAB relation seem to have interesting and subtle structure, but they do not naturally form a group with respect to the product operation, as we show below.

Consider the product of f and the reverse of g, which has the form:

$$f * g^{-1} = (x_1, x_2, \dots, x_9, x_{10}, x_1, x_0, x_9, \dots, x_2, x_1).$$

Note that  $f * g^{-1}$  is nullhomotopic, using only TAB loops as intermediate steps. The first step of the nullhomotopy is as follows:

$$(x_1, x_2, \dots, x_9, x_{10}, x_1, x_0, x_9, \dots, x_2, x_1)$$
 to  
 $(x_1, x_2, \dots, x_9, x_9, x_0, x_0, x_9, \dots, x_2, x_1),$ 

and then the loop deforms continuously to a constant map  $(x_1, x_1, \ldots, x_1)$  in an obvious way.

Since f and g are not TAB equivalent, but  $f * g^{-1}$  is pointed nullhomotopic, the TAB relation, which is finer than the equivalence used in  $\Pi_1(X, x_1)$ , cannot be used to define a group. Nevertheless the TAB equivalence provides subtle and interesting information about loops in our space.

#### 4. A NEW FORMULATION OF THE FUNDAMENTAL GROUP

The equivalence relation of Definition 2.5 used to define the fundamental group relies on trivial extensions, which are often cumbersome to handle. In this section we give an equivalent definition of the fundamental group which does not require trivial extensions. Our construction instead is based on *eventually constant paths*. Let  $\mathbb{N} = \{1, 2, ...\}$  denote the natural numbers, and  $\mathbb{N}^* = \{0\} \cup \mathbb{N}$ . We consider  $\mathbb{N}^*$  to be a digital image with 2-adjacency.

**Definition 4.1.** Given a digital image X, a continuous function  $f : \mathbb{N}^* \to X$  is called an *eventually constant path* or *EC path* if there is some point  $c \in X$  and some  $N \ge 0$  such that f(x) = c whenever  $x \ge N$ . When

convenient we abbreviate the latter by  $f(\infty) = c$ . The *endpoints* of an EC path f are the two points f(0) and  $f(\infty)$ . If f is an EC path and  $f(0) = f(\infty)$ , we say f is an EC loop, and f(0) is called the basepoint.  $\Box$ 

We say that a homotopy H between EC paths is an EC homotopy when the function  $H_t : \mathbb{N}^* \to X$  defined by  $H_t(s) = H(s,t)$  is an EC path for all  $t \in [0,k]_{\mathbb{Z}}$ . To indicate an EC homotopy, we write  $f \simeq^{EC} g$ , or  $f \simeq^{EC}_{\kappa} g$  if it is desirable to state the adjacency  $\kappa$  of X. We say an EC homotopy Hholds the endpoints fixed when  $H_t(0) = f(0) = g(0)$  and there is a  $c \in \mathbb{N}^*$ such that  $n \ge c$  implies  $H_t(n) = f(n) = g(n)$  for all t.  $\Box$ 

Not all homotopies of EC paths are EC homotopies, as the following example shows.

**Example 4.2.** Let  $f, g: \mathbb{N}^* \to [0,1]_{\mathbb{Z}}$  be defined by f(0) = g(0) = 0, f(n) = g(n) = 1 for n > 0. Let  $H: \mathbb{N}^* \times [0,2]_{\mathbb{Z}} \to [0,1]_{\mathbb{Z}}$  be defined by  $H_0 = H_2 = f = g$ ,  $H_1(s) = 0$  if s is even,  $H_1(s) = 1$  if s is odd. Then H is a homotopy from f to g that is not an EC homotopy.

*Proof.* It is easy to see that H is a homotopy. However,  $H_1$  is not an EC path. The assertion follows.

A familiar argument shows that EC homotopy is an equivalence relation.

**Proposition 4.3.** *EC* homotopy and *EC* homotopy holding the endpoints fixed are equivalence relations among *EC* paths.

*Proof.* We give a proof without the assumption of endpoints being held fixed. The same argument can be used with obvious modifications to obtain the assertion for endpoints held fixed.

Reflexive: Given an EC path  $f : \mathbb{N}^* \to X$ , clearly the function  $H : \mathbb{N}^* \times \{0\} \to X$  given by H(x, 0) = f(x) shows  $f \simeq^{EC} f$ .

Symmetric: If  $H : \mathbb{N}^* \times [0, m]_{\mathbb{Z}} \to X$  is an EC homotopy from f to g, then it is easy to see that the function  $H' : \mathbb{N}^* \times [0, m]_{\mathbb{Z}} \to X$  defined by

$$H'(x,t) = H(x,m-t),$$

shows  $g \simeq^{EC} f$ .

Transitive: Suppose  $H : \mathbb{N}^* \times [0, m_1]_{\mathbb{Z}} \to X$  is an EC homotopy from f to g, and  $K : \mathbb{N}^* \times [0, m_2]_{\mathbb{Z}} \to X$  is an EC homotopy from g to h. Then the function  $L : \mathbb{N}^* \times [0, m_1 + m_2]_{\mathbb{Z}} \to X$  defined by

$$L(x,t) = \begin{cases} H(x,t) & \text{if } 0 \le t \le m_1; \\ K(x,t-m_1) & \text{if } m_1 \le t \le m_2 \end{cases}$$

is an EC homotopy from f to h.

Homotopy of trivial extensions of loops can be easily stated in terms of EC homotopy of the corresponding EC loops. The latter formulation is preferable since it does not require trivial extensions, which obviates the need for several technical lemmas. For example the proof given below for Proposition 4.13 is much easier than the corresponding statement for trivial extensions (see [1, Proposition 4.8], which has only a sketch of a proof from [16]); and the proof given below for Theorem 5.3 is somewhat simpler, being based on EC homotopy, than it would have been had we had to construct trivial extensions.

Given a path  $f: [0,m]_{\mathbb{Z}} \to X$ , we denote by  $f_{\infty}: \mathbb{N}^* \to X$  the function defined by

$$f_{\infty}(n) = \begin{cases} f(n) & \text{if } 0 \le n \le m; \\ f(m) & \text{if } n \ge m. \end{cases}$$

Given an EC path  $g: \mathbb{N}^* \to X$ , let

$$N_g = \min\{m \in \mathbb{N}^* \mid n \ge m \text{ implies } g(n) = g(m)\}$$

and let  $g_{-}: [0, N_g]_{\mathbb{Z}} = g|_{[0, N_g]_{\mathbb{Z}}}$ . We have the following.

# **Proposition 4.4.** Let X be a digital image.

a) Let  $f : \mathbb{N}^* \to X$  be an EC path. Then  $(f_-)_{\infty} = f$ .

b) Let  $f : [0,m]_{\mathbb{Z}} \to X$  be a path in X. Then f is a trivial extension of  $(f_{\infty})_{-}$ . We have  $f = (f_{\infty})_{-}$  if and only if either m = 0 or m > 0 and  $f(m-1) \neq f(m)$ .

*Proof.* These assertions are immediate consequences of the definitions above.  $\hfill \square$ 

**Lemma 4.5.** Let  $f, g : [0, m]_{\mathbb{Z}} \to X$  be paths with  $f \simeq g$ . Then  $f_{\infty} \simeq^{EC} g_{\infty}$ . If the homotopy from f to g holds the endpoints fixed, then so does the induced EC homotopy from  $f_{\infty}$  to  $g_{\infty}$ .

*Proof.* Let  $H : [0,m]_{\mathbb{Z}} \times [0,k]_{\mathbb{Z}} \to X$  be a homotopy of f to g. Consider  $G : \mathbb{N}^* \times [0,k]_{\mathbb{Z}} \to X$ , defined as follows:

$$G(s,t) = \begin{cases} H(s,t) & \text{if } s \le m \\ H(m,t) & \text{if } s > m. \end{cases}$$

Clearly G is an EC homotopy of  $f_{\infty}$  to  $g_{\infty}$ . Further, G holds the endpoints fixed if H does so.

**Lemma 4.6.** Let f and g be EC homotopic EC paths in X. Then  $f_{-}$  and  $g_{-}$  have homotopic trivial extensions. If f and g are homotopic holding the endpoints fixed, then  $f_{-}$  and  $g_{-}$  have trivial extensions that are homotopic holding the endpoints fixed.

*Proof.* Let  $N_f, N_g$  be as defined above. Without loss of generality,  $N_f \leq N_g$ . Let  $H : \mathbb{N} \times [0, m]_{\mathbb{Z}} \to X$  be a homotopy from f to g. Let  $H' : [0, N_g] \times [0, m]_{\mathbb{Z}} \to X$  be the restriction of H to  $[0, N_g] \times [0, m]_{\mathbb{Z}}$ . It is easily seen that H' is a homotopy between a trivial restriction f' of  $f_-$  and the function  $g_-$ , where  $f' : [0, N_g]_{\mathbb{Z}} \to X$  is defined by

$$f'(n) = \begin{cases} f(n) = f_{-}(n) & \text{if } 0 \le n \le N_f; \\ f(N_f) & \text{if } N_f \le n \le N_g. \end{cases}$$

Further, if H holds the endpoints fixed, then so does H'.

**Lemma 4.7.** Let  $f : [0,m]_{\mathbb{Z}} \to X$  be a loop based at  $x_0 \in X$  and  $\overline{f} : [0,n]_{\mathbb{Z}} \to X$  be a trivial extension of f. Then  $f_{\infty}$  and  $\overline{f}_{\infty}$  are EC homotopic with fixed endpoints.

*Proof.* We will prove the Lemma in the case that  $\overline{f}$  is obtained from f by inserting a single trivial loop. The full result follows by induction. Specifically, let  $f = f_1 * f_2$  and  $\overline{f} = f_1 * c * f_2$ , where c is a trivial loop. Say that  $f_1 : [0,m]_{\mathbb{Z}} \to X$  and  $f_2 : [0,n]_{\mathbb{Z}} \to X$  and  $c : [0,k]_{\mathbb{Z}} \to X$ . Then consider  $H : \mathbb{N}^* \times [0,k]_{\mathbb{Z}} \to X$  given by:

$$H(s,t) = \begin{cases} f_1(s) & \text{if } 0 \le s \le m; \\ c(s-m) & \text{if } m \le s \le m+t; \\ f_2(s-(m+t)) & \text{if } m+t \le s \le m+t+n; \\ x_0 & \text{if } m+t+n \le s. \end{cases}$$

At time stage t we have  $H_t = (f_1 * c_{|_{[0,t]_z}} * f_2)_\infty$ , so H is an EC homotopy of  $f_\infty$  to  $\overline{f}_\infty$  as desired. Further, H fixes the endpoints, since  $H(0,t) = f_1(0)$  for all t and  $H(x,t) = f_2(n)$  for all  $x \ge m + t + n$  and all t.

**Theorem 4.8.** Let f and g be loops in X having some common basepoint p. Then there are trivial extensions  $\overline{f}, \overline{g}$  of f, g respectively with  $\overline{f} \simeq \overline{g}$  with fixed endpoints if and only if  $f_{\infty}$  and  $g_{\infty}$  are EC homotopic with fixed endpoints.

*Proof.* First we assume that there are trivial extensions  $\bar{f}, \bar{g}$  with  $\bar{f} \simeq \bar{g}$  fixing endpoints. Then by Lemmas 4.7 and 4.5 we have  $f_{\infty} \simeq^{EC} \bar{f}_{\infty} \simeq^{EC} \bar{g}_{\infty} \simeq^{EC} g_{\infty}$  and all homotopies fix the endpoints as desired.

For the converse assume that  $f_{\infty} \simeq^{EC} g_{\infty}$  with fixed endpoints. Let  $H : \mathbb{N}^* \times [0, k]_{\mathbb{Z}} \to X$  be the EC homotopy. Since H fixes the endpoints (at p) and has only finitely many stages, there must be some M such that H(s, t) = p for all  $s \geq M$  and for all t.

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Let  $\overline{f}, \overline{g} : [0, M]_{\mathbb{Z}} \to X$  be the restrictions of  $f_{\infty}, g_{\infty}$  respectively to  $[0, M]_{\mathbb{Z}}$ . Then  $\overline{f} = f * c$  is a trivial extension of f, where c is a trivial loop at p. Similarly  $\overline{g}$  is a trivial extension of g.

Let  $\overline{H} : [0, M]_{\mathbb{Z}} \times [0, k]_{\mathbb{Z}} \to X$  be the restriction of H to  $[0, M]_{\mathbb{Z}} \times [0, k]_{\mathbb{Z}}$ . Then H is a homotopy of  $\overline{f}$  to  $\overline{g}$  fixing the endpoints as desired.  $\Box$ 

It is natural to overload the \* notation as follows.

**Definition 4.9.** For  $x_0 \in X$ , let  $f_0, f_1 : \mathbb{N}^* \to X$  be  $x_0$ -based EC loops in X. Define  $f_0 * f_1 : \mathbb{N}^* \to X$  by

$$f_0 * f_1(n) = \begin{cases} f_0(n) & \text{if } 0 \le n \le N_{f_0}; \\ f_1(n - N_{f_0}) & \text{if } N_{f_0} \le n. \end{cases}$$

It is easily seen that  $f_0 * f_1$  is well defined and is an EC loop in X. The \* operator on EC loops has the following properties.

### Proposition 4.10.

- Let  $f, g: \mathbb{N}^* \to X$  be  $x_0$ -based EC loops, for some  $x_0 \in X$ . Then  $f_- * g_- = (f * g)_-$ .
- Let  $f: [0,m]_{\mathbb{Z}} \to X$ ,  $g: [0,n]_{\mathbb{Z}} \to X$  be  $x_0$ -based EC loops, for some  $x_0 \in X$ . Then  $f_{\infty} * g_{\infty} = (f * g)_{\infty}$ .

*Proof.* These properties are simple consequences of Definition 4.9.  $\Box$ 

**Lemma 4.11.** Let f, g, g' be EC loops in X at a common basepoint, with  $g \simeq^{EC} g'$  holding the endpoints fixed. Then  $f * g \simeq^{EC} f * g'$  holding the endpoints fixed.

*Proof.* Let  $H : \mathbb{N}^* \times [0, m] \to X$  be the EC homotopy from g to g', and let  $L : \mathbb{N}^* \times [0, m] \to X$  be given by

$$L(s,t) = (f * H_t)(s).$$

Then L is a EC homotopy from f \* g to f \* g' holding the endpoints fixed.  $\Box$ 

In order to prove Proposition 4.13 below, we must take care in how we mimic the proof of Lemma 4.11 on the first factors of the \* products, as shown by the following.

**Example 4.12.** Let  $f, g : \mathbb{N}^* \to [0, 1]_{\mathbb{Z}}$  be defined by

$$f(n) = g(n) = \begin{cases} n & \text{if } n \in \{0, 1, 2\}; \\ 1 & \text{if } n = 3; \\ 0 & \text{if } n > 3. \end{cases}$$

Then there is an EC homotopy  $H : \mathbb{N}^* \times [0,2]_{\mathbb{Z}} \to [0,1]_{\mathbb{Z}}$  from f to f such that the function  $K : \mathbb{N}^* \times [0,2]_{\mathbb{Z}} \to [0,1]_{\mathbb{Z}}$  defined by  $K(n,t) = H_t(n) * g(n)$  is not continuous in t, where  $H_t : \mathbb{N}^* \to [0,1]_{\mathbb{Z}}$  is the induced function  $H_t(n) = H(n,t)$ .

*Proof.* Define 
$$H(n,t)$$
 by  $H(n,0) = f(n) = g(n) = H(n,2)$ ,  
$$H(n,1) = \begin{cases} f(n) & \text{if } n \neq 5; \\ 1 & \text{if } n = 5. \end{cases}$$

It is easy to see that H is a homotopy. However,  $K = H_0 * g = H_2 * g$ and  $L = H_1 * g$  are represented respectively by the sequences

$$(K(0), K(1), K(2), \ldots) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 0, \ldots)$$

 $(L(0), L(1), L(2), \ldots) = (0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 0, \ldots).$ 

In particular,  $H_0 * g(6) = 2$  and  $H_1 * g(6) = 0$ , so at n = 6,  $H_t * g$  is not continuous in t.

**Proposition 4.13.** Let f, f', g, g' be EC loops in X at a common basepoint such that  $f \simeq^{EC} f'$  and  $g \simeq^{EC} g'$  with both homotopies holding the endpoints fixed. Then we have  $f * g \simeq^{EC} f' * g'$  holding the endpoints fixed.

*Proof.* By Lemma 4.11 we have  $f * g \simeq^{EC} f * g'$  holding the endpoints fixed.

By an argument similar to that of the proof of Lemma 4.11 we will show that  $f * g' \simeq^{EC} f' * g'$ . Example 4.12 shows that  $H_t * g'$  will not necessarily be continuous in t; however, this is easily fixed by inserting an extra constant segment in the first factor. In particular, let  $H : \mathbb{N}^* \times [0, m]_{\mathbb{Z}} \to X$  be an EC homotopy from f to f' that holds the endpoints fixed. Let  $M = \max\{N_{H_t} \mid t \in [0, m]_{\mathbb{Z}}\}$ . For each  $t \in [0, m]_{\mathbb{Z}}$ , let  $c_t : [0, M - N_{H_t}]_{\mathbb{Z}} \to \{x_0\}$  be a constant function. Then the function  $K : \mathbb{N}^* \times [0, m]_{\mathbb{Z}} \to X$ defined by  $K(n, t) = (H_t * c_t * g')(n)$  is an EC homotopy from f \* g' to f' \* g' that holds the endpoints fixed.

Thus by transitivity of EC homotopy we have  $f * g \simeq^{EC} f' * g'$ , holding endpoints fixed.

Let  $G(X, x_0)$  be the set of all EC homotopy classes of EC loops in X based at  $x_0$ .

**Proposition 4.14.**  $G(X, x_0)$  with the  $\cdot$  operation defined by  $[f] \cdot [g] =$ [f \* g] is a group.

*Proof.* By Proposition 4.13, the  $\cdot$  operation is closed and well defined on  $G(X, x_0)$ . Clearly, the EC pointed homotopy class of the constant map  $c(n) = x_0$  for all  $n \in \mathbb{N}^*$  is the identity element. Given an  $x_0$ -based EC loop  $f: \mathbb{N}^* \to X$ , the function  $g: \mathbb{N}^* \to X$  defined by

$$g(n) = \begin{cases} f(N_f - n) & \text{if } 0 \le n \le N_f; \\ x_0 & \text{if } n \ge N_f, \end{cases}$$
for [f]

gives an inverse for |f|.

We have the following analog of Theorem 2.10.

**Theorem 4.15.** Suppose  $F: (X, \kappa, x_0) \to (Y, \lambda, y_0)$  is a pointed continuous function. Then F induces a homomorphism  $F_*: G(X, x_0) \to G(Y, y_0)$ defined by  $F_*([f]) = [F \circ f].$ 

*Proof.* Given  $x_0$ -based EC loops  $f, g : \mathbb{N} \to X$ , we have, by using Propositions 4.4 and 4.10,

$$F([f * g]) = [F \circ (f * g)] = [F \circ ((f * g)_{-})_{\infty}] = [((F \circ f_{-}) * (F \circ g_{-}))_{\infty}]$$
$$= [(F \circ f_{-})_{\infty} * (F \circ g_{-})_{\infty}] = [(F \circ f) * (F \circ g)].$$
The assertion follows.

The assertion follows.

The main result of this section is the following.

**Theorem 4.16.** Given a digital image X and a point  $x_0 \in X$ , the groups  $G(X, x_0)$  and  $\Pi_1(X, x_0)$  are isomorphic.

*Proof.* Let  $F: \Pi_1(X, x_0) \to G(X, x_0)$  be defined by  $F([f]_X) = [f_\infty]_X$ , where  $[f_{\infty}]_X$  is the set of EC loops that are  $x_0$ -based in X and are EC homotopic in X to  $f_{\infty}$  holding the endpoints fixed.

From Lemma 4.6, F is one-to-one. Also, F is onto, since given an  $x_0$ -based EC loop f, we have  $[f] = F([f_-])$ . From Proposition 4.13, F is a homomorphism. The assertion follows. 

# 5. Homotopy Equivalence and Fundamental Groups

In the paper [3], it is asserted that digital images that are (unpointed) homotopy equivalent have isomorphic fundamental groups. However, the proof of this assertion is incorrect. Roughly, the flaw in the argument given in [3] is that insufficient care was given to making sure that a certain homotopy between two loops holds the endpoints fixed. In this section, we give a correction.

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**Theorem 5.1.** [2] Let  $(X, \kappa)$  be a digital image and let p, r be points of the same  $\kappa$ -component of X. Let q be a  $\kappa$ -path in X from p to r. Then the induced function  $q_{\#}: \Pi_1^{\kappa}(X, p) \to \Pi_1^{\kappa}(X, r)$  defined by  $q_{\#}([f]) = [q^{-1} * f * q]$  is an isomorphism.

Theorem 5.1 was proven in [2] for the version of the fundamental group based on finite loops. However, essentially the same argument makes Theorem 5.1 valid for the version of the fundamental group based on EC loops, stated below.

**Corollary 5.2.** Let  $(X, \kappa)$  be a digital image and let p, r be points of the same  $\kappa$ -component of X. Let q be a  $\kappa$ -path in X from p to r. Then the induced function  $q_{\#}: \prod_{1}^{\kappa}(X, p) \to \prod_{1}^{\kappa}(X, r)$  defined for a p-based EC loop f in X by  $q_{\#}([f]) = [(q^{-1})_{\infty} * f * q_{\infty}]$ , is an isomorphism.  $\Box$ 

**Theorem 5.3.** Suppose  $(X, \kappa)$  and  $(Y, \lambda)$  are (not necessarily pointed) homotopy equivalent digital images. Let  $F : X \to Y$ ,  $G : Y \to X$  be homotopy inverses. Let  $p \in X$ . Then  $\Pi_1^{\kappa}(X, p)$  and  $\Pi_1^{\lambda}(Y, F(p))$  are isomorphic groups.

*Proof.* Let  $F_* : \Pi_1^{\kappa}(X,p) \to \Pi_1^{\lambda}(Y,F(p))$  be the homomorphism induced by F according to Theorem 4.15. Let  $r = (G \circ F)(p)$ . Let  $G_* : \Pi_1^{\lambda}(Y,F(p)) \to \Pi_1^{\kappa}(X,r)$  be the homomorphism induced by G according to Theorem 4.15. Let  $H : X \times [0,m]_Z \to X$  be a homotopy from  $1_X$  to  $G \circ F$ . Let q be the path from p to r defined by q(t) = H(p,t).

For  $s \in [0,m]_{\mathbf{Z}}$ , let  $q_s : [0,m]_{\mathbf{Z}} \to X$  be the path from q(0) = p to q(s) = H(p,s) given by  $q_s(t) = q(\min\{s,t\})$ . For a *p*-based EC loop *f* in *X*, let  $K : \mathbb{N}^* \times [0,m]_{\mathbb{Z}} \to X$  be defined by

$$K(n,t) = (q_t * (H_t \circ f_-) * (q_t)^{-1})_{\infty}(n).$$

Since  $q_t$  is a path from r to  $q(t) = H(p,t) = H_t(f(0)) = H_t(f_-(N_f)) = (q_t)^{-1}(0)$ , K is well defined and, for each t, the induced function  $K_t$  is a EC loop based at p. Also, if we let  $\overline{p}$  denote the constant EC loop at p, then

$$K(n,0) = ((q_0) * (H_0 \circ f_-) * (q_0)^{-1})_{\infty}(n) =$$
$$(\overline{p} * f_- * \overline{p})_{\infty}(n) = f(n)$$

and

$$K(n,m) = (q_m * (H_m \circ f_-) * (q_m)^{-1}))_{\infty}(n) =$$
$$(q * (G \circ F \circ f_-) * q^{-1})_{\infty}(n).$$

Therefore, K is a EC homotopy from f to

$$(q * (G \circ F \circ f_{-}) * q^{-1})_{\infty} = q_{\infty} * (G \circ F \circ f_{-})_{\infty} * (q^{-1})_{\infty} = q_{\infty} * (G \circ F \circ f) * (q_{\infty})^{-1}$$

that keeps the endpoints fixed.

Let  $q_{\#}: \Pi_1^{\kappa}(X, p) \to \Pi_1^{\kappa}(X, r)$  be defined by  $q_{\#}([f]) = [q_{\infty} * f * (q_{\infty})^{-1}]$ . By the conclusion of the previous paragraph, the function  $q_{\#} \circ G_* \circ F_*$  is the identity map on  $\Pi_1^{\kappa}(X, p)$ . We know from Corollary 5.2 that  $q_{\#}$  is an isomorphism. It follows that  $F_*$  is onto and  $G_*$  is one-to-one. A similar argument shows that  $G_*$  is onto and  $F_*$  is one-to-one. Therefore,  $F_*$  is an isomorphism.  $\Box$ 

#### 6. FURTHER REMARKS

We have given the first example of two digital images with  $c_u$ -adjacencies that are homotopy equivalent but not pointed homotopy equivalent. We have introduced a variant of the loop equivalence, based on the notion of tight at the basepoint (TAB) pointed homotopy, and have explored properties of this notion. We have given an alternate but equivalent approach to the digital fundamental group based on EC loops that offers the advantage of avoiding the often-clumsy use of trivial extensions. We have provided a correction to the faulty proof of [3] that (unpointed) homotopy equivalent digital images have isomorphic fundamental groups.

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